Nonquantum Entanglement Resolves a Basic Issue in Polarization Optics

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The issue raised in this Letter is classical, not only in the sense of being nonquantum, but also in the sense of being quite ancient: which subset of 4×4 real matrices should be accepted as physical Mueller matrices in polarization optics? Nonquantum entanglement or inseparability between the polarization and spatial degrees of freedom of an electromagnetic beam whose polarization is not homogeneous is shown to provide the physical basis to resolve this issue in a definitive manner.

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Entanglement is traditionally studied almost exclusively in the context of quantum systems. However, this notion is basically kinematic, and so is bound to present itself whenever and wherever the state space of interest is the tensor product of two (or more) vector spaces. Polarization optics of paraxial electromagnetic beams happens to have precisely this kind of a setting, and so one should expect entanglement to play a role in this situation. It turns out that entanglement in this nonquantum setup is not just a matter of academic curiosity. We shall show in this Letter that consideration of this nonquantum entanglement resolves a fundamental issue in classical polarization optics. And it will appear that this issue could not have been resolved without explicit consideration of entanglement. We begin by outlining the structure of classical polarization optics [1–5].

The Mueller-Stokes Formalism.—Traditional Mueller-Stokes formalism applies to plane electromagnetic waves or, more generally, to uniformly polarized or elementary beams (see below). If the wave propagates along the positive z axis, the complex-valued components E_1, E_2 of the transverse electric field along the x and y directions can be arranged into a column vector

$$\boldsymbol{E} \equiv \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \tag{1}$$

called the Jones vector. (A scalar factor of the form $\exp[i(kz - \omega t)]$ has been suppressed). While $\boldsymbol{E}^\dagger \boldsymbol{E} = |E_1|^2 + |E_2|^2$ is (a measure of) the intensity, the ratio $\gamma = E_1/E_2$ specifies the state of polarization.

When $\boldsymbol{E}$ is not deterministic, the state of polarization is described by the polarization matrix (once called coherency matrix) [1–3]

$$\Phi \equiv \langle \boldsymbol{E}\boldsymbol{E}^\dagger \rangle = \begin{bmatrix} \langle E_1 E_1^* \rangle & \langle E_1 E_2^* \rangle \\ \langle E_2 E_1^* \rangle & \langle E_2 E_2^* \rangle \end{bmatrix}, \tag{2}$$

where $\langle \cdots \rangle$ denotes ensemble average. The two defining properties of the polarization matrix are Hermiticity, $\Phi^\dagger = \Phi$, and non-negativity, $\Phi \geq 0$: every 2×2 matrix obeying these two conditions is a valid polarization matrix. It is clear that the intensity corresponds to trΦ, and fully polarized (pure) states describable by Jones vectors $\boldsymbol{E}$ correspond to det$\Phi = 0$. Partially polarized or mixed states correspond to det$\Phi > 0$.

An alternative, but equivalent, way of describing the polarization state of a plane wave is by means of the so-called Stokes parameters. These are four real numbers, traditionally denoted S_i, $(i = 0, \ldots, 3)$ and connected to the polarization matrix by the linear relations

$$\begin{aligned} S_0 &= \Phi_{11} + \Phi_{22}; \qquad S_1 = \Phi_{11} - \Phi_{22}; \\ S_2 &= \Phi_{12} + \Phi_{21}; \qquad S_3 = i(\Phi_{12} - \Phi_{21}). \end{aligned} \tag{3}$$

We note in passing that the Stokes parameters turn out to be the coefficients of the expansion of Φ into a sum of Pauli matrices (plus the unit matrix) [1]. The parameters S_i, $(i = 0, \ldots, 3)$, are measurable quantities and define the components of the Stokes vector $S \in \boldsymbol{R}^4$. It is seen, in particular, that the intensity equals $S_0 = \mathrm{tr}\Phi$. While Hermiticity of Φ is equivalent to reality of the Stokes vector S, the non-negativity conditions tr$\Phi > 0$, det$\Phi \geq 0$ read $S_0 > 0$, and $S_0^2 - S_1^2 - S_2^2 - S_3^2 \geq 0$, respectively. The space of all real vectors $S \in \boldsymbol{R}^4$ satisfying these two conditions will be denoted by $\Omega^{(\mathrm{pol})}$:

$$\Omega^{(\mathrm{pol})} = \{S \in \boldsymbol{R}^4 | S_0 > 0, S_0^2 - S_1^2 - S_2^2 - S_3^2 \geq 0\}. \tag{4}$$

It is the traditional state space for polarization optics.

Typical systems of interest in polarization optics are spatially homogeneous (in the transverse plane), in the sense that their action is independent of the coordinates (x, y). If such a system is deterministic and acts linearly on the field amplitude, it is described by a complex 2×2

numerical matrix J, the Jones matrix of the system [1–3]:

$$J\colon \boldsymbol{E} \to \boldsymbol{E}' = J\boldsymbol{E} \Leftrightarrow \Phi \equiv \langle \boldsymbol{E}\boldsymbol{E}^\dagger \rangle \to \Phi' = \langle \boldsymbol{E}'\boldsymbol{E}'^\dagger \rangle = J\Phi J^\dagger. \tag{5}$$

Anisotropic optical elements can be absorbing. Accordingly, the intensity $S_0 = \mathrm{tr}\Phi$ need not be preserved and hence J need not be unitary. It is clear that Jones systems map pure states ($\det\Phi = 0$) into pure states.

We can go from a pair of indices, each running over 1 and 2, to a single index running over 0 to 3 and vice versa, according to the following correspondence rule:

$$\begin{bmatrix} \tilde{\Phi}_0 \\ \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \\ \tilde{\Phi}_3 \end{bmatrix} = \begin{bmatrix} \Phi_{11} \\ \Phi_{12} \\ \Phi_{21} \\ \Phi_{22} \end{bmatrix}. \tag{6}$$

The one-to-one relationship in Eq. (3) between S and Φ may thus be written as the vector equation

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_0 \\ \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \\ \tilde{\Phi}_3 \end{bmatrix}. \tag{7}$$

Linear optical systems of interest can be more general than the ones described by Jones matrices. Such a general system acts directly on the Stokes vector rather than through the Jones vector. It is specified by a numerical 4×4 real matrix called the Mueller matrix, transforming the Stokes vectors linearly:

$$M\colon S \to S' = MS. \tag{8}$$

Mueller matrix of a Jones system specified by a matrix J will be called Mueller-Jones matrix $M(J)$.

Since M produces a linear transformation on S, the linear invertible relationship (3) or (7) between S and Φ implies that M will induce a linear transformation $H^{(M)}$ on Φ. Such a transformation may be written as [4]:

$$H^{(M)}\colon \Phi \to \Phi', \qquad \Phi'_{ij} = \sum_{k\ell} H^{(M)}_{ik,j\ell}\Phi_{k\ell}. \tag{9}$$

That Φ' needs to be Hermitian for all Hermitian Φ demands that the map $H^{(M)}$, viewed as a 4×4 matrix with ik (going over 0 to 3) labeling the rows and $j\ell$ labeling the columns, be Hermitian. It can be seen that this correspondence between real matrices M and Hermitian matrices $H^{(M)}$ is one-to-one [4]. Elements of $H^{(M)}$ in terms of those of M can be found in Eq. (8) of Ref. [4].

If the system described by M is a Jones system with Jones matrix J, it is not difficult to see [4] that $H^{(M)} = \tilde{J}\tilde{J}^\dagger$, where $\tilde{J}$ is the column vector associated with the 2×2 matrix J according to a correspondence rule similar to that of Eq. (6). Thus, we arrive at the following result of fundamental importance [4].

Proposition 1.—A Mueller matrix M represents a Jones system if and only if the associated Hermitian matrix $H^{(M)}$ is a one-dimensional projection. If $H^{(M)}$ is such a projection $\tilde{J}\tilde{J}^\dagger$, then $M = M(J)$, J being the 2×2 matrix associated with the column vector $\tilde{J}$.

As a consequence, we have [4,5]

Proposition 2.—A real matrix M can be realized as a positive sum (ensemble) of Mueller-Jones matrices if and only if the associated Hermitian matrix $H^{(M)}$ is positive semidefinite. If $H^{(M)} = \sum_k \tilde{J}^{(k)}\tilde{J}^{(k)\dagger}$, then $M = \sum_k M(J^{(k)})$ where $M(J^{(k)})$ is the Mueller-Jones matrix associated with $J^{(k)}$.

With this brief outline, we are ready to describe the fundamental issue being addressed in the present Letter.

The Issue.—The Mueller-Stokes formalism takes $\Omega^{(\mathrm{pol})}$ as the state space. Thus, given a 4×4 real matrix M, in order that it qualifies to be a Mueller matrix one should demand that it maps the state space $\Omega^{(\mathrm{pol})}$ into itself. Let us denote by $\mathcal{M}$ the collection of all such matrices. We shall further denote by $\mathcal{M}^{(+)}$ the collection of M matrices which can be realized as positive sum of Mueller-Jones matrices $M(J)$. It is clear that $\mathcal{M}^{(+)}$ is contained in $\mathcal{M}$. The structure of $\mathcal{M}^{(+)}$ is fairly simple: we know from Propositions 1 and 2 that elements of $\mathcal{M}^{(+)}$ are in one-to-one correspondence with non-negative 4×4 matrices $H^{(M)}$ [4,5]. But the structure of $\mathcal{M}$ is considerably more involved. Owing to a sequence of developments [6–10], which are surprisingly recent in relative terms, a complete characterization of $\mathcal{M}$ is presently available.

That elements of $\mathcal{M}^{(+)}$ are Mueller matrices is clear, for they are realizable as positive sums of Jones systems. That M matrices which fall outside $\mathcal{M}$ are not Mueller matrices is also clear, for they fail to map the state space $\Omega^{(\mathrm{pol})}$ into itself. Thus the issue is really one about the gray domain "in between"—the complement of $\mathcal{M}^{(+)}$ in $\mathcal{M}$: are these M matrices physical Mueller matrices?

By definition, members of this domain cannot be realized as positive sums of Jones systems. But they map $\Omega^{(\mathrm{pol})}$ into itself. There exists, of course, no known scheme to realize them physically. On the other hand there are Mueller matrices, extracted from actual experiments reported, which fall deep into this gray domain (we shall consider later an example from Ref. [11]).

There are two difficulties in simply dismissing these matrices as unphysical: first, the experimenters did not realize them as positive sums of Jones systems, and so the fact that they fall outside $\mathcal{M}^{(+)}$ cannot be enough reason to dismiss them; and second, within the Mueller-Stokes formalism there seems to exist no additional qualification we can demand of a Mueller matrix, over and above the requirement that it should map $\Omega^{(\mathrm{pol})}$ into itself.

In this Letter we present a compelling physical ground which judges every M matrix which is not an element of $\mathcal{M}^{(+)}$ as unphysical; it comes from consideration of entanglement or inseparability between the polarization and spatial degrees of freedom of light beams.

Nonquantum entanglement.—Let us now go beyond plane waves and consider paraxial electromagnetic beams. The simplest beam field has, in a transverse plane $z =$ constant described by coordinates $(x, y) \equiv \boldsymbol{\rho}$, the form $\boldsymbol{E}(\boldsymbol{\rho}) = (E_1\hat{\boldsymbol{x}} + E_2\hat{\boldsymbol{y}})\psi(\boldsymbol{\rho})$, where E_1, E_2 are complex constants, and the scalar-valued function $\psi(\boldsymbol{\rho})$ may be assumed to be square-integrable over the transverse plane: $\psi(\boldsymbol{\rho}) \in L^2(\boldsymbol{R}^2)$. We denoted by $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ the unit vectors, respectively, along the x, y axes. It is clear that the polarization part $(E_1\hat{\boldsymbol{x}} + E_2\hat{\boldsymbol{y}})$ and the spatial dependence or modulation part $\psi(\boldsymbol{\rho})$ of such an uniformly polarized beam are well separated, allowing one to focus attention on one aspect at a time. When one is interested in only the modulation aspect, the part $(E_1\hat{\boldsymbol{x}} + E_2\hat{\boldsymbol{y}})$ may be suppressed, thus leading to "scalar optics." On the other hand, if the spatial part $\psi(\boldsymbol{\rho})$ is suppressed we are led to the traditional polarization optics or Mueller-Stokes formalism for plane waves.

Beams whose polarization and spatial modulation separate in the above manner will be called elementary beams. Suppose we superpose or add two such elementary beam fields $(a\hat{\boldsymbol{x}} + b\hat{\boldsymbol{y}})\psi(\boldsymbol{\rho})$ and $(c\hat{\boldsymbol{x}} + d\hat{\boldsymbol{y}})\zeta(\boldsymbol{\rho})$. The result is not of the elementary form $(e\hat{\boldsymbol{x}} + f\hat{\boldsymbol{y}})\phi(\boldsymbol{\rho})$, for any e, f, $\phi(\boldsymbol{\rho})$, unless either (a, b) is proportional to (c, d) so that one gets committed to a common polarization, or $\psi(\boldsymbol{\rho})$ and $\zeta(\boldsymbol{\rho})$ are proportional so that one gets committed to a fixed spatial mode. Thus, the set of elementary fields is not closed under superposition.

Since superposition principle is essential for optics, we are led to consider beam fields of the more general form $\boldsymbol{E}(\boldsymbol{\rho}) = E_1(\boldsymbol{\rho})\hat{\boldsymbol{x}} + E_2(\boldsymbol{\rho})\hat{\boldsymbol{y}}$, and consequently to pay attention to the implications of inseparability or entanglement of polarization and spatial variation. This more general form is obviously closed under superposition. We may write $\boldsymbol{E}(\boldsymbol{\rho})$ as a (generalized) Jones vector

$$\boldsymbol{E}(\boldsymbol{\rho}) = \begin{bmatrix} E_1(\boldsymbol{\rho}) \\ E_2(\boldsymbol{\rho}) \end{bmatrix}, \qquad E_1(\boldsymbol{\rho}), \qquad E_2(\boldsymbol{\rho}) \in L^2(\boldsymbol{R}^2). \quad (10)$$

The intensity at location $\boldsymbol{\rho}$ corresponds to $|E_1(\boldsymbol{\rho})|^2 + |E_2(\boldsymbol{\rho})|^2$. This field is of the elementary or separable form if and only if $E_1(\boldsymbol{\rho})$ and $E_2(\boldsymbol{\rho})$ are linearly dependent (proportional to one another). Since such a proportionality is rather exceptional, it is to be expected that, in a typical electromagnetic beam, polarization and spatial modulation are inseparably entangled.

We can handle fluctuating beams, by means of the so-called beam-coherence-polarization (BCP) matrix $\Phi(\boldsymbol{\rho}; \boldsymbol{\rho}') \equiv \langle \boldsymbol{E}(\boldsymbol{\rho})\boldsymbol{E}(\boldsymbol{\rho}')^\dagger\rangle$ [12]:

$$\Phi(\boldsymbol{\rho}; \boldsymbol{\rho}) = \begin{bmatrix} \langle E_1(\boldsymbol{\rho})E_1(\boldsymbol{\rho}')^*\rangle & \langle E_1(\boldsymbol{\rho})E_2(\boldsymbol{\rho}')^*\rangle \\ \langle E_2(\boldsymbol{\rho})E_1(\boldsymbol{\rho}')^*\rangle & \langle E_2(\boldsymbol{\rho})E_2(\boldsymbol{\rho}')^*\rangle \end{bmatrix}. \quad (11)$$

As the name suggests, the BCP matrix describes both the coherence and polarization properties. It is a generalization of the numerical matrix of Eq. (2), to the case of (possibly inhomogeneously polarized) beam fields.

It is clear from the very definition in Eq. (11) of BCP matrix that this matrix kernel, viewed as an operator, is Hermitian non-negative:

$$\begin{gathered}\Phi_{k\ell}(\boldsymbol{\rho}; \boldsymbol{\rho}') = \Phi^*_{\ell k}(\boldsymbol{\rho}'; \boldsymbol{\rho}), \qquad \ell, k = 1, 2; \\ Q(\boldsymbol{F}) = \int d^2\rho d^2\rho' \boldsymbol{F}(\boldsymbol{\rho})^\dagger \Phi(\boldsymbol{\rho}; \boldsymbol{\rho}')\boldsymbol{F}(\boldsymbol{\rho}') \geq 0,\end{gathered} \quad (12)$$

for any (well behaving) vector $\boldsymbol{F}(\boldsymbol{\rho})$. These are the defining properties of the BCP matrix: every 2×2 matrix of two-point functions $\Phi_{k\ell}(\boldsymbol{\rho}; \boldsymbol{\rho}')$ meeting just these two conditions is a valid BCP matrix of some beam of light.

Resolution of the issue.—In the BCP matrix of Eq. (11), each of the four elements $\Phi_{k\ell}(\boldsymbol{\rho}; \boldsymbol{\rho}') = \langle E_k(\boldsymbol{\rho})E_\ell(\boldsymbol{\rho}')^*\rangle$ is an (infinite-dimensional) operator $L^2(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}^2)$. For the issue on hand, however, it proves sufficient to limit ourselves to a much more restricted class of beams, corresponding to a two-dimensional rather than infinite-dimensional space. More precisely, let us consider those (pure) beams that can be expressed as

$$\boldsymbol{E}(\boldsymbol{\rho}) = \psi_1(\boldsymbol{\rho})\hat{\boldsymbol{x}} + \psi_2(\boldsymbol{\rho})\hat{\boldsymbol{y}}, \quad (13)$$

where ψ_1 and ψ_2 are orthonormal functions. It should be stressed that fields of the form of Eq. (13) are necessarily entangled (inhomogeneously polarized).

The elements of the BCP matrix corresponding to $\boldsymbol{E}(\boldsymbol{\rho})$ in Eq. (13) take the form $\Phi_{k\ell}(\boldsymbol{\rho}, \boldsymbol{\rho}') = \psi_k(\boldsymbol{\rho})\psi^*_\ell(\boldsymbol{\rho}')$. According to Eq. (9), the transformed BCP matrix elements are

$$\Phi'_{ij}(\boldsymbol{\rho}, \boldsymbol{\rho}') = \sum_{k\ell} H^{(M)}_{ik,j\ell}\psi_k(\boldsymbol{\rho})\psi^*_\ell(\boldsymbol{\rho}'). \quad (14)$$

The quadratic form Q appearing in Eq. (12) reads

$$Q(\boldsymbol{F}) = \sum_{ij} \iint \Phi'_{ij}(\boldsymbol{\rho}, \boldsymbol{\rho}')F^*_i(\boldsymbol{\rho})F_j(\boldsymbol{\rho}')d^2\rho d^2\rho'. \quad (15)$$

Let us assume F_i, $(i = 1, 2)$, to have the form

$$F_i(\boldsymbol{\rho}) = \sum_\alpha c_{i\alpha}\psi_\alpha(\boldsymbol{\rho}), \qquad (i = 1, 2), \quad (16)$$

where the (complex) constants $c_{i\alpha}$, $(i,\ \alpha = 1, 2)$, can be chosen at will. On inserting Eqs. (14) and (16) into Eq. (15) and using the orthonormality of ψ_1 and ψ_2, we obtain

$$Q(\boldsymbol{F}) = Q(\{c_{i\alpha}\}) = \sum_{ij}\sum_{\alpha\beta} H^{(M)}_{i\alpha,j\beta}c^*_{i\alpha}c_{j\beta}. \quad (17)$$

Thus, the physical requirement $Q(\{c_{i\alpha}\}) \geq 0$ for any choice of the $c_{i\alpha}$'s demands $H^{(M)}$ to be non-negative.

This leads to our final result. Suppose $H^{(M)}$ is not non-negative. Then, Φ', the result of M acting on the inhomogeneously polarized Jones vector (13), fails to be non-negative and hence is unphysical, showing in turn that M could not have been physical. Thus $H^{(M)} \geq 0$ is a necessary condition for M to be a physical Mueller matrix. On the other hand, we have seen that if $H^{(M)} \geq 0$ then M can be physically realized as a positive sum of Jones systems, showing that $H^{(M)} \geq 0$ is a sufficient condition for M to be a Mueller matrix. We thus have

Theorem: The necessary and sufficient condition for M to be a physical Mueller matrix is that the associated Hermitian matrix $H^{(M)} \geq 0$. Every physical Mueller matrix is a positive sum of Mueller-Jones matrices.

Thus, M matrices which map $\Omega^{(\mathrm{pol})}$ into itself should be called pre-Mueller matrices rather than Mueller matrices. For, to be promoted to the status of Mueller matrices they need to meet the stronger condition $H^{(M)} \geq 0$ arising from consideration of entanglement.

Discussion.— For a simple illustration of the gray region between $\mathcal{M}^{(+)}$ and $\mathcal{M}$, the focus of this work, let us consider M matrices of the diagonal form $\mathrm{diag}(d_0, d_1, d_2, d_3)$. It is clear that such an M will map $\Omega^{(\mathrm{pol})}$ into $\Omega^{(\mathrm{pol})}$, and hence be in $\mathcal{M}$, iff $|d_k/d_0| \leq 1$, $k = 1, 2, 3$. In the Euclidean space $\boldsymbol{R}^3$ spanned by the parameters $(d_1/d_0,\ d_2/d_0,\ d_3/d_0)$ this corresponds to the solid cube with vertices at $(\pm 1, \pm 1, \pm 1)$. Now $H^{(M)}$ associated with $\mathrm{diag}(d_0, d_1, d_2, d_3)$, as computed from Eq. (8) of Ref. [4], is

$$H^{(M)} = \frac{1}{2}\begin{bmatrix} d_0 + d_1 & 0 & 0 & d_2 + d_3 \\ 0 & d_0 - d_1 & d_2 - d_3 & 0 \\ 0 & d_2 - d_3 & d_0 - d_1 & 0 \\ d_2 + d_3 & 0 & 0 & d_0 + d_1 \end{bmatrix}. \tag{18}$$

We see $H^{(M)} \geq 0$ iff $-d_1 - d_2 - d_3 \leq d_0$, $-d_1 + d_2 + d_3 \leq d_0$, $d_1 + d_2 - d_3 \leq 1$, and $d_1 - d_2 + d_3 \leq d_0$. i.e., iff $(d_1/d_0,\ d_2/d_0,\ d_3/d_0)$ is in the solid tetrahedron with vertices at $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$, occupying one-third the volume of the cube.

Since Mueller-Jones matrices are physical, it is clear that the matrix $M(J)MM(J')$, where $M(J)$ and $M(J')$ are invertible Mueller-Jones matrices, is a physical Mueller matrix iff M is. As shown in Ref. [9], the symmetric Mueller matrix reported by van Zyl *et al.* [11] corresponds to $(d_0, d_1, d_2, d_3) = (0.9735, 0.9112, 0.4640, -0.3838)$. This clearly corresponds to a point inside our cube since $|d_k| < d_0$, $k = 1, 2, 3$. But it sits well outside the tetrahedron; indeed, the third inequality $d_1 + d_2 - d_3 \leq d_0$ reads $1.759 \leq 0.9735$, a substantial violation.

Before we conclude, some remarks concerning use of the term entanglement are in order. First, entanglement encountered in quantum mechanics generically couples variables of two (or more) physical systems. However, entanglement could be between two different attributes or degrees of freedom of one and the same system. An example of the latter is the spin and spatial degrees of freedom of an electron in d-dimensional space [13]. Our problem of inhomogeneous polarization of optical beams has the same kinematic or mathematical structure as that of the quantum electron, with $d = 2$.

Second, entanglement carries, in the quantum context, a rich variety of implications such as nonlocality and Bell's inequality violation. Some aspects of entanglement are purely kinematic and arise directly from the superposition principle in the tensor product of two or more Hilbert spaces, and such aspects can be expected to manifest in classical wave optics as well. There are others which arise from quantum measurement and the associated collapse of states, but classical optics does not share in this luxury and richness. The phrase nonquantum entanglement has been used here to stress that certain situations in polarization optics need the same type of mathematical description that applies to quantum entanglement, not to suggest that quantum phenomena have necessarily a correspondence within classical optics.

Finally, it has been appreciated for long [14] that the applicability of one and the same type of mathematics to two different physical realms often leads to beneficial advancements in one field, on the basis of knowledge acquired in the other. In the present Letter, we have seen just one important example of this.

In view of the rapidly growing current interest in an unified approach to coherence and polarization in optics, it is hoped that our result will stimulate further research into the kinematic role of entanglement in classical optics.

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