Notes on Superconductivity - 1

A short note on the two fluid model

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This short note introduces, in a simple form, a few topics on the so-called two-fluid model. It does not replace a book chapter *at all*: it should be intended as a guide to the Lectures. An (incomplete) list of appropriate readings is reported in the Bibliography.
Where symbols are not defined, they refer to commonly intended significance. The basics of superconductivity, such as the London equations, are assumed to be known. Please note that ξ indicates here the Ginzburg-Landau coherence length.
It is commonly found that the ac response of a superconductor has a major importance. Depending on the audience, one may refer to important information about the microscopic state (e.g., measurements of the gap amplitude), to important features for applications (e.g., transmission lines, accelerating cavities, resonators for signal processing,...), and even to key experiments to assess the nature of unconventional superconductors (e.g., the discussion about "s-wave" or "d-wave" state in cuprates). Purpose of this Note is to give some element to understand the seemingly simple two-fluid model, and to bring to the attention of the student some of the links between the trivial treatment of the two-fluid model, and the underlying microscopic theory.

1.1 A few results from the microscopic theory

In this Section we report some results of the microscopic theory that will be needed later on. We do not demonstrate the relations here reported, and we refer to the Bibliography.

When a specimen is in the superconducting state, it attains the lowest free-energy level. In other words, the ground state is the superconducting state and any "normal" carrier appears as an excitation of such a state. Such quasiparticles (QPs) have some very relevant differences with respect to normal crystal electrons, most notably the gap in the excitation spectrum.[1]

[1] In this Section we are concerned with conventional, s-wave superconductors. See

The gap appears in many familiar quantities, such as the spectrum of the excitation and the density of states.

For what concerns the spectrum of excitations, we report in Fig. 1.1 the energy of the QPs vs. the corresponding energy for a hypotetical crystal electron.[2] The fact that the plot is *not* a straight line implies that the two objects are different: in fact, at low energy the QP spectrum exhibits a *gap* Δ_0. The fact that the plot *approaches* a straight line at high energies, implies that well above the Fermi level QPs behave like crystal electrons: at high energies, the presence of the gap has no strong relevance. The fact that there are two branches ("hole-like" and "electron-like") is a consequence of the paired state ($\{\mathbf{k}, -\mathbf{k}\}$) over which excitations appear. The energies of the QPs are related to the crystal electron spectrum by the following relation:

$$E_k = \sqrt{(\epsilon_k - E_F)^2 + \Delta_0^2} \tag{1.1}$$

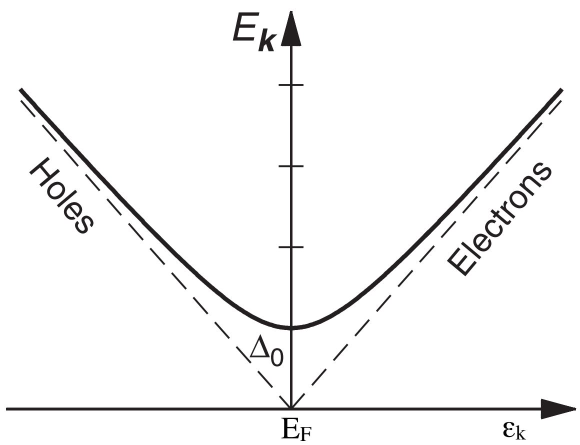

Figure 1.1: Quasiparticle spectrum: the energy of excited states (quasiparticles, QPs) E_k are plotted vs. the corresponding crystal-electron energy, ϵ_k. Should the two kinds of charge carriers be the same object, straight lines should be plotted. The gap clearly characterizes the QPs.

The second important quantity that we will need in the following is the QP density of states. In general, the crystal electron density of states (DOS, $N(E)$) is a varying function of the energy (we refer to 3D metals here).

further on for d-wave superconductors

[2]Please note that the QP energy is referred to the Fermi energy.

However, close to E_F, that is the region in which we are mostly interested, the energy variation of the DOS is not strong, and it can be taken as a constant (for simplicity). One finds, for the QP DOS, the relation:

$$N_{qp}(E) = \begin{cases} N(E_F)\frac{E_k}{\sqrt{E_k^2-\Delta_0^2}} & \text{for } E_k > \Delta_0 \\ 0 & \text{for } E_k < \Delta_0 \end{cases} \tag{1.2}$$

which is plotted in figure 1.2. Note once more that the QP energy is measured from the Fermi energy E_F.

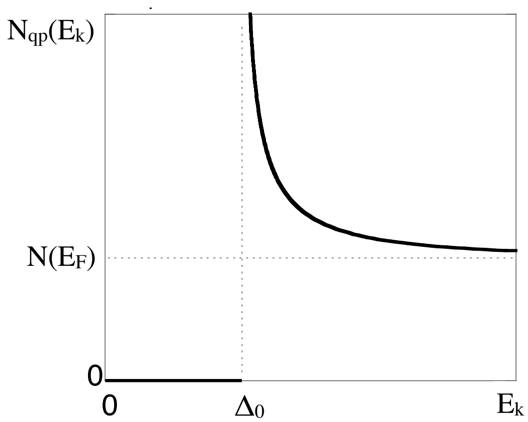

Figure 1.2: QP density of states in a s-wave superconductor in the superconducting state.

1.2 Two fluid model

The basic assumption of the two-fluid model is that the total current (density) can be written as a sum of two separate components, given by the current brought by Cooper pairs ("supercurrent", $\vec{J}_s$) and quasiparticles ("normal current", $\vec{J}_n$):

$$\vec{J} = \vec{J}_s + \vec{J}_n \tag{1.3}$$

This assumption implies that *two* kinds of charge carriers act *independently*, without interaction between them. In terms of the three relevant length, λ, ξ_0, ℓ (ξ_0 is the microscopic correlation length, giving a measure of the Cooper pair spatial correlation - "size" of the Cooper pair), one can safely apply the two-fluid model when the response is:

1. *local*, $\lambda \gg \xi_0, \ell$, so that the current at a certain location is not influenced

by the electric field at different locations, and
2. *clean*, $\ell \gg \xi_0$, so that the scattering processes (involved in QP response) do not affect the behavior of the Cooper pair.

We will see later an explicit calculation in this limit.

Thus, when the two-fuid model can be applied, the problem reduces to separately obtaining the "normal" and "superfluid" conductivities, σ_n and σ_s, respectively, that link the electric field to the respective current component: $\vec{J}_{n,s} = \sigma_{n,s}\vec{E}$. Once the conductivities have been obtained, from Eq.(1.3) one has the total conductivity $\sigma = \sigma_n + \sigma_s$.

The quasiparticle conductivity can be easily obtained from the Drude model. Let us take the simplest, trivial form of the Drude model as a simple example. We start from the equation of motion of n_n quasiparticles (per unit volume) of charge q. They give rise to a current $\vec{J}_n = n_n q \vec{v}_n$. The equation of motion has to include the scattering, and the simplest expression takes the form:[3]

$$m\frac{d\vec{v}_n}{dt} = q\vec{E} - \frac{m}{\tau}\vec{v}_n \tag{1.4}$$

where the drag term $\frac{m}{\tau}\vec{v}_n$ is due do QP scattering. In the harmonic regime, where $\vec{J}_n(t) = \vec{J}\mathrm{e}^{\mathrm{i}\omega t}$ and $\vec{E}(t) = \vec{E}\mathrm{e}^{\mathrm{i}\omega t}$, one finds the well-known Drude conductivity:

$$\sigma_n = \frac{n_n q^2}{m}\frac{1}{\mathrm{i}\omega + 1/\tau} \tag{1.5}$$

The superfluid conductivity directly comes from the first London equation:

$$\frac{d\vec{J}_s}{dt} = \frac{n_s q^2}{m}\vec{E} \tag{1.6}$$

which can be easily derived from the fluxoid quantization [1], or from the free acceleration of charge carriers (see Eq.(1.4) with $\tau \to \infty$).[4] Using the same harmonic regime as above, one gets

$$\sigma_s = -\mathrm{i}\frac{n_s q^2}{m\omega} = -\frac{\mathrm{i}}{\mu_0 \omega \lambda^2} \tag{1.7}$$

where we have used the definition of the London penetration depth, and we identified the superfluid charge carrier density with the bulk order parameter, $n_s = |\psi_\infty|^2$.

It will prove useful to introduce the total carrier density, n, and the quasiparticle and superfluid fractions, x_n and x_s, so that $x_n + x_s = 1$ and $n = n_s + n_n = n(x_s + x_n)$.

[3] This exceedingly simple model is presented for illustrative purposes. In general, one has to perform the appropriate integration in $\vec{k}$ space, e.g. using the Boltzmann equation formalism.

[4] There are several caveats to this approach. The interested reader can be referred to [2] for a discussion.

In the Literature one finds the two-fluid conductivity as written in many slightly different ways. We will report for future reference the expressions that we will refer to in the following. The total conductivity can be written as:

$$\sigma = \sigma_s + \sigma_n = \sigma_1 - \mathrm{i}\sigma_2 \tag{1.8}$$

and the term $\omega\tau$ in σ_n can often be neglected (but one should be careful: this assumption, almost straightforward for normal metals in the sub-GHz range, may be not justified in cuprate superconductors below T_c, or at frequencies near the THz regime -a typical combination of the recent research).

One might argue that the most neutral (with respect to further assumptions) expression is

$$\sigma = \frac{n_n q^2}{m}\frac{1}{\mathrm{i}\omega + 1/\tau} - \frac{\mathrm{i}}{\mu_0\omega\lambda^2} \tag{1.9}$$

In this case, n_n is the concentration of QP, $q = e$ and m are the electron charge and QP effective mass, respectively, and $\mathrm{i}\omega + 1/\tau$ can be taken as appropriate average over the QP spectrum. λ is a measurable quantity of the superconducting state. With $\omega\tau \ll 1$, Eq.(1.9) acquires the immediate circuital analogous of two parallel impedances: a resistance (resistivity) $\sigma_n^{-1} = \sigma_1^{-1} = m/n_n q^2\tau$ and an inductance (inductivity) $\mu_0\lambda^2 = 1/\omega\sigma_2$. This consideration justifies the commonly employed, albeit slightly imprecise, sentence “the superconducting electrons short-circuit the normal electrons at low frequencies”.

Introducing x_n and x_s with the sum rule:

$$x_n + x_s = 1 \tag{1.10}$$

(thus implicitly considering the superfluid charge carriers as “superlectrons”, with the same charge as QPs), introducing the dc Drude conductivity $\sigma_0 = nq^2\tau/m$, one can also write

$$\sigma = \sigma_0\left[\frac{x_s}{\mathrm{i}\omega\tau} + \frac{x_n}{1+\mathrm{i}\omega\tau}\right] \tag{1.11}$$

In terms of characteristic lengths, one introduces the so-called skin depth:

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma_0}} \tag{1.12}$$

(see also below), and the two-fluid conductivity can be manipulated to yield:

$$\sigma = \frac{1}{\omega\mu_0}\left[-\frac{\mathrm{i}}{\lambda^2} + \frac{2}{\delta^2}\frac{x_n}{1+\mathrm{i}\omega\tau}\right] \tag{1.13}$$

Note that in the whole two-fluid treatment the microscopic aspects have been carefully "hidden" in:
1. the temperature dependence of x_n and x_s
2. the QP relaxation time τ, which does not necessarily coincide with the relaxation time in the normal state.

Since $\lambda^2(T) = \lambda^2(0)/x_s(T)$,[5] a measure of the temperature dependence of λ might yield information on the microscopic details, through x_s. Thus, a connection between microscopic and 'two-fluid" parameters would prove useful.

1.3 A connection between the two-fluid conductivity and the microscopic parameters

The two-fluid conductivity is a very common tool to analyse the experimental data. Thus, we provide here an example (in the local and clean limit) on how a seemingly classical analysis can yield insights into the microscopic mechanisms.

We start from a result (that we do not demonstrate here) of the BCS theory (see [3]). In the clean limit, the QP conductivity (σ_n, in our notation) can be written explicitly to yield:

$$\sigma_n = \frac{nq^2}{m} \int_{-\infty}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E}\right) \frac{N_{qp}(E)}{N(E_F)} \frac{1}{\mathrm{i}\omega + 1/\tau} dE \tag{1.14}$$

Where $f_{FD}$is the Fermi-Dirac distribution. Multiplying and dividing by $\int_{-\infty}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E}\right) N_{qp}(E) dE$, one gets:

$$\sigma_n = \frac{nq^2}{m} \int_{-\infty}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E}\right) \frac{N_{qp}(E)}{N(E_F)} dE \left\langle \frac{1}{\mathrm{i}\omega + 1/\tau} \right\rangle \tag{1.15}$$

where $\langle\rangle$ indicates the average over the distribution $\left(-\frac{\partial f_{FD}}{\partial E}\right) \frac{N_{qp}(E)}{N(E_F)}$. Eq.(1.15) can be brought to the following familiar Drude-like form:

$$\sigma_n = \frac{nq^2\tau_n}{m} x_n \left\langle \frac{\tau/\tau_n}{1 + \mathrm{i}\omega\tau} \right\rangle = \sigma_0 x_n \left\langle \frac{\tau/\tau_n}{1 + \mathrm{i}\omega\tau} \right\rangle \tag{1.16}$$

once one makes the (somewhat intuitive) identification:

$$x_n = \int_{-\infty}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E}\right) \frac{N_{qp}(E)}{N(E_F)} dE \tag{1.17}$$

(so that $n_n = nx_n$), *and* the QP relaxation time τ has been normalized by the *normal state* relaxation time τ_n. Remember that QP are not strictly

[5] Whence the name "superfluid fraction" for the measurable quantity $\lambda^2(0)/\lambda^2(T)$.

"normal" crystal electrons, and as such they can undergo different scattering processes (an example will be given in the analysis of cuprates).

The important link between BCS theory and two-fluid is given by Eq.(1.17): in the assumption that the two fluids are uncorrelated, using the sum rule, Eq.(1.10) yields an explicit expression for the temperature dependence of the superfluid fraction:

$$x_s = 1 - x_n = 1 - \int_{-\infty}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E} \right) \frac{N_{qp}(E)}{N(E_F)} dE \tag{1.18}$$

At sufficiently low temperatures, one finds (the calculation is reported in Sec.1.4) the important relation:

$$x_s(T) \simeq 1 - \sqrt{\frac{2\pi\Delta(0)}{kT}} \mathrm{e}^{-\Delta(0)/kT} \tag{1.19}$$

where $\Delta(0)$ is the gap extrapolated at zero temperature. Thus, a measurement of λ at low T yields the size of the gap, if an activated process is detected. Otherwise, if activated behavior is not detected, it may contribute to evidence for nonconventional superconductivity (see Sec.1.7).

1.4 Derivation of Eq.(1.19)

In this Section we derive Eq.(1.19). We will refer all the energies to the Fermi level, $E - E_F \to E$. We indicate with Δ the gap. To evaluate the integral, we recall Eq.(1.2), that allows to write (note that $N_{qp}(E) = N_{qp}(-E)$):

$$\int_{-\infty}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E} \right) \frac{N_{qp}(E)}{N(E_F)} dE = 2 \int_{\Delta}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E} \right) \frac{N_{qp}(E)}{N(E_F)} dE \tag{1.20}$$

Inserting $N_{qp}(E) = E/\sqrt{E^2 - \Delta^2}$, one writes:

$$\begin{aligned} 2 \int_{\Delta}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E} \right) \frac{E}{\sqrt{E^2 - \Delta^2}} dE &= 2 \int_{\Delta}^{+\infty} \left(-\frac{\partial f_{FD}}{\partial E} \right) \frac{d}{dE} \sqrt{E^2 - \Delta^2} dE \\ &= 2 \int_{\Delta}^{+\infty} \left(\frac{\partial^2 f_{FD}}{\partial E^2} \right) \sqrt{E^2 - \Delta^2} dE \end{aligned}$$

where the last equality has been obtained by integrating by parts, and recalling that $\partial f_{FD}/\partial E \to 0$ ($E \to \infty$). Now, if a gap exists, the QP energy is necessarily larger than the gap itself (remember that we are measuring the gap from E_F, and there are no accessible states between E_F and $E_F + \Delta$, see figure 1.1, this is explicitly stated with Δ as the lower limit of integration), so that at sufficiently low T one can approximate $\partial^2 f_{FD}/\partial E^2 \approx (-1/kT)^2 \exp -E/kT$. Thus, one can write

$$2\int_{\Delta}^{+\infty}\left(\frac{\partial^2 f_{FD}}{\partial E^2}\right)\sqrt{E^2-\Delta^2}dE \approx 2\int_{\Delta}^{+\infty}\left(-\frac{1}{kT}\right)^2 \mathrm{e}^{-E/kT}\sqrt{E^2-\Delta^2}dE$$
$$= 2\int_0^{\infty}\mathrm{e}^{-\delta}\mathrm{e}^{-x}\sqrt{x(x+2\delta)}dx$$

where we have changed the integration variable as $x=(E-\Delta)/kT$, and we have defined $\delta=\Delta/kT$. The latter integral has an explicit expression, that can be found in the tables [4]. The analytical result reads:

$$2\int_0^{\infty}\mathrm{e}^{-\delta}\mathrm{e}^{-x}\sqrt{x(x+\delta)}dx = 2\frac{1}{\sqrt{\pi}}\frac{2\Delta}{kT}\Gamma\left(\frac{3}{2}\right)K_{-1}(\delta)$$

Where Γ is the Euler Gamma function, $\Gamma(3/2)=\sqrt{\pi}/2$, and K_{-1} is the modified Bessel function of order -1. Using the fact that $K_{-x}=K_x$ when $x\in\mathbb{R}$, and recalling that we are in the limit of low temperature, so that $\Delta/kT=\delta\ll 1$ and $\Delta\approx\Delta(0)$, we use the asymptotic expression [4] $K_1(\delta)\approx\sqrt{\frac{\pi}{2\delta}}\mathrm{e}^{-\delta}$ to obtain the desired result:

$$x_n = \int_{-\infty}^{+\infty}\left(-\frac{\partial f_{FD}}{\partial E}\right)N_{qp}(E)dE \simeq \sqrt{\frac{2\pi\Delta(0)}{kT}}\mathrm{e}^{-\Delta(0)/kT} \quad (1.21)$$

which, together with the sum rule, Eq.(1.10), demonstrates Eq.(1.19).

1.5 Surface impedance: normal metals.

When a specimen is much thicker than the electromagnetic (e.m. in the following) screening length (to be specified later), the e.m. field decays almost exponentially in the specimen, and the measured e.m. response is not directly the conductivity. We introduce in the following the concept of surface impedance, limiting the treatment to (i) the local limit, (ii) plane wave e.m. field, (iii) normal incidence of the e.m. field over a planar surface of a semi-infinite specimen, and (iv) harmonic regime, $\mathrm{e}^{\mathrm{i}\omega t}$. Many details are not worked out, and can be found in any treatise of classical electromagnetism (e.g., [5])

Assume a semiinfinite specimen, occupying the $\{x>0\}$ half space, and a magnetic field $\vec{H}\parallel\hat{z}$. The Maxwell equations:

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \quad (1.22)$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad (1.23)$$

can be combined (making the curl of the first) with the constitutive equations (we assume a nonmagnetic material, $\mu = \mu_0$)

$$\vec{D} = \epsilon \vec{E} \tag{1.24}$$

$$\vec{J} = \sigma \vec{E} \tag{1.25}$$

to yield, in the harmonic regime ($\vec{H}(t) = \vec{H} \mathrm{e}^{\mathrm{i}\omega t}$), the decay of the magnetic field and the complex attenuation constant:

$$\vec{H}(x) = \vec{H}(0) \mathrm{e}^{-\gamma x} \tag{1.26}$$

$$\gamma = \mathrm{i}\omega \sqrt{\mu_0 \epsilon \left(1 - \mathrm{i}\frac{\sigma}{\omega\epsilon}\right)} \tag{1.27}$$

When $\sigma \gg \omega\epsilon$, appropriate to a good conductor, one can approximate

$$\gamma \simeq \mathrm{i}\omega \sqrt{\mu_0 \epsilon \left(-\mathrm{i}\frac{\sigma}{\omega\epsilon}\right)} = (1+\mathrm{i})\sqrt{\frac{\mu_0 \omega \sigma}{2}} = \frac{1+\mathrm{i}}{\delta} \tag{1.28}$$

Where the last equality defines the complex skin depth δ. When a real conductivity can be considered, one recovers Eq.(1.12).

Due to the exponential decay of the e.m. field inside the (super)conductor, the response is given by an integration over the propagation direction (here, x). One defines the appropriate response function, the *surface impedance*, as:

$$Z_s = \frac{E_{\|}}{\int_0^{\infty} J(x) dx} = \frac{E_{\|}}{H_{\|}} = R_s + \mathrm{i}X_s \tag{1.29}$$

where the subscript "$\|$" indicates the component of the appropriate vector parallel to the surface of the sample, the second equality comes from Eq.s (1.22) and (1.26), and the last equality defines the *surface resistance* R_s and the *surface reactance* X_s. In the harmonic regime and the geometry here proposed, from Eq. (1.23) and the fact that $\vec{E}(x) = \vec{E}(0)\mathrm{e}^{-\gamma x}$ one finds $-\gamma E_{\|} = -\mathrm{i}\omega\mu_0 H_{\|}$, so that :

$$Z_s = \sqrt{\frac{\mu_0}{\epsilon}} \frac{1}{\sqrt{1 - \mathrm{i}\sigma/\omega\epsilon}} \simeq (1+\mathrm{i})\sqrt{\frac{\mu_0 \omega}{2\sigma}} = \frac{1+\mathrm{i}}{\sigma\delta} \tag{1.30}$$

where the approximate equality holds in good conductors.

When $\sigma \in \mathbb{R}$, one speaks of the normal skin effect. In this case one finds:

$$R_s = X_s = \sqrt{\frac{\mu_0 \omega}{2\sigma}} = \frac{1}{\sigma\delta} \tag{1.31}$$

An important feature resides in the frequency dependence: in conventional metals in the normal skin effect regime one has the so-called *square root scaling*,

$$R_s = X_s \propto \omega^{1/2} \tag{1.32}$$

This frequency scaling is often observed to a good approximation in normal metals. As we will see, superconductors behave differently.

1.6 Surface impedance: superconductors.

In a superconductor, while one can safely assume $\sigma \gg \omega\epsilon$, the conductivity is complex, as given by Eq.(1.8). Thus, in general, the complex surface impedance takes the form

$$Z_s = \sqrt{\frac{\mathrm{i}\mu_0\omega}{\sigma_1 - \mathrm{i}\sigma_2}} \tag{1.33}$$

However, one can easily find[6] that even at temperatures as close to the transition as $T = 0.95T_c$, one has $\sigma_2 \gg \sigma_1$. Thus, one can expand Eq.(1.33) to first order in the small parameter $\mathrm{i}\sigma_1/\sigma_2$, to obtain the following approximate relations:

$$Z_s \simeq \mathrm{i}\sqrt{\frac{\mu_0\omega}{\sigma_2}} + \frac{\sqrt{\mu_0\omega}}{2}\frac{\sigma_1}{\sigma_2^{3/2}} \tag{1.34}$$

Using $\sigma_2 \simeq 1/mu_0\omega\lambda^2$, and $\sigma_1 \simeq \sigma_0(\tau/\tau_n)x_n$, one finds

$$X_s \simeq \mu_0\omega\lambda \tag{1.35}$$

$$R_s \simeq \frac{\mu_0^2\sigma_0}{2}\frac{\tau}{\tau_n}\omega^2\lambda^3 x_n \tag{1.36}$$

In conventional superconductor it is customary to take $\tau/\tau_n = 1$. However, this factor is of significant importance in cuprates, so it is useful to write it down explicitly in order to keep track of the approximations.

Eq.s (1.35) and (1.36) yield an essential tool for the interpretation of the experiments: in a wide temperature range (not too close to T_c), a measure of the surface reactance is a direct measurement of the London penetration depth, in particular for what concerns the temperature dependence. Moreover, the simple two-fluid model predicts (again, in a wide temperature range) the so called *quadratic frequency scaling* (ω^2) in the surface resistance. This point is relevant both for fundamental physics, and for applications: with lowering frequency, the losses drop much faster than in a normal conductor.

The same Eq.s (1.35) and (1.36) can be exploited further in the low-T limit, where Eq.s (1.19) and (1.21) hold. In that case, one can take $\lambda \approx \lambda(0)$ in Eq.(1.36) (low T, λ saturates), and find:

$$R_s \simeq A\mathrm{e}^{-\Delta(0)/kT} \tag{1.37}$$

where $A \sim T^{-1/2}$ depends weakly on temperature, so that the low temperature surface resistance is dominated by the exponential. Thus, one has immediately $\log R_s = \log A - \frac{\Delta(0)}{kT}$. A plot of $\log R_s$ vs. $1/T$ should yield a straight line, whose slope is a direct measure of the superconducting gap.

[6] The reader is invited to check this assertion as an exercise, by choosing some superconducting material and respective material parameters.

1.7 Two-fluid model and d-wave superconductors.

We present here an extremely succint overview of the behavior of the two-fluid conductivity in a d-wave superconductor.

A d-wave superconductor has, as one of the most relevant features, that the superconducting gap is not fully developed in momentum space, and it has a $\vec{k}$ dependence. The specific shape of $\Delta_{\mathbf{k}}$ arises from the specific pairing mechanism and heavily influences the physical properties of the superconductor. In conventional superconductors the experiments showed an isotropic (s-wave) gap symmetry which is naturally explained within BCS phonon coupling. On the other hand, in cuprates the experimental results point to an anisotropic gap for which the most widely accepted form is the $d_{x^2-y^2}$, described in momentum space by the function:[7]

$$\Delta_{\mathbf{k}} = \Delta_0 \cos(2\phi) \tag{1.38}$$

where the constant $\Delta_0 > 0$ is the maximum gap amplitude and ϕ is the angle that the vector $\mathbf{k}$ in the k_x-k_y momentum plane makes with the k_x axis. A plot of equation (1.38) in the k_x-k_y plane is provided in figure 1.3 and compared to the isotropic s-wave (fully gapped) gap.

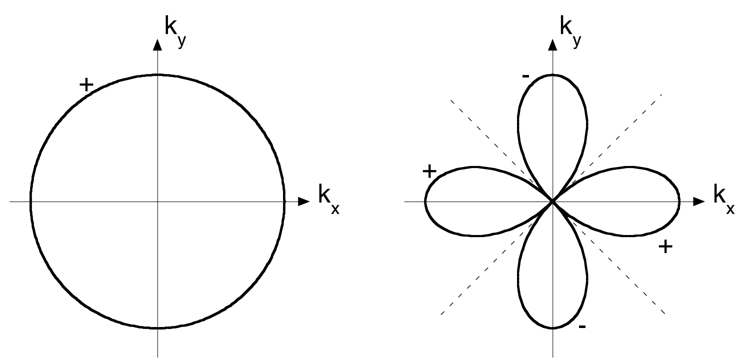

Figure 1.3: Isotropic s-wave and $d_{x^2-y^2}$ gap symmetries in momentum space.

A key feature of the $d_{x^2-y^2}$ gap amplitude, besides its in-plane anisotropy, is the presence of zeroes (*nodes*) in specific regions of the Fermi surface. This fact can be appreciated by computing the averaged density of states $\bar{N}(E)$, which is the quantity that enters effectively in defining many physical properties. It can be shown that the average QP density of states takes the

[7]It should be emphasized that nodal QP with a DOS given by equation 1.4 are common to many gap symmetries, provided the existence of nodal lines. The present choice of a $d_{x^2-y^2}$ symmetry is for illustrative purposes only.

form:

$$\bar{N}(E) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\bar{N}(E,\phi)}{N(E_F)} d\phi$$

where:

$$\bar{N}(E) = \begin{cases} N(E_F)\frac{E}{[E^2-\Delta_0^2\cos^2(2\phi)]^{1/2}}, & E > \Delta_0(\phi) \\ 0, & E < \Delta_0(\phi) \end{cases}$$

so that it it turns out:

$$\bar{N}(E) = \begin{cases} \frac{2}{\pi}\frac{E}{\Delta_0}K\left(\frac{E}{\Delta_0}\right) & \text{for } 0 \le E < \Delta_0 \\ \frac{2}{\pi}K\left(\frac{\Delta_0}{E}\right) & \text{for } E > \Delta_0 \end{cases}$$

where K is the complete elliptic integral function of the first kind [4].

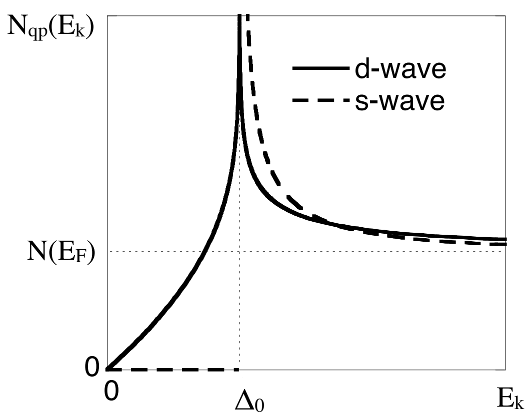

Figure 1.4: DOS of a d-wave superconductor, with isotropic s-wave DOS given for comparison.

The effective DOS is plotted in figure 1.4 together with the result for the isotropic s-wave gap discussed in Sec.1.1. The presence of nodes in the gap has a dramatic impact on the physics of d-wave superconductors. Besides the different divergences at $E = \Delta_0$ (logarithmic and square root for d- and s- wave, respectively), the main fundamental feature is the presence of a non-zero DOS down to the Fermi level. Moreover, the DOS varies linearly with $\frac{E}{\Delta_0}$ close to E_F, at $E \ll \Delta_0$. Due to the features of the the Fermi-Dirac function, the states close to the Fermi level (in the region of linear E dependence) are the most relevant in the determination of the physical properties. Thus, it is mostly the low energy excitations that dictate the physics of cuprates: a very different situation with respect to fully gapped, s-wave superconductors.

The low-energy states states allow for the existence of zero-energy excitations with respect to the superconducting ground state. These excitations, the so-called nodal quasiparticles (QP), can therefore be created by any energy perturbation of the superconducting system, however small. The most dramatic effect is the removal of the activated behaviors at low T of many thermal and electrodynamic properties, typical of conventional superconductors, replaced by power law dependences.

The existence of nodes in the gap can be inferred from various types of measurements. An elementary, semi-qualitative argument is given in the following. Let us reconsider Eq.(1.17) in the light of the QP DOS, Eq.(1.39) and figure 1.4. For energies above the gap ($E > \Delta_0$) one does not expect significant variation in x_n with respect to the fully-gapped bcs result. However, the low-energy part is dramatically different: in the integral (Eq.(1.17)) we need to consider the energy range $\Delta_0 > E > 0$. Thus, we can write approximately, for the d-wave QP fraction, $x_{n,d} \approx x_n(T) + x_{n,lowE}(T)$. At low temperature we still expect an activated behavior for x_n as in Eq.(1.21), possibly with some different numerical coefficient. However, at low T one has to add the low energy term, taking into account the states below the gap. Accordingly to Eq.(1.20), one writes:

$$x_{n,lowE} = 2\int_0^{\Delta_0} \left(-\frac{\partial f_{FD}}{\partial E}\right) \bar{N}(E) dE \tag{1.39}$$

$$\approx 2\int_0^{\Delta_0} \left(-\frac{\partial f_{FD}}{\partial E}\right) \alpha E dE \simeq 2\int_0^{\infty} \left(-\frac{\partial f_{FD}}{\partial E}\right) \alpha E dE \tag{1.40}$$

$$= 2\alpha \int_0^{\infty} f_{FD} dE = 2kT\alpha \ln(2) = CT \tag{1.41}$$

where from (1.39) to (1.40) we made use of the fact that the Fermi function drops rapidly above $E = 0$ (remember that energies are here measured from E_F), so that it acts as a very rapid cutoff in the integration. Thus, we have (i) linearly approximated $\bar{N}(E) \approx \alpha E$ in the full range of integration, and (ii) extended the integration to $+\infty$, without significantly affecting the overall result. From (1.40) to (1.41) we first integrated by parts, using $f_{FD}(\infty) = 0$, and then made use of [6] $\int_0^\infty f_{FD}(E) dE = kT \int_0^\infty (\mathrm{e}^u + 1)^{-1} du = kT \ln(2)$.

The final result is that, in a d-wave superconductor,[8] quasiparticles are excited at any temperature, proportionally to the absolute temperature: in addition to the activated process, there exist a nonactivated, linear excitation process of QP. This feature yields enormous changes in the temperature dependence of the London penetration depth and of the surface resistance.

[8] In general, in a superconductor with lines of nodes in the gap.

To exemplify, consider the temperature dependence of λ. This quantity is directly measurable by the surface reactance, Eq.(1.35). Thus, one gets $\lambda^{-2}(T) = \lambda^{-2}(0)x_s(T)$. In a d-wave superconductor at low T one should clearly observe $\lambda^{-2}(T) = A - BT$, as experimentally found in several cuprates.

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