# Partially correlated thin annular sources. The vectorial case.

**M. Santarsiero,**[1] **V. Ramírez-Sánchez,**[2] **and R. Borghi**[3]

[1] *Dipartimento di Fisica,*
*Università degli Studi "Roma Tre", and CNISM*
*via della Vasca Navale,*
*84; 00146 Rome, Italy*

[2] *Departamento de Óptica,*
*Facultad de Ciencias Físicas,*
*Universidad Complutense de Madrid,*
*28040 Madrid, Spain*

[3] *Dipartimento di Elettronica Applicata,*
*Università degli Studi "Roma Tre", and CNISM*
*via della Vasca Navale,*
*84; 00146 Rome, Italy*

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We develop an electromagnetic analysis for partially correlated thin annular sources. The elements of the correlation matrix are assumed to depend only on the angular distance between two typical points. For any such source, we show how the modal expansion can be found. Correlation changes upon free propagation are discussed. Further, examples and possible synthesis schemes are presented.



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## 1. Introduction

This is the second in a series of two papers in which we find the modal expansions for a class of partially correlated thin annular sources. In the first paper [1] we have developed the scalar treatment based on the use of the mutual intensity [2] while the extension to the electromagnetic case will be discussed in the present paper.

Diffraction-free and $J_0$-correlated fields epitomize the type of radiation emitted by a thin annular source under coherent and incoherent illumination, respectively [3, 4]. In the first paper [1] (dealing with scalar sources), it was shown that these fields constitute limiting cases of beams generated by a more general class of annular sources with partial correlation. Under the hypothesis of angularly shift-invariant correlation functions, the modal expansion for such sources was found and it allowed us to study in a simple way how the correlation properties of the fields radiated by them change upon propagation. Possible experimental procedures for synthesizing scalar shift-invariant correlation functions along the annulus were also proposed.

In recent years, there has been an increasing interest in correlation properties of vectorial optical sources [5–28]. Here, we will extend the analysis of Ref. 1 to the case of vectorial sources and discuss correlation properties of electromagnetic annular sources, concentrating our attention on the angular dependence of the correlation functions. For describing the vectorial spatial correlation properties we shall make use of the beam coherence-polarization (BCP) matrix [6, 7]. A formally equivalent treatment could be performed in the space-frequency domain by means of the so-called CSD matrix [14, 30].

Analogously to their scalar counterparts, the class of fields produced by annular sources of the the kind considered here embraces, as limiting cases, perfectly coherent diffraction-free beams and $J_0$-correlated electromagnetic secondary sources [25]. In their most general form, vectorial partially coherent thin annular sources can be used as the starting point for the synthesis of electromagnetic partially coherent diffraction-free beams [28].

This work is arranged as follows. In Section 2 we will show that the modal analysis [15, 16] of electromagnetic angularly shift-invariant sources can be accomplished by means of elementary Fourier analysis. Changes in the vectorial correlation properties of the field radiated by the annular source in free-space propagation will be analyzed in Section 3. Furthermore, some relevant examples are considered and possible experimental procedures for synthesizing shift-invariant BCP matrices along the annulus are proposed in Sections 4 and 5, respectively. Main results will be summarized in the last section.

## 2. Modal analysis of angularly shift-invariant sources

Let us recall that the BCP matrix, to be denoted by $\widehat{J}$, is a 2×2 matrix, whose elements $J_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$, $(\alpha = x, y;\ \beta = x, y)$, are given by [6, 7]

$$J_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \langle E_\alpha(\boldsymbol{\rho}_1, z, t)\, E^*_\beta(\boldsymbol{\rho}_2, z, t)\rangle, \tag{1}$$

where $E_\alpha(\boldsymbol{\rho}, z, t)$ is the fluctuating $\alpha$-component of the electric field at transverse position $\boldsymbol{\rho}$, longitudinal coordinate $z$, and time $t$. The angular brackets stand for a temporal average. When $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$ the BCP matrix reduces to the (local) polarization matrix [30].

With the aim of extending the results of Ref. 1 to the vectorial case, we consider electromagnetic sources in the form of an infinitely thin annulus of radius $a$, placed across the plane $z = 0$. The pertinent BCP matrix can be written as

$$\widehat{J}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, 0) = K\, \delta(\rho_1 - a)\, \delta(\rho_2 - a)\, \widehat{J}_a(\varphi_1 - \varphi_2), \tag{2}$$

where $\boldsymbol{\rho}_j = (\rho_j, \varphi_j)$, $(j = 1, 2)$, are polar coordinates across the source plane, $\delta(\cdot)$ is the Dirac function, and $K$ is a positive constant having dimensions of an area. The shift-invariant matrix $\widehat{J}_a(\varphi_1 - \varphi_2)$ accounts for the angular correlation along the annulus and will be loosely referred to as the BCP matrix of the source.

Notice that the polarization matrix across the source, which corresponds to $\widehat{J}(\boldsymbol{\rho}, \boldsymbol{\rho}, 0)$, does not depend on the position along the annulus. This means that all the local properties of the source, such as the polarization state, the polarization degree, and the irradiance, which is proportional to $\mathrm{Tr}[\widehat{J}_a(0)]$, with $\mathrm{Tr}[\cdot]$ denoting the trace, are uniform along the annulus.

Since all the elements of $\widehat{J}_a$ depend on $\varphi_1 - \varphi_2$ only, they can be expanded into Fourier series. The BCP matrix then takes the form

$$\widehat{J}_a(\varphi_1 - \varphi_2) = \sum_{n=-\infty}^{\infty} \widehat{\gamma}_n\, \mathrm{e}^{\mathrm{i}n(\varphi_1 - \varphi_2)}\ , \tag{3}$$

where $\widehat{\gamma}_n$ $(n = 0, \pm 1, \pm 2, \dots)$ are $2 \times 2$ matrices containing the Fourier coefficients of the elements of $\widehat{J}_a$. The elements of $\widehat{\gamma}_n$ cannot take arbitrary values because of some very general constraints about the form of any plausible BCP matrices. First of all, since $\widehat{J}(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1, 0) = \widehat{J}^\dagger(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, 0)$, with the dagger denoting Hermitian conjugation, the matrices $\hat{\gamma}_n$ must be

Hermitian for any value of $n$. Moreover, since the diagonal elements of the BCP matrix, namely, $J_{\alpha\alpha}$, $(\alpha = x, y)$, are themselves possible scalar mutual intensities (because they represent the source obtained on filtering the original one by means of a linear polarizer whose transmission axis is aligned along the $\alpha$-axis), the result found for the scalar case [1] applies and the following condition has to be satisfied by the diagonal elements of the $\widehat{\gamma}_n$ matrices:

$$\gamma_{n,\alpha\alpha} \geq 0, \quad \forall n, \quad (\alpha = x, y). \tag{4}$$

We deal now with the non-negative definiteness issue [7]. Accordingly, we have to consider the quadratic form, say $Q$, defined by

$$Q = \iint \boldsymbol{g}^{\dagger}(\varphi_1)\, \widehat{J}_a(\varphi_1 - \varphi_2)\, \boldsymbol{g}(\varphi_2)\, \mathrm{d}\varphi_1 \mathrm{d}\varphi_2\,, \tag{5}$$

where proportionality factors have been omitted. Here, all the integrals are extended over a $2\pi$ interval and $\boldsymbol{g}$ is an arbitrary (well behaving) vector function, written as a column vector. For any choice of $\boldsymbol{g}$, $Q$ must be non-negative. If we express $\boldsymbol{g}$ through its Fourier expansion, i.e.,

$$\boldsymbol{g}(\varphi) = \sum_{n=-\infty}^{\infty} \boldsymbol{\eta}_n \exp(\mathrm{i}n\varphi)\,, \tag{6}$$

and insert from Eqs. (3) and (6) into Eq. (5), we obtain the following non-negativeness condition:

$$\sum_{n=-\infty}^{\infty} \boldsymbol{\eta}_n^{\dagger} \widehat{\gamma}_n \boldsymbol{\eta}_n \geq 0\,, \tag{7}$$

which must be valid for arbitrary choices of the vectors $\eta_n$. It should be noted that Eq. (7) can be interpreted as the non-negativeness condition for the infinite diagonal block matrix $\mathrm{diag}(\ldots, \widehat{\gamma}_{-1}, \widehat{\gamma}_0, \widehat{\gamma}_1, \ldots)$ which will be satisfied iff *each* $2 \times 2$ diagonal block $\widehat{\gamma}_n$ $(n = 0, \pm 1, \pm 2, \ldots)$ is non-negative definite [29]. In particular, due to their above remainded Hermiticity property, this requires that their trace and determinant be non-negative. As far as the former is concerned, its non-negativity follows from Eq. (4), while the same condition on the determinant leads to

$$|\gamma_{n,xy}|^2 \leq \gamma_{n,xx}\, \gamma_{n,yy}, \quad \forall n. \tag{8}$$

Let us now pass to the problem of determining the vector modes [15] of the source. To this aim, we recall that the matrices $\widehat{\gamma}_n$ can always be expressed through their spectral decomposition [31], which in the present case reads

$$\widehat{\gamma}_n = \sum_{i=1}^{2} \sigma_n^{(i)} \mathbf{\Phi}_n^{(i)} \mathbf{\Phi}_n^{(i)\dagger} \ . \tag{9}$$

Here, $\sigma_n^{(i)}$ are the two eigenvalues of $\widehat{\gamma}_n$ and $\mathbf{\Phi}_n^{(i)}$ are column vectors representing the corresponding eigenvectors. For a positive semidefinite matrix, the eigenvalues are nonnegative and the eigenvectors are (or can be chosen as, in case of degeneracy) orthonormal. The latter property implies that, when the eigenvectors are thought of as the Jones vectors of two electromagnetic fields, such fields have mutually orthogonal polarization states. On inserting from Eq. (9) into Eq. (3), the following expression is obtained:

$$\widehat{J}_a(\varphi_1 - \varphi_2) = \sum_{n=-\infty}^{+\infty} \sum_{i=1}^{2} \lambda_n^{(i)} \ \mathbf{\Psi}_n^{(i)}(\varphi_1) \ \mathbf{\Psi}_n^{(i)\dagger}(\varphi_2) \ , \tag{10}$$

where

$$\lambda_n^{(i)} = 2\pi \, \sigma_n^{(i)} \ , \tag{11}$$

and

$$\mathbf{\Psi}_n^{(i)}(\varphi) = \frac{1}{\sqrt{2\pi}} \ \mathbf{\Phi}_n^{(i)} \ \mathrm{e}^{\mathrm{i}n\varphi} \ . \tag{12}$$

Since the vector fields in Eq. (12) are mutually orthonormal, i.e.,

$$\int \mathbf{\Psi}_n^{(i)\dagger}(\varphi) \ \mathbf{\Psi}_m^{(j)}(\varphi) \ \mathrm{d}\varphi = \delta_{ij} \, \delta_{nm} \ , \tag{13}$$

the expansion in Eq. (10) just corresponds to the modal expansion of the BCP of the annular source, with eigenvalues and modes given by $\lambda_n^{(i)}$ and $\mathbf{\Psi}_n^{(i)}(\varphi)$, respectively.

It is to be noted that the $n$th term of the expansion in Eq. (10) is constituted by the superposition of two mutually uncorrelated fields, having the same angular dependence. Since the weights of these two fields are $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$, such term corresponds, in general, to a partially polarized source with a (uniform) polarization matrix given by [see Eq. (9)]

$$\widehat{P}_n(\boldsymbol{\rho}, 0) = \widehat{\gamma}_n \ , \tag{14}$$

and whose degree of polarization can be expressed as [30]

$$p_n = \frac{|\lambda_n^{(1)} - \lambda_n^{(2)}|}{\lambda_n^{(1)} + \lambda_n^{(2)}} = \frac{|\sigma_n^{(1)} - \sigma_n^{(2)}|}{\sigma_n^{(1)} + \sigma_n^{(2)}} \ , \tag{15}$$

because the two mode in each term have orthogonal polarizations. Furthermore, it should be stressed that each of such terms corresponds to a field that would be considered as perfectly coherent from the spatial point of view if no anisotropic elements were used to measure its coherence properties. This is quantified by the (*equivalent*) degree of coherence [8, 14] which, when applied to the angular part of the BCP matrix, reads (with $\varphi_{12} = \varphi_1 - \varphi_2$)

$$\mu_{\rm eq}(\varphi_{12}) = \frac{\mathrm{Tr}[\widehat{J}_a(\varphi_{12})]}{\mathrm{Tr}[\widehat{J}_a(0)]} \ . \tag{16}$$

Another parameter used to specify the coherence properties of a vectorial field is the *electromagnetic* degree of coherence [13], which in the present case is

$$\mu_{\rm em}(\varphi_{12}) = \frac{\sqrt{\mathrm{Tr}[\widehat{J}_a(\varphi_{12})\widehat{J}_a^{\dagger}(\varphi_{12})]}}{\mathrm{Tr}[\widehat{J}_a(0)]} \ . \tag{17}$$

For each of the above partially polarized component fields of the decomposition in Eq. (9), i.e., for $\hat{J}_a(\varphi_{12}) = \hat{\gamma}_n \exp[in(\varphi_{12})]$, Eq. (16) gives $\mu_{\rm eq} = \exp(in\varphi_{12})$, so that it is of unitary modulus, while the electromagnetic degree of coherence turns out to be $\mu_{\rm em} = [(1+p_n^2)/2]^{1/2}$, i.e., it takes a constant value.

On the other hand, the limiting case of spatially incoherent annular source is reached when all $\widehat{\gamma}_n$ matrices in Eq. (3) equal the polarization matrix of the source, apart from a common proportionality factor [25].

Finally, we give the explicit expressions of eigenvalues and eigenvectors of the matrices $\widehat{\gamma}_n$. They are obtained by solving the corresponding eigenvalue equations, i.e.,

$$\widehat{\gamma}_n \, \mathbf{\Phi}_n = \sigma_n \, \mathbf{\Phi}_n \ , \tag{18}$$

whose associated secular equations read

$$\sigma_n^2 - T_n \sigma_n + D_n = 0 \ , \tag{19}$$

where $T_n$ and $D_n$ stand for the trace and the determinant of $\widehat{\gamma}_n$, respectively. For what said before, $D_n$ is a non-negative quantity. Further, it is easily seen that $4D_n/T_n^2$ does not exceed one. Then, the two solutions turn out to be

$$\sigma_n^{(1,2)} = \frac{T_n}{2}\left(1 \pm \sqrt{1 - \frac{4D_n}{T_n^2}}\right) , \tag{20}$$

and are both non-negative, as was expected. If $|\gamma_{n,xy}| \neq 0$, the corresponding eigenvectors can be specified by the vectors

$$\mathbf{\Phi}_n^{(i)} = N_n^{(i)} \begin{bmatrix} 1 \\ \\ \dfrac{\gamma_{n,xx} - \sigma_n^{(i)}}{\gamma_{n,xy}} \end{bmatrix} , \tag{21}$$

where the normalization factor, $N_n^{(i)}$, is given by

$$N_n^{(i)} = \left( 1 + \left| \frac{\gamma_{n,xx} - \sigma_n^{(i)}}{\gamma_{n,xy}} \right|^2 \right)^{-1/2} . \tag{22}$$

If $|\gamma_{n,xy}| = 0$, the eigenvectors turn out to be

$$\mathbf{\Phi}_n^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad \mathbf{\Phi}_n^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \tag{23}$$

with eigenvalues $\sigma_n^{(1)} = \gamma_{n,xx}$ and $\sigma_n^{(2)} = \gamma_{n,yy}$, respectively.

## 3. Propagation

In this section we will study how the vectorial correlation properties of the field radiated by our annular sources change upon free propagation. As in the scalar case, we shall limit ourselves to paraxial approximation. The reader should be cautioned that the results can be applied beyond a certain distance from the source [32, 33].

Let us recall that the elements of the BCP matrix change upon propagation according to the same law holding for the mutual intensity, so that

$$\begin{aligned} \widehat{J}(\boldsymbol{r}_1, \boldsymbol{r}_2, z) = \frac{1}{\lambda^2 z^2} \iint \widehat{J}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, 0) \\ \times \exp\left\{ \frac{\mathrm{i}k}{2z} \left[ (\boldsymbol{r}_1 - \boldsymbol{\rho}_1)^2 - (\boldsymbol{r}_2 - \boldsymbol{\rho}_2)^2 \right] \right\} \mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2 \, . \end{aligned} \tag{24}$$

where $\boldsymbol{r}_j = (r_j, \vartheta_j)$, $(j = 1, 2)$, are polar coordinates across the transverse plane at a distance $z$ from the soure. On inserting from Eq. (2) into Eq. (24) we obtain

$$\begin{aligned} \widehat{J}(\boldsymbol{r}_1, \boldsymbol{r}_2, z) = \frac{Ka^2}{\lambda^2 z^2} \exp\left[ \frac{\mathrm{i}k}{2z} (r_1^2 - r_2^2) \right] \\ \times \iint \widehat{J}_a(\varphi_1 - \varphi_2) \exp\left\{ -\mathrm{i} \, \frac{ka}{z} \left[ r_1 \cos(\vartheta_1 - \varphi_1) \right.\right. \\ \left.\left. - r_2 \cos(\vartheta_2 - \varphi_2) \right] \right\} \, \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_2 \, . \end{aligned} \tag{25}$$

Using the Mercer expansion in Eq. (10) together with the definition in Eq. (12), and proceeding as in the scalar case [1], the following result is obtained:

$$\widehat{J}(\boldsymbol{r}_1,\boldsymbol{r}_2,z)=\sum_{i=1}^{2}\sum_{n=-\infty}^{+\infty}\lambda_n^{(i)}\,\boldsymbol{\Psi}_n^{(i)}(\boldsymbol{\rho}_1,z)\,\boldsymbol{\Psi}_n^{(i)\dagger}(\boldsymbol{\rho}_2,z)\;, \tag{26}$$

with

$$\begin{aligned}\boldsymbol{\Psi}_n^{(i)}(\boldsymbol{\rho},z)&=\alpha_z\sqrt{\frac{K}{2\pi}}\,J_n\left(\alpha_z r\right)\\ &\times\exp\left(\mathrm{i}\,\frac{kr^2}{2z}\right)\,\boldsymbol{\Phi}_n^{(i)}\,\mathrm{e}^{\mathrm{i}n\vartheta}\quad,(i=1,2),\end{aligned} \tag{27}$$

where $\alpha_z=ka/z$ and $J_n(\cdot)$ is the Bessel function of the first kind and order $n$ [34].

The series on the right-hand side of Eq. (26) represents, within the paraxial approximation, the modal expansion of the field propagated away from the annular electromagnetic source, with the modes given in Eq. (27).

The local polarization matrix is obtained by letting $\boldsymbol{r}_1=\boldsymbol{r}_2=\boldsymbol{r}$ in Eq. (26). Then, on recalling Eqs. (9) and (11), the following expression is derived for the propagated polarization matrix:

$$\widehat{P}(\boldsymbol{r},z)=K\alpha_z^2\sum_{n=-\infty}^{+\infty}\widehat{\gamma}_n\,J_n^2\left(\alpha_z r\right)\;, \tag{28}$$

which is a function of the radial coordinate $r$. As a consequence, both the state and the degree of polarization of the field may change across the observation plane. This may appear a little surprising, because the irradiance and the polarization properties of the source are supposed to be uniform along the annulus. Indeed, this is a consequence of the different propagation properties of the modes, whose polarization is different from one another.

As far as the dependence on the longitudinal coordinate is concerned, Eq. (28) states that the transverse irradiance and polarization patterns simply increase their size during propagation without changing their shape. It is then understood how, by using suitable collimating lenses, such annular sources can give rise to partially coherent electromagnetic diffraction-free beams [28].

## 4. Examples

As a first simple example we consider the following BCP matrix:

$$\widehat{J}_a(\varphi_{12})=\frac{I_0}{2}\begin{bmatrix}1&0\\0&\mathrm{e}^{\mathrm{i}\varphi_{12}}\end{bmatrix}\;, \tag{29}$$

with $I_0$ being a positive constant. On evaluating such matrix for $\varphi_{12} = 0$, it is found that the field associated to the source is completely unpolarized. The only nonvanishing matrices of the expansion in Eq. (3) are

$$\widehat{\gamma}_0 = \frac{I_0}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \; ; \quad \widehat{\gamma}_1 = \frac{I_0}{2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} , \tag{30}$$

so that the source can be thought of as the uncorrelated superposition of a uniphase and uniform field polarized along the $x$-axis, and a uniform field polarized along the $y$-axis whose phase varies linearly from zero to $2\pi$ along the annulus. The two fields carry the same power, $I_0/2$.

The polarization matrix of the field propagated at a distance $z$ is readily obtained from Eqs. (28) and (30), and turns ot to be

$$\widehat{P}(\boldsymbol{r}, z) = \frac{K I_0 \alpha_z^2}{2} \begin{bmatrix} J_0^2\left(\alpha_z r\right) & 0 \\ & \\ 0 & J_1^2\left(\alpha_z r\right) \end{bmatrix} , \tag{31}$$

with degree of polarization given by

$$p(\boldsymbol{r}, z) = \frac{\left|J_0^2\left(\alpha_z r\right) - J_1^2\left(\alpha_z r\right)\right|}{J_0^2\left(\alpha_z r\right) + J_1^2\left(\alpha_z r\right)} . \tag{32}$$

Plots of the propagated irradiance and degree of polarization are shown in Fig. 1 as functions of $\alpha_z r$.

For another simple example, let us consider a BCP of the form

$$\widehat{J}_a(\varphi_{12}) =$$

$$I_0 \begin{bmatrix} \cos\varphi_{12} - \mathrm{i}\beta \sin\varphi_{12} & -\sin\varphi_{12} - \mathrm{i}\beta\cos\varphi_{12} \\ & \\ \sin\varphi_{12} + \mathrm{i}\beta\cos\varphi_{12} & \cos\varphi_{12} - \mathrm{i}\beta\sin\varphi_{12} \end{bmatrix} , \tag{33}$$

where $\beta$ is a real quantity such that $|\beta| \leq 1$. The degrees of coherence in Eqs. (16) and (17) pertaining to such a source are found to be

$$\mu_{\mathrm{eq}}(\varphi_{12}) = \cos\varphi_{12} - \mathrm{i}\beta\sin\varphi_{12} \tag{34}$$

and

$$\mu_{\mathrm{em}}(\varphi_{12}) = \sqrt{\frac{1+\beta^2}{2}} \,, \tag{35}$$

respectively, while the degree of polarization is obtained from the local polarization matrix, i.e., by letting $\varphi_{12} = 0$ in Eq. (33), and equals $|\beta|$ at any point of the annulus. This means that, on changing $\beta$, both the coherence and the polarization properties of the source can be varied. In particular, when $|\beta|$ ranges from 0 to 1, the field passes from completely unpolarized to perfectly polarized. Correspondingly, the modulus of its equivalent degree of coherence goes from $|\cos\varphi_{12}|$ to 1, while the electromagnetic degree of coherence goes from 0.5 to 1.

The modes of the source described in Eq. (33) can be easily found. Only the values $n = 1$ and $n = -1$ are involved in the expansion in Eq. (3). For each of them, there is only one eigenvalue different from zero, namely, $\sigma_{\pm 1} = I_0(1 \mp \beta)$, which the following two vector modes correspond to:

$$\boldsymbol{\Psi}_{\pm 1}(\varphi) = \frac{I_0}{\sqrt{4\pi}} \begin{bmatrix} 1 \\ \mp \mathrm{i} \end{bmatrix} \mathrm{e}^{\pm \mathrm{i}\varphi} . \tag{36}$$

In conclusion, the overall field is the superposition of two uncorrelated modes having counter-rotating circular polarization states and generally different powers. It is easy to verify that the propagated field keeps the inital polarization state at any distance $z$, at any point of the transverse plane.

A third example is provided by the BCP matrix

$$\widehat{J}_a(\varphi_{12}) = I_0 \begin{bmatrix} \mathrm{tri}\left(\frac{\varphi_{12}}{\pi}\right) & \mathrm{tri}\left(\frac{\varphi_{12}}{\pi} - 1\right) \\ \mathrm{tri}\left(\frac{\varphi_{12}}{\pi} + 1\right) & \mathrm{tri}\left(\frac{\varphi_{12}}{\pi}\right) \end{bmatrix} , \tag{37}$$

with

$$\mathrm{tri}(t) = \begin{cases} 1 - |t| & (|t| \le 1) \; ; \\ 0 & (|t| > 1) \; . \end{cases} \tag{38}$$

By Fourier transforming the above elements, the $\widehat{\gamma}_n$ matrix turns out to be

$$\widehat{\gamma}_n = \frac{I_0}{2} \, \mathrm{sinc}^2\left(\frac{n}{2}\right) \begin{bmatrix} 1 & (-1)^n \\ (-1)^n & 1 \end{bmatrix} , \tag{39}$$

where $\mathrm{sinc}(t) = \sin(\pi t)/(\pi t)$. For each value of $n$, such matrix has only one eigenvalue, namely,

$$\sigma_n = I_0 \, \mathrm{sinc}^2\left(\frac{n}{2}\right) , \tag{40}$$

and the corresponding vector mode turns out to be

$$\boldsymbol{\Psi}_n = \frac{1}{\sqrt{4\pi}} \begin{bmatrix} 1 \\ (-1)^n \end{bmatrix} \mathrm{e}^{\mathrm{i}n\varphi} . \tag{41}$$

In this case, the polarization matrix of the field propagated at a distance $z$ turns ot to be, after some calculations,

$$\begin{aligned} \widehat{P}(\boldsymbol{r}, z) = & \frac{K \, I_0 \, \alpha_z^2}{2} \\ & \times \begin{bmatrix} J_0^2\left(\alpha_z r\right) + F\left(\alpha_z r\right) & J_0^2\left(\alpha_z r\right) - F\left(\alpha_z r\right) \\ \\ J_0^2\left(\alpha_z r\right) - F\left(\alpha_z r\right) & J_0^2\left(\alpha_z r\right) + F\left(\alpha_z r\right) \end{bmatrix} , \end{aligned} \tag{42}$$

with the function $F_z(r)$ defined as

$$F(t) = \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{J_{2n+1}^2(t)}{(2n+1)^2} , \tag{43}$$

and the degree of polarization is given by

$$p(\boldsymbol{r}, z) = \frac{\left|J_0^2\left(\alpha_z r\right) - F\left(\alpha_z r\right)\right|}{J_0^2\left(\alpha_z r\right) + F\left(\alpha_z r\right)} . \tag{44}$$

As we shall see in Sec. 5, the latter two examples refer to cases that can be realized in the laboratory by means of simple experimental tools.

## 5. Synthesis

In this section, we discuss possible experimental procedures for synthesizing shift-invariant BCP matrices along the annulus.

One of the techniques proposed for the synthesis of scalar annular sources consists in a rotating transparency, placed in front of an annular aperture, illuminated by a plane wave impinging orthogonally [1]. We wonder whether, for our present purposes, the rotating transparency of the scalar case can be replaced by a rotating anisotropic element. It is to be noted that in the scalar case the value of the transmission function at a fixed point of the transparency does not depend on the rotation angle. It is this feature that gives rise to the shift-invariance of the mutual intensity. When the transparency is replaced by an anisotropic element the role of the transmission function is played by the 2×2 Jones matrix characterizing the element [2]. Generally speaking, such matrix depends on the rotation angle. As a consequence, the resulting elements of the BCP matrix will not be shift-invariant. A notable exception is given by a rotator, whose Jones matrix does not change if the angular position of the rotator is varied.

Suppose now that an anisotropic element is constituted by a variable rotator. More precisely, we assume that, when the rotator is held fixed, the rotation angle it introduces is a function of the angular position $\varphi$ on the element. Denoting by $\eta$ the rotation angle introduced by the rotator, we then have

$$\eta = \eta(\varphi). \tag{45}$$

As an example, suppose $\eta$ to be given by

$$\eta(\varphi) = \varphi. \tag{46}$$

Suppose further that such element is illuminated by a plane wave with arbitrary, but uniform, polarization state. Let us write the electric field of such a wave as

$$\boldsymbol{E}_{\text{in}} = \begin{bmatrix} A_x \\ A_y \mathrm{e}^{\mathrm{i}\alpha} \end{bmatrix}, \tag{47}$$

with $A_x$, $A_x$, and $\alpha$ positive constants. Then, at a typical point of the annulus, the electric field will be

$$\boldsymbol{E}_{\text{out}}(\varphi) = \begin{bmatrix} \cos\varphi\, A_x - \sin\varphi\, A_y \mathrm{e}^{\mathrm{i}\alpha} \\ \\ \sin\varphi\, A_x + \cos\varphi\, A_y \mathrm{e}^{\mathrm{i}\alpha} \end{bmatrix}. \tag{48}$$

Now, let the anisotropic element rotate at angular speed $\omega$, so that the angle $\varphi$ in Eq. (48) is replaced by $\varphi - \omega t$. Then, we can evaluate the elements of the BCP matrix taking a time average. It is easily found that

$$
\widehat{J}_a(\varphi_{12}) = \frac{A^2}{2} \times \begin{bmatrix} \cos\varphi_{12} - \mathrm{i}\varepsilon\sin\alpha\,\sin\varphi_{12} & -\sin\varphi_{12} - \mathrm{i}\varepsilon\sin\alpha\,\cos\varphi_{12} \\ & \\ \sin\varphi_{12} + \mathrm{i}\varepsilon\sin\alpha\,\cos\varphi_{12} & \cos\varphi_{12} - \mathrm{i}\varepsilon\sin\alpha\,\sin\varphi_{12} \end{bmatrix}, \tag{49}
$$

where $A^2 = (A_x^2 + A_y^2)$ and

$$
\varepsilon = \frac{2A_xA_y}{A^2}\,. \tag{50}
$$

The obtained BCP matrix is of the form specified by Eq. (33) in Sec. 4, with $I_0 = A^2/2$ and $\beta = \varepsilon\sin\alpha$, and all considerations done in that case hold. In particular, both the polarization degree and the coherence degree of the output radiation can be varied on changing the polarization state of the input one.

From the experimental point of view, realizing a rotator whose rotation angle varies continuously from 0 to $2\pi$ may be a demanding task. A discretized version of such an element, however, can be realized following the approach used for the generation of azimuthally or radially polarized beams (see, for instance, [35] and references therein). In one of such techniques [36], use is made of an optical element divided into four quadrants. In each of such quadrants the element basically acts as a half-wave plate, whose fast axis is rotated by 45° with respect to the adjacent one. Now, if such a tool is superimposed to a spatially uniform half-wave plate, a four-quadrant rotator is obtained, which in each sector rotates the incident polarization by a different angle [37].

To have an idea of the elecromagnetic source one would obtain by using a discretized rotator into the above experimental scheme, we consider a transparent disk made of just two halves. One of them (say, the upper part) does not impart any rotation to the incoming field, whereas the other produces a rotation of $\pi/2$. A plane wave of amplitude $A$, linearly polarized along the $x$ axis, illuminates the disk put in front of the annular aperture. At a typical point of the annulus, the Cartesian components of the electric field will be

$$
\begin{aligned}
E_x(\varphi) &= A\ \mathrm{rect}\left(\frac{\varphi - \pi/2}{\pi}\right), \\
E_x(\varphi) &= A\ \mathrm{rect}\left(\frac{\varphi + \pi/2}{\pi}\right),
\end{aligned} \tag{51}
$$

where

$$\mathrm{rect}(t) = \begin{cases} 1 \;\; (|t| \leq 1/2) \; ; \\ \\ 0 \;\; (|t| > 1/2) \; . \end{cases} \tag{52}$$

On proceeding as for the previous case, the BCP matrix turns out to be the same as that of Eq. (37) in Sec. 4, with $I_0 = \pi |A|^2/\omega$.

The above device can be made more elaborated if the disk is divided into more than two sectors, as for the four-quadrant tool quoted above. Alternatively, one could maintain only two sectors but of different angular extent. As could be easily verified, this would differentiate $J_{xx}$ from $J_{yy}$.

## 6. Summary

In this paper the properties of a class of electromagnetic partially correlated thin annular sources have been analyzed in the framework of the BCP formalism. Our analysis generalizes that presented, for the scalar case, in Ref. [1]. In the limiting cases of perfect coherence and complete incoherence, these sources are shown to produce diffraction-free and $J_0$-correlated electromagnetic beams, respectively.

When the vectorial correlation properties of the radiation across the sources are assumed to be dependent only on the angular distance between two points on the annulus, the modal expansion of the sources can be found by means of elementary Fourier analysis. The knowledge of the modes, in particular, allowed us to study, in a simple way, changes in the correlation properties of the electromagnetic fields propagating away from such sources.

Examples have been presented of electromagnetic annular sources with very different coherence and polarization properties, and possible experimental schemes, based on ordinary anysotropic optical elements, have eventually been proposed for the synthesis of such sources.

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## References

1. F. Gori, M. Santarsiero, R. Borghi and C. Li "Partially correlated thin annular sources: the scalar case," J. Opt. Soc. Am. A **25,** 2826-2832 (2008)

2. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics*, (Cambridge U. Press,Cambridge, 1995).
3. J. Durnin, J.J. Miceli, J.H. Eberly "Diffraction-free beams," Phys. Rev. Lett. **58,** 1499-1501 (1987)
4. F. Gori, G. Guattari, C. Padovani "Modal expansion of $J_0$-correlated Schell-model sources," Opt. Comm. **64,** 311-316 (1987)
5. D. F. V. James, "Change of polarization of light beam on propagation in free-space," J. Opt. Soc. Am. A **11,** 1641-1643 (1994).
6. F. Gori, "Matrix treatment for partially polarized, partially coherent beams," Opt. Lett. **23,** 41-43 (1998).
7. F. Gori, M. Santarsiero, S. Vicalvi, R. Borghi, and G. Guattari, "Beam coherence-polarisation matrix," Pure Appl. Opt. **7,** 941-951 (1998)
8. F Gori, M Santarsiero, R Borghi, G Guattari, "The irradiance of partially polarized beams in a scalar treatment," Opt. Commun. **163**, 159-163 (1999).
9. S. R. Seshadri, "Partially coherent Gaussian Schell-model electromagnetic beam," J. Opt. Soc. Am. A **16,** 1373-1380 (1999).
10. F. Gori, M. Santarsiero, R. Borghi, and G. Piquero, "Use of the van Cittert-Zernike theorem for partially polarized sources," Opt. Lett. **25,** 1291-1293 (2000).
11. F. Gori, M. Santarsiero, G. Piquero, R. Borghi, A. Mondello, and R. Simon, "Partially polarized Gaussian Schell-model beams," J. Opt. A: Pure Appl. Opt. **3,** 1-9 (2001).
12. G. Piquero, R. Borghi, A. Mondello, and M. Santarsiero, "Far field of beams generated by quasi-homogeneous sources passing through polarization gratings," Opt. Commun. **195,** 339-350 (2001).
13. J. Tervo, T. Setälä and A.T. Friberg, "Degree of coherence for electromagnetic fields," Opt. Express **11,** 1137-1143 (2003)
14. E. Wolf, "Unified theory of coherence and polarization of statistical electromagnetic beams," Phys. Lett. **A312,** 263-267 (2003).
15. F. Gori, M. Santarsiero, R. Simon, G. Piquero, R. Borghi, and G. Guattari, "Coherent-mode decomposition of partially polarized, partially coherent sources," J. Opt. Soc. Am. A **20,** 78-84 (2003).
16. J. Tervo, T. Setälä, and A. T. Friberg, "Theory of partially coherent electromagnetic fields in the space-frequency domain," J. Opt. Soc. Am. A **21,** 2205-2215 (2004).

17. R. Martínez-Herrero, G. Piquero, and P. Mejías, "Parametric characterization of the spatial structure of partially coherent and partially polarized beams," J. Opt. A: Pure Appl. Opt. **6,** S67-S71 (2004)
18. O. Korotkova, M. Salem, E. Wolf, "Beam conditions for radiation generated by an electromagnetic Gaussian Schell-model source," Opt. Lett. **29,** 1173-1175 (2004).
19. P. Réfrégier and F. Goudail, "Invariant degrees of coherence of partially polarized light," Opt. Express **13,** 6051-6060 (2005)
20. H. Roychowdhury and O. Korotkova, "Realizability conditions for electromagnetic Gaussian Schell-model sources," Opt. Commun. **249,** 379-385 (2005).
21. T. Shirai, "Polarization properties of a class of electromagnetic Gaussian Schell-model beams which have the same far-zone intensity distribution as a fully coherent laser beam," Opt. Commun. **256,** 197-209 (2005).
22. F. Gori, M. Santarsiero, R. Borghi and E. Wolf, "Effects of coherence on the degree of polarization in Young interference pattern," Opt. Lett. **31,** 688-690 (2006).
23. A. Luis, "Ray picture of polarization and coherence in a Young interferometer," J. Opt. Soc. Am. A **23,** 2855-2860 (2006).
24. M. Santarsiero, F. Gori, R. Borghi and G. Guattari, "Vector-mode analysis of symmetric two-point sources," J. Opt. A: Pure Appl. Opt. **9,** 593-602 (2007).
25. F. Gori, M. Santarsiero, R. Borghi, "Modal expansion for $J_0-$correlated electromagnetic sources," Opt. Lett. **33,** 1857-1859 (2008).
26. S. N. Volkov, D. F. V. James, T. Shirai and E. Wolf, "Intensity fluctuations and the degree of cross-polarization in stochastic electromagnetic beams," J. Opt. A: Pure Appl. Opt. **10,** 055001 (2008).
27. J. Tervo, T. Setl, A. Roueff, P. Rfrgier, A. T. Friberg, "Two-point Stokes parameters: interpretation and properties," OL **34,** 3074-3076 (2009).
28. R. Borghi, F. Gori, S. A. Ponomarenko, "On a class of electromagnetic diffraction-free beams," J. Opt. Soc. Am. A **26,** 2275-2281 (2009).
29. R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambrdge U. P., Cambridge, 1985)
30. E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light*, (Cambridge U. Press, Cambridge, 2007).
31. S. K. Berberian, *Introduction to Hilbert Space* (Oxford U. Press, Oxford, UK, 1961).
32. G. S. Agarwal, G. Gbur, and E. Wolf, Opt.Lett. **29** 459 (2004).

33. F. Gori, "Far-zone approximation for partially coherent sources," Opt. Lett. **30,** 2840-2842 (2005).

34. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products,* 6th ed., A. Jeffrey and D. Zwillinger, eds. (Academic, New York, 2000).

35. Q. W. Zhan. "Cylindrical vector beams: from mathematical concepts to applications," Advances in Optics and Photonics **1,** 1-57 (2009).

36. S. Quabis, R. Dorn and G. Leuchs, "Generation of a radially polarized doughnut mode of high quality," Appl. Phys. B **81**, 597600 (2005)

37. Q. W. Zhan and J. R. Leger, "Focus shaping using cylindrical vector beams," Opt Express **10,** 324-331 (2002)

**Figure captions**

**Figure 1:** Normalized irradiance distribution (dashed) and polarization degree (full) of the field emitted by the source described in Eq. (29).

**Figure 2:** Normalized irradiance distribution (dashed) and polarization degree (full) of the field emitted by the source described in Eq. (37).

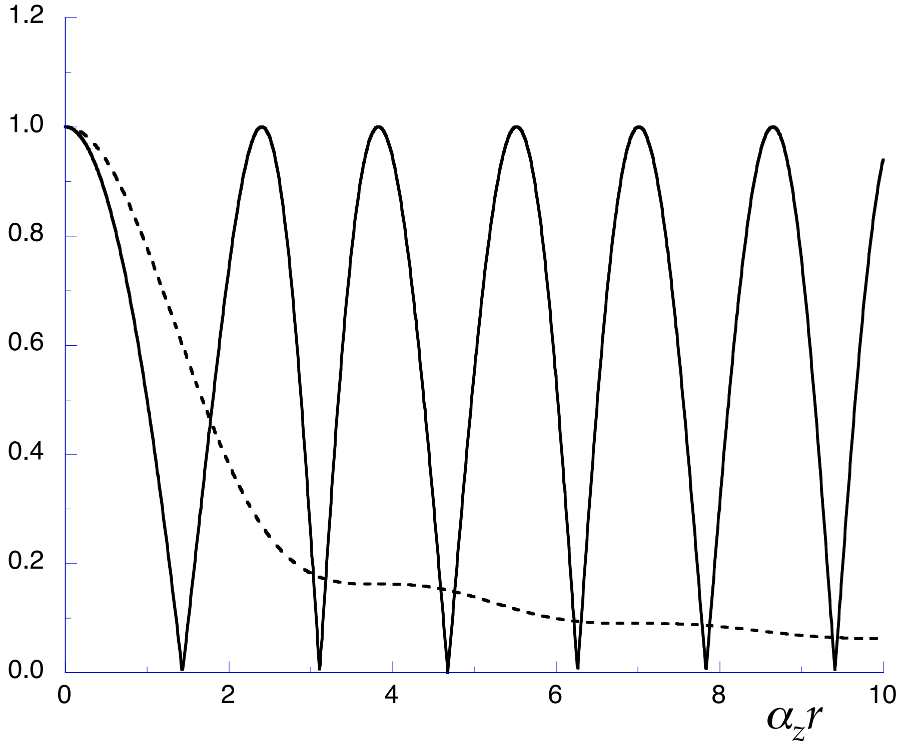


Fig. 1. Normalized irradiance distribution (dashed) and polarization degree (full) of the field emitted by the source described in Eq. (29).

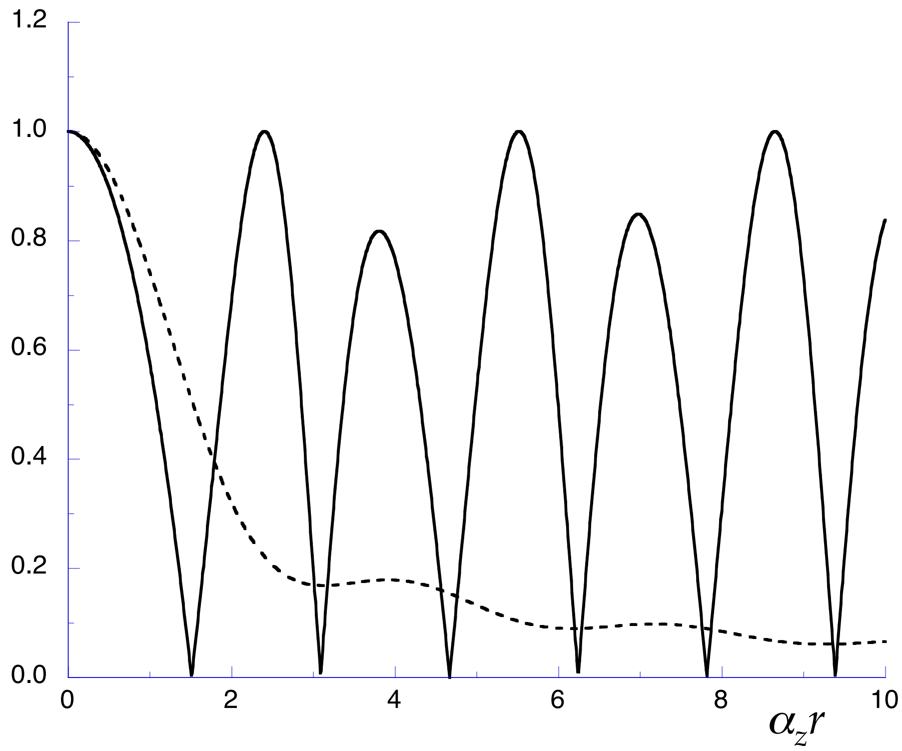


Fig. 2. Normalized irradiance distribution (dashed) and polarization degree (full) of the field emitted by the source described in Eq. (37).