

A short note on flux lines and flux motion

E. Silva, N. Pompeo, *Università Roma Tre*

This short note introduces, in a simple form, a few topics on Vortex motion in Type-II superconductors. It does not replace a book chapter *at all*: it should be intended as a guide to the Lectures. An (incomplete) list of appropriate readings is reported in the Bibliography.

Where symbols are not defined, they refer to commonly intended significance. The basics of vortex lines and of an Abrikosov vortex lattice are assumed to be known. The word “fluxon” and “vortex” will be used interchangeably. Please note that ξ indicates here the Ginzburg-Landau coherence length.

Purpose of this Note is to bring to the attention of the student the following important points:

1. An external current (density) \vec{J} originates a force on the vortices, proportional to J . This force can then set in motion the vortices.
2. Moving fluxons give rise to an electromotive force, revealed by an electric field \vec{E} .
3. $\vec{E} \cdot \vec{J} \neq 0$ in general. Thus, moving fluxons determine a net *dissipation* in the superconductor.
4. Defects, or in general local depressions of the order parameter $|\psi|^2$, may *pin* the vortices, thus hindering vortex motion and ultimately bringing to (nearly) zero the dissipation.
5. The new dissipationless (pinned) state can be sustained until a maximum current (density) J_c (*depinning* critical current density) is reached.
6. A finite J_c (finite pinning) gives rise to *irreversible* processes in the flux penetration and exit, and to trapped flux.

1.1 The Abrikosov state

Type II superconductors, identified by values of the Ginzburg-Landau (GL) parameter¹ $\kappa = \lambda/\xi > 1/\sqrt{2}$, have a magnetic susceptibility $\chi > -1$ when

¹The GL theory, strictly valid near T_c , gives a T -independent κ . Far from T_c , experimental κ acquires a slight temperature variation.

the applied magnetic field H is $H_{c1} < H < H_{c2}$, with H_{c1} and H_{c2} the lower and upper critical fields, respectively. The flux of the magnetic induction $\Phi(\mathbf{B})$ partially penetrates in the superconductor volume as flux “tubes”, called *fluxons*. This behavior can be explained in the framework of the GL theory: it can be shown that a normal/superconducting interface has a net energy arising from the difference between magnetic and condensation energies. These energies vary spatially on the different scales λ and ξ , respectively, so that in type II superconductors with $\lambda > \xi/\sqrt{2}$ their net difference is negative and proliferation of interfaces is energetically favored. Abrikosov [1] showed that in this “mixed state” a regular distribution of cylindrical “normal” regions² with $\psi = 0$ on the axis, each one carrying a flux quantum $\Phi_0 = h/(2e) = 2.07 \cdot 10^{-15} \text{ T m}^2$, gave the lowest energy. In the ideal isotropic material, fluxons are round cylinders with the axis parallel to the externally applied magnetic field and are surrounded by circular supercurrents, which gave fluxons the alternative name of *vortices*.

The structure of a single vortex can be determined within the GL framework. We consider an isolated vortex. We use a cylindrical coordinate frame (r, θ, z) with the z -axis coincident with the fluxon axis. Let ψ_∞ the bulk value of the order parameter. In presence of an interface one has $\psi(\mathbf{r}) = \psi_\infty f(r) e^{i\theta}$, with $f(r) \approx \tanh(r/\xi)$ [2]. The important point resides in the vanishing of the order parameter, $\psi(0) = 0$, along the fluxon axis. The flux carried by a single fluxon is $\Phi_0 = h/2e$, as a consequence of the simultaneous requirements of flux quantization and maximum number of interfaces. We stress that a flux line is then a phase structure: the phase of the order parameter changes by 2π over a circle around the vortex axis. This property will be important in the following.

For $\kappa \gg 1$ and for $r > \xi$ the field profile is [3]:

$$\mathbf{B}(\mathbf{r}) = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right) \hat{\mathbf{z}} \quad (1.1)$$

being K_0 the zeroth order Bessel function. For $r \gg \lambda$ one has $B \sim e^{-r/\lambda}$. These profiles are plotted in figure 1.1.

An important quantity is the free energy per unit length ϵ_{fl} , which is also called the vortex line tension. This quantity can be determined by considering the kinetic contribution of the vortex supercurrents and the field energy, and neglecting the condensation energy which involves only the small area of radius $\propto \xi$:

$$\epsilon_{fl} = \frac{\mu_0}{2} \int [B(r)^2 + \lambda |\nabla \times \mathbf{B}(r)|^2] dS \quad (1.2)$$

²more correctly, quasiparticle states are excited above the ground (superconducting) state.

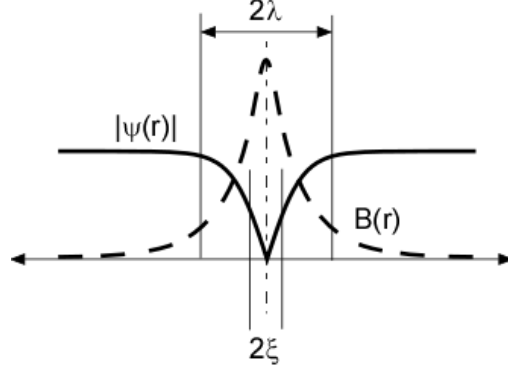


Figure 1.1: Structure of a single vortex centered in $r = 0$: $\psi(r)$ and $B(r)$ profiles, with length scales ξ and λ , respectively.

Using expression (1.1) for the field profile, the above equation yields a free energy density per unit length [3]:

$$\epsilon_{fl} = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} \ln \kappa \quad (1.3)$$

The neglected contribution due to the condensation energy lost in the vortex core changes only slightly the result. One has [3]:

$$\epsilon_{fl} = \frac{\Phi_0^2}{4\pi\mu_0\lambda^2} (\ln \kappa + o(0.1)) \quad (1.4)$$

where $o(0.1)$ represents a constant of the order of 0.1. Within this framework the lower critical field H_{c1} can be easily determined by equating the energy cost (per unit length) of a fluxon, using ϵ_{fl} from (1.3), with the energy (per unit length) gained by the appearance of a single vortex at a magnetic field $H = H_{c1}$ [2], equal to $H_{c1}\Phi_0$, yielding:

$$H_{c1} = \frac{\Phi_0}{4\pi\mu_0\lambda^2} \ln(\kappa) = H_c \frac{\ln(\kappa)}{\sqrt{2\kappa}} \quad (1.5)$$

being H_c the thermodynamic critical field which has been defined by the condensation energy.

The application of field $H > H_{c1}$ determines the creation of many vortices that arrange on a regular pattern. Abrikosov identified the possible regular patterns as square or hexagonal basic unit cells (see figure 1.2), with intervortex distance a_{FL} equal to [2]:

$$a_{FL} = a_0 \sqrt{\frac{\Phi_0}{B}}, \quad a_0 = \begin{cases} 1 & \text{for square cell} \\ \sqrt[4]{4/3} & \text{for hexagonal cell} \end{cases} \quad (1.6)$$

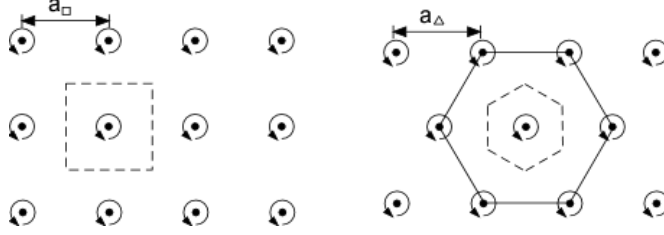


Figure 1.2: Square (*left panel*) and hexagonal (*right panel*) fluxon lattice, viewed along a section normal to vortices lattices. Vortex supercurrents are sketched by round arrows.

The hexagonal (“triangular”) lattice has the lowest energy and gives a stable configuration, as confirmed by experiments.³ The lattice is found to be stable against perturbations (“elasticity of the vortex lattice”).

An important point is the effective interactions between vortices. Let us take two fluxons with their axis parallel to z , and placed at \mathbf{r}_1 and \mathbf{r}_2 . The overall energy can be computed through (1.2) using the overall field $\mathbf{B}(\mathbf{r} - \mathbf{r}_1) + \mathbf{B}(\mathbf{r} - \mathbf{r}_2)$, with $\mathbf{B}(r)$ as in (1.1). The result consists in two self-energy terms ϵ_{fl} as (1.3) plus the following positive interaction term:

$$\epsilon_{int} = \frac{\Phi_0^2}{2\pi\mu_0\lambda^2} K_0 \left(\frac{|\mathbf{r} - \mathbf{r}_1|}{\lambda} \right) \quad (1.7)$$

which corresponds to a repulsive interaction. The symmetry of the Abrikosov fluxon lattice ensures that the overall \mathbf{J} is zero, thus fluxons remain motionless. The intervortex interaction also opposes to vortex displacements from their equilibrium position, giving elastic properties to the Abrikosov lattice.

The magnetic induction profile in the vortex state deserves a short discussion. In general, if N is the total number of fluxons in the specimen of cross section S , $\langle B \rangle = N\Phi_0/S$ is the average induction, $n = N/S$ is the areal density of flux lines, taking for simplicity a square lattice one has for the intervortex spacing:

$$a \simeq \sqrt{\frac{1}{n}} = \sqrt{\frac{\Phi_0}{\langle B \rangle}} \quad (1.8)$$

When $\langle B \rangle = \mu_0 H_{c1} \simeq \Phi_0/4\pi\lambda^2$ (we stress that $\langle B \rangle = \mu_0 H_{c1}$ means $H > H_{c1}$), $a \sim \lambda$, so that the field profiles “touch” themselves. It turns out that the field profile in the mixed state, unless the applied field is very close to H_{c1} , resembles a modulation over an average induction only slightly smaller than the applied field $\mu_0 H$ (see figure 1.3) is then $\mathbf{B} \approx \mu_0 \mathbf{H}$, apart

³Depending on the temperature range, magnetic field orientation, disorder contributions and material anisotropy, a square lattice can be also observed.

a small ripple. However, their cores are still well separated. This is the so-called London limit.⁴

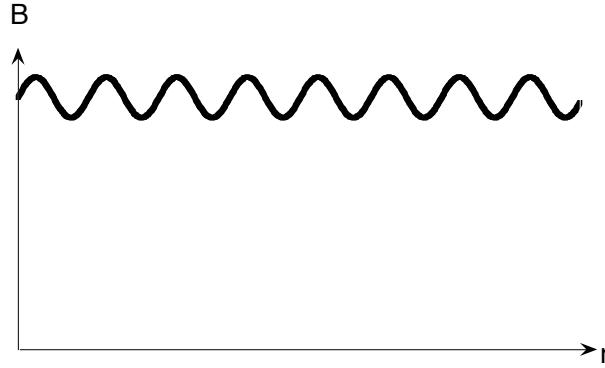


Figure 1.3: $B(r)$ resulting from overlapping vortices.

In high κ superconductors, the London approximation is valid in the great part of the mixed state region (see magnetization curves in figure 1.4).

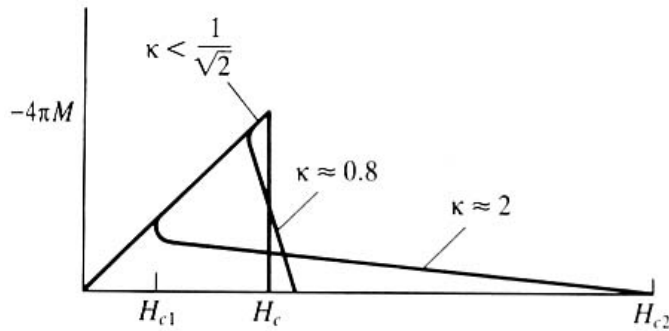


Figure 1.4: Magnetization curve for type II superconductors, compared with type I superconductors (from [2]).

By still increasing the applied field, vortex cores eventually overlap and the sample recovers the normal state. An estimate of the upper critical field $H_{c2} \approx \Phi_0/(2\pi\xi^2\mu_0)$ can be obtained considering vortex cores of section $\approx \pi\xi^2$ which cover the whole superconductor and give the uniform induction $\Phi_0/(2\pi\xi^2) = \mu_0 H_{c2}$. An exact calculation gives for H_{c2} the following expression [2]:

⁴As an important consequence, demagnetization effects due to the geometry are not relevant in this case since they are confined to H values comparable to H_{c1} .

$$H_{c2} = \frac{\Phi_0}{2\pi\mu_0\xi^2} = H_c\sqrt{2}\kappa \quad (1.9)$$

1.2 The Lorentz force

We here show that a current exerts a force over a fluxon.

The standard formulation, closely related to the GL theory, starts from the formulation of the two-vortex interaction energy per unit length. Assuming the validity of Eq.1.7, the force per unit length exerted by fluxon 2 on fluxon 1 can be obtained by taking the derivative of ϵ_{int} . In vector form it yields [2]:

$$\mathbf{f}_{L,21} = \mathbf{J}_2(\mathbf{r}_1) \times \Phi_0 \quad (1.10)$$

where Φ_0 is a vector of modulus Φ_0 , directed along the vortex axis, and \mathbf{J} is the current density of the fluxon 2. For an arbitrary distribution of vortices, the total force on any vortex results from the sum of all the current densities (which give a total \mathbf{J} on the vortex site):

$$\mathbf{f}_L = \mathbf{J} \times \Phi_0 \quad (1.11)$$

This force is called the ‘‘Lorentz force’’ acting on vortices.

A second, macroscopic derivation can be obtained from the Maxwell equations as follows. Let us consider a flux lattice. The fluxons are oriented along the positive z axis. A uniform current $\vec{J} = J\hat{y}$ is passed along the positive y axis. From $\nabla \times \vec{B} = \mu_0\vec{J}$ we get

$$-\frac{\partial B_z}{\partial x} = \mu_0 J_y \quad (1.12)$$

Thus, a gradient of the induction exists. Such gradient is not due to a continuous field, but to the different density of fluxons: *the spacing increases with increasing x* , that is the lattice becomes less and less dense as x increases. It is intuitive that, like all kind of particles, vortices from the denser region will move toward the rarified region: this is nothing else than a force on vortices,⁵ directed in this specific case along \vec{x} . It is not difficult to be more quantitative: the density of magnetic energy is $u = B^2/2\mu_0$. The force per unit volume *over the lattice* is then $\vec{F}_{lat} = -\nabla u$. In the present case, one has only the x component:

$$F_{lat,x} = -\frac{1}{2\mu_0} 2B_z \frac{\partial B_z}{\partial x} = B_z J_y = n\Phi_0 J_y \quad (1.13)$$

Reinstating all vectors, and distributing the total force equally on each of the N fluxons, one has again that on a single vortex there exists a force *per unit length* given by Eq.(1.11)

⁵Note that to use the classical argument we had to assume a system of many fluxons.

1.3 The electric field of moving fluxons

We show here that moving fluxons produce an electric field. We assume that vortices move with velocity \vec{v} .

The most convincing argument comes from the joint consideration that (i) fluxons have a phase structure and (ii) when a phase difference ϕ exists between two points in a superconductor, the (second) Josephson relation takes place:

$$V = \frac{\hbar}{2e} \dot{\phi} = \frac{\Phi_0}{2\pi} \dot{\phi} \quad (1.14)$$

where $\dot{\phi}$ is the time derivative of the phase difference between two points, and V is the voltage appearing between the same two points.

Now, consider the phase distribution around a vortex (figure 1.5, upper panel): the points P_1 and P_2 experience a variation of their phase difference ϕ equal to 2π for each fluxon crossing the line connecting P_1 to P_2 (figure 1.5, lower panels). Thus, even a single moving fluxon gives rise to a voltage. Given a fluxon density $n = B/\Phi_0$, if fluxons move perpendicularly to the segment $\overline{P_1P_2}$ of length l with velocity v , the average number of 2π phase slips in the time interval δt is $nlv\delta t$, yielding:

$$\delta\phi = 2\pi \frac{B}{\Phi_0} lv\delta t \quad (1.15)$$

so that the voltage is:

$$V = \frac{\Phi_0}{2\pi} \frac{\delta\phi}{\delta t} = Blv \quad (1.16)$$

and the electric field is $E = V/l = Bv$. In vector form:

$$\mathbf{E} = \mathbf{B} \times \mathbf{v} \quad (1.17)$$

which yields the local electric field \mathbf{E} “induced” by fluxons moving with velocity \mathbf{v} . A more exhaustive analysis of this treatment can be found in the specialized literature [5, 6].

We propose also a classical argument [1] that, at the expense of hiding somewhat the physics behind vortex motion, is extremely simple.

Let us take a vortex lattice, thus a magnetic induction \vec{B} inside the superconductor. Let vortices be aligned with the z axis, and move along the positive x axis with velocity \vec{v} . Assume invariance of the reference frame: the superconducting specimen moves with velocity $-\vec{v}$ in the fixed induction \vec{B} . Thus, perpendicular to the “specimen” motion there exists a Lorentz force on the charge carriers of charge q : $\vec{F}_L = q(-\vec{v} \times \vec{B})$, that is an electric field $\vec{E} = \vec{F}_L/q = -\vec{v} \times \vec{B} = \vec{B} \times \vec{v}$.

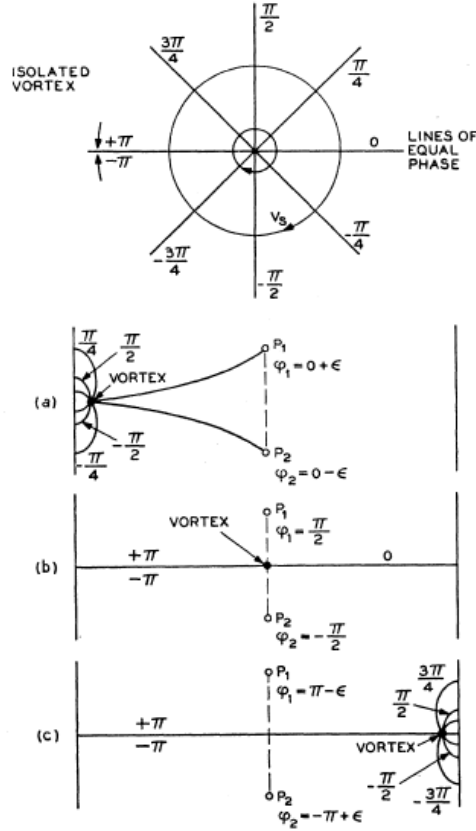


Figure 1.5: *Upper panel*: phase distribution around a vortex. *Lower panels*: phase slip determined by a moving vortex (from [5]).

1.4 Origin of dissipation

Thus, when a current density \vec{J} is passed through the superconductor, on a single fluxon there exists the force per unit length $\vec{F} = \vec{J} \times \vec{\Phi}_0$, and the force per unit volume on the whole lattice $\vec{F} = \vec{J} \times \vec{B}$ (the fact that both forces, dimensionally different, are commonly indicated by the same symbol and same name - ‘‘Lorentz force’’, can be confusing). Under the effect of the Lorentz force, the fluxon(s) can move and they give rise to an electric field $\vec{E} \parallel \vec{J}$ (in the approximations here used). Thus, a finite power dissipation $\vec{E} \cdot \vec{J}$ exists. Before studying the different regimes of fluxon motion, we wish to qualitatively describe the possible sources of such dissipation [7].

Assume a moving vortex. Consider the following mechanisms.

1. The arising electric field acts on all charge carriers present. In particular, excitations in the vortex core undergo conventional scattering processes. Such scattering processes excite phonons, and ultimately give rise to a Joule

loss.

2. A second process is a nonequilibrium process. Along the path of the moving fluxon there is a continuous conversion between Cooper pairs to quasiparticles (at the front of the moving fluxon: Cooper pairs convert to quasiparticle to enter the fluxon core), and quasiparticles to Cooper pairs (at the back of the fluxon, where quasiparticles convert back to the condensate when the fluxon moved away). Remember that the ground state is the superconducting state: it requires a certain energy (that is then *absorbed*) to convert Cooper pairs to quasiparticles. By contrast, the recondensation of quasiparticles into Cooper pairs *releases* energy (e.g., as heat). If the motion of the fluxon is slow enough, such processes are equilibrium processes. However, if the fluxon moves sufficiently fast (with respect to typical relaxation times), the process at the front of the fluxon (absorption of energy) takes place in a high magnetic field (the one of the fluxon) and, thus, require less energy than in zero field. However, the process at the back of the fluxon is a transition to the ground state in zero field (if the fluxon velocity is high enough), with the maximum possible energy jump. Thus, more energy is released than absorbed: this energy must be supplied by the passing current.

1.5 Pinning

In real materials fluxon motion is prevented by the so-called pinning: there exist pinning sites (to be specified later) that exert an attractive volume pinning force, F_p , over the lattice. Thus, until $F < F_p$, the fluxons do not move and dissipationless regime is restored. As a consequence, with dc currents and nonzero pinning, a finite resistance appears only for current densities J greater than a given critical value (the depinning current density) J_c , defined by $F = J_c B > F_p$.

In general, J_c can depend upon several variables (temperature, magnetic induction), and can also vary locally.

The origin of the phenomenon of pinning is directly connected with the concept of condensation energy, and it can be understood as follows. As we mentioned above, the creation of a vortex requires a certain amount of energy per unit length, ϵ . In order to evaluate ϵ one can proceed as follows. Let L and A be the thickness and cross-section, respectively, of the superconducting specimen. To have N fluxons requires the free energy $\Delta G_\Phi \approx N\epsilon L = n\epsilon AL$ (since $N = nA$). However, the magnetic energy gained by the reduction of the expulsion of the magnetic flux is, at B_{c1} , $\Delta G_B = AL\mu_0 H_{c1} \Delta M = ALH_{c1} n\Phi_0$, where AL is the volume of the specimen, and $\Delta M = n\Phi_0/\mu_0$ is the change in magnetization with respect to the Meissner state. At H_{c1} , one must have $\Delta G_\Phi = \Delta G_B$. Thus $\epsilon = H_{c1}\Phi_0$.

A fluxon is an elongated object: if part of its section crosses a region

where the condensation energy is already lost (because, e.g., a defect, a non-superconducting region, or even a microhole), its effective length is reduced by a certain amount, the loss of condensation energy is smaller, and overall it is favorable to “sit” over the defect. Since the effective length is reduced, the fluxon can also tolerate some bending to adapt over several defects (whence the name “tension” for ϵ).

As an exercise to the reader, we propose the following questions: assume a perfectly rigid lattice. Then:

1. Is the lattice pinned by a single defect?
2. Is the lattice pinned by a dense, random distribution of defects?

Finite temperatures can introduce thermally activated processes, giving rise to the thermally-induced motion of fluxons around their equilibrium positions (*flux creep, thermally activated flux flow*). By contrast, in absence of pinning or for temperatures or driving forces high enough, one has a steady motion of the vortex lattice (*flux flow*). For strong pinning, one can even have important irreversibility phenomena. For some detail on the dissipative regimes and on the trapped flux phenomena we refer the reader to the Bibliography (e.g., [9] chapter 8)

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