Synthesis of electromagnetic Schell-model sources

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1. INTRODUCTION

Schell-model (SM) sources were introduced in the scalar theory of coherence [1,2] as a fundamental tool to describe partially coherent light sources having shift-invariant degree of spatial coherence. They found application in the study of the radiation emitted by several natural sources [3,4] and, since they can always be synthesized in the laboratory starting from spatially incoherent sources [5], they soon became a typical tool for experimentally validating several results of the scalar theory of coherence [6–14].

The generalization of SM sources to the vectorial case has been considered in recent times, and, as well as their scalar counterparts, electromagnetic SM (ESM) sources [15] are playing an important role in the development of modern coherence theory. In particular, great attention has been devoted to the so-called electromagnetic Gaussian SM (EGSM) sources, which have been introduced [16–18] within the framework of the electromagnetic theory of coherence as the natural generalization to the vectorial case of scalar GSM sources [2,19]. Since their introduction, EGSM sources have proved to be useful in revealing interesting and intriguing new coherence and polarization features in the study of random electromagnetic beams [20–33].

On the other hand, synthesis of ESM sources is still far from being a common task in optics laboratories, and, until very recent times, experimental procedures had been proposed only for some specific classes of ESM sources [34,35]. The first experimental arrangement was presented some years ago [34] for the synthesis of a particular type of EGSM source. These ESGM sources were generated by exploiting the van Cittert–Zernike (vCZ, henceforth) theorem for electromagnetic sources [36], starting from two spatially incoherent primary sources that had suitable intensity profiles and were linearly polarized along orthogonal axes and that eventually superimposed through a Mach–Zehnder interferometer (MZI, henceforth). Later, a different method for generating EGSM beams, which does not use the vCZ theorem, was proposed [35]. According to it, the source is synthesized by superimposing two spatially coherent beams with orthogonal polarizations, their phases modulated by two mutually correlated liquid-crystal spatial light modulators and their intensities modulated by two Gaussian-shaped amplitude filters. Even in this case, however, obtainable sources constitute a subset of the most general class of EGSM sources.

In a recent paper, however, a procedure that represents the extension to the vectorial case of the well-established technique used in the scalar case has been suggested for the synthesis of the most general ESM source [37]. It has been shown, in fact, that any ESM source can be synthesized by using the vCZ theorem, starting from a primary planar spatially incoherent source that is characterized by a suitable position-dependent polarization matrix.

Controlling the intensity distribution and the polarization state of a spatially incoherent source is a task that is far from trivial from an experimental point of view. In the present paper we propose a possible arrangement for producing such a source, to be used in the synthesis of a general ESM source. The technique exploits the fact that any polarization matrix can be synthesized starting from two perfectly polarized fields having orthogonal polarizations, incoherently superimposed with suitable powers. The analytical properties of such fields suggest the use of an experimental setup based on a MZI, endowed with amplitude transmittances placed into its arms, and anisotropic optical elements. As we shall see, in many cases of practical interest only conventional optical elements are needed.

The paper is arranged as follows. In Section 2, ESM...
sources are described in the framework of the so-called beam-coherence–polarization (BCP) matrix, and realizability conditions for such sources are given. In Sections 3 and 4 the general synthesis scheme and the procedure to generate the primary incoherent source are given. As an example, the proposed method is applied to the particular case of EGSM sources in Section 5. Finally, the main conclusions are summarized in Section 6.

2. ELECTROMAGNETIC SCHELL-MODEL SOURCES

We shall describe an ESM source by means of the BCP matrix [38,39]. Let us recall that this is a $2 \times 2$ matrix, whose elements are defined through the relations

$$J_{ab}(r_1,r_2) = \langle E_a(r_1,t)E_b^*(r_2,t) \rangle; \quad (\alpha,\beta = x,y),$$

(1)

where $E_a(r,t)$ ($a=x,y$) is the analytic signal associated with the electric field component along the $a$ axis of a quasi-monochromatic wave at position $r$ and time $t$. The angular brackets denote time averages. The extension to a space–frequency approach, where the BCP matrix is replaced by the cross-spectral density tensor [2,3,15], is straightforward and will not be considered here.

The most general ESM source is characterized by a BCP matrix whose elements are of the following form [2,15]:

$$J_{ab}(r_1,r_2) = s_a(r_1)s_b^*(r_2)J_{ab}(r_1-r_2).$$

(2)

Its diagonal elements ($J_{aa}, \alpha=x,y$) correspond to the mutual intensities that would characterize the source if the $\beta$ component of the field ($\beta \neq \alpha$) were eliminated, for example, by a linear polarizer. In the present case, they represent scalar SM sources and the functions $J_{aa}$ are the corresponding degrees of spatial coherence [2], while $s_a(r)$ ($\alpha=x,y$), defined as

$$s_a(r) = \sqrt{J_{aa}(r,r)},$$

(3)

are nonnegative functions. The analytical forms of such degrees of spatial coherence cannot be chosen at will, because they have to ensure that the functions $J_{aa}$ represent nonnegative definite kernels, as required for spatial correlation functions [2]. It can be proved, however, that the nonnegativeness condition is satisfied if, and only if, $J_{aa}$ have nonnegative Fourier transforms, i.e. [8],

$$\tilde{J}_{aa}(\nu) \geq 0 \quad (\alpha=x,y)$$

(4)

for any $\nu$, where the tilde denotes Fourier transformation and $\nu$ is the vector position across the Fourier plane.

Functions $J_{xx}$ and $J_{yy}$, on the other hand, account for the correlations that exist at two distinct points between the $x$ and the $y$ component of the field. In particular, since $J_{xx}(r_1,r_2) = J_{yy}^*(r_2,r_1)$, Eqs. (2) and (3) lead to

$$\tilde{J}_{yy}(r_1-r_2) = \tilde{J}_{xx}(r_2-r_1),$$

(5)

or, equivalently, in the Fourier domain,

$$\tilde{J}_{yy}(\nu) = \tilde{J}_{xx}^*(\nu).$$

(6)

As far as the functional form of $J_{xx}$ is concerned, it cannot be arbitrary either, because of the nonnegativity con-
where \( P_{\alpha\beta} \) are the elements of \( \hat{P} \). It should be noted that, apart from curvature factors, an optical system of this kind corresponds to a free propagation in Fraunhofer conditions, so that Eq. (9) reduces to the vCZ theorem for electromagnetic sources [36].

On using Eqs. (8) and (9), together with the definition in Eq. (3), it is not difficult to show that the ESM source in Eq. (2) can be generated as the output of the optical system in Fig. 1, starting from a primary, spatially incoherent source characterized by a position-dependent polarization matrix, say, \( \hat{P}(\mathbf{u}) \), whose elements are given by

\[
P_{\alpha\beta}(\mathbf{u}) = \frac{1}{\lambda^2 f^2} j_{\alpha\beta} \left( \frac{\mathbf{u}}{\lambda f} \right),
\]

and using an amplitude mask across the plane \( \Pi_0 \) of the form

\[
t_\alpha(r) = s_\alpha(r).
\]

It is interesting to note that the requirement that the incoherent source across \( \Pi_1 \) be physically realizable is automatically satisfied if the condition in Eq. (7) is met. In fact, in order for the incoherent source to be realizable, the matrix \( \hat{P}(\mathbf{u}) \) must be a \textit{bona fide} polarization matrix; i.e., it must be Hermitian, positive semidefinite, and with nonnegative diagonal elements at any source point. Hermiticity of \( \hat{P}(\mathbf{u}) \) follows directly from Eqs. (10) and (6), while the positivity of its diagonal elements comes from Eq. (4). On the other hand, semipositivity of \( \hat{P} \) implies that its determinant must be nonnegative, i.e.,

\[
P_{xx}(\mathbf{u})P_{yy}(\mathbf{u}) \geq |P_{xy}(\mathbf{u})|^2
\]

for any \( \mathbf{u} \). Taking Eq. (10) into account, the latter condition reduces to the following one:

\[
\frac{j_{xx}\left( \frac{\mathbf{u}}{\lambda f} \right) j_{yy}\left( \frac{\mathbf{u}}{\lambda f} \right)}{j_{xy}\left( \frac{\mathbf{u}}{\lambda f} \right)} \geq \frac{j_{xx}\left( \frac{\mathbf{u}}{\lambda f} \right)}{j_{xy}\left( \frac{\mathbf{u}}{\lambda f} \right)}^2,
\]

which is equivalent to that in Eq. (7). This means, in turn, that any ESM source can be realized through a synthesis scheme of this kind [37].

4. SYNTHESIS OF THE INCOHERENT SOURCE

It is easily realized that a critical point of the above procedure is represented by the synthesis of the primary incoherent source, whose position-dependent polarization matrix is defined in Eq. (10). In some simple cases this may not represent a serious problem [34], but in general one might need to control both the intensity and the polarization state of the source point by point, which is a rather challenging experimental problem. As we are going to show, however, this problem can be solved in the most general case by noting that the polarization matrix can always be expressed through its spectral decomposition [40], which in the present case reads

\[
\hat{P} = \mu_+ \mathbf{U} \mathbf{U}_+^\dagger + \mu_- \mathbf{U} \mathbf{U}_-^\dagger.
\]

Here, \( \mu_\pm \) are the two eigenvalues of \( \hat{P} \), \( \mathbf{U}_\pm \) are the corresponding eigenvectors, represented by column vectors, and the dagger denotes Hermitian conjugation. Of course, for a \textit{bona fide} polarization matrix the eigenvalues are nonnegative and the eigenvectors are (or can be chosen as) orthonormal.

Physically, the expansion in Eq. (14) states that any polarization matrix can be synthesized starting from two perfectly polarized fields having orthogonal polarizations, represented by the Jones vectors \( \mathbf{U}_\pm \), incoherently superimposed with powers given by \( \mu_\pm \).

Eigenvalues and eigenvectors of \( \hat{P} \) can be readily evaluated and turn out to be

\[
\mu_\pm = \frac{1}{2} [(P_{xx} + P_{yy}) \pm \sqrt{(P_{xx} - P_{yy})^2 + 4|P_{xy}|^2}],
\]

and

\[
\mathbf{U}_\pm = \frac{1}{\sqrt{1 + \eta}} \begin{bmatrix} \eta e^{i\phi} \\
1
\end{bmatrix},
\]

respectively, where

\[
\eta = \frac{(P_{xx} - P_{yy}) + \sqrt{(P_{xx} - P_{yy})^2 + 4|P_{xy}|^2}}{2|P_{xy}|},
\]

and \( \phi \) is the argument of the complex number \( P_{xy} \). In the above expressions we omitted, for brevity, the explicit dependence on \( \mathbf{u} \), but it should be kept in mind that the eigenvalues, the quantity \( \eta \), and the phase \( \phi \) depend in general on the position across the source. Furthermore, it should be stressed that \( \eta \) is nonnegative for any value of \( \mathbf{u} \), as is evident from Eq. (17), but it may assume any value, including, as limiting values, zero and infinity, corresponding to modes polarized along \( x \) or \( y \).

The symmetry of the mode structure evidenced in Eq. (16) suggests a practical way for producing the spatially incoherent source to be used in the synthesis of the ESM source. The scheme is depicted in Fig. 2. Basically, it relies on a MZI fed by two independent collimated monochromatic laser beams (\( \ell_z \)), linearly polarized at \( \pi/4 \) with respect to the horizontal, entering the interferometer from two orthogonal directions.

Consider first Fig. 2(a). The laser beam \( \ell_z \), after being expanded and collimated, passes through a transparency characterized by the real transmission function \( t_z = A \sqrt{\mu_+} \), placed at the transverse plane \( \Pi_1 \). The parameter \( A \) is an arbitrary amplitude factor and can be chosen at will, for instance, to set to 1 the maximum value of \( t_z \).

The polarizing beam splitter (PBS) splits the initial polarization state in such a way that the reflected component (\( y \) polarized) is sent to the upper branch of the MZI, while the transmitted component (\( x \) polarized) goes through the lower branch. The lens \( L_z \) images the field emerging from \( \Pi_1 \) onto the planes \( \Pi_2 \) and \( \Pi_3 \), in the lower
and in the upper path, respectively. At planes $\Pi_L$ and $\Pi_d$, two transparencies, having transmission functions given by $\eta_x = \eta_x / \sqrt{1 + \eta_x^2}$ and $\eta_y = 1 / \sqrt{1 + \eta_y^2}$, respectively, are placed, so that the $x$ component of the field is multiplied by $\eta_x$ and the $y$ component by $\eta_y$. Note that both such functions take values between 0 and 1, so that the corresponding transparencies consist in amplitude-only masks. The collecting, nonpolarizing beam splitter BS eventually provides the superposition of the images of the fields emerging from $\Pi_L$ and $\Pi_d$ onto the plane $\Pi_i$, through the lens $L$. We shall assume, without loss of generality, that the optical lengths of the two arms are identical (or differ by an integer number of wavelengths).

At the plane $\Pi_i$, the phase difference $\phi$ between the $x$ and the $y$ component of the mode is introduced by means of a suitable anisotropic optical element (denoted by $\Phi$ in Fig. 2). The latter may consist, for instance, of a plate made of a birefringent material having locally varying thickness. The radiation emerging from the plate is even-

![Fig. 2. (Color online) A MZI fed by two independent laser beams: $\ell_+$ (a) and $\ell_-$ (b).](image)

...at the same plane, a converging lens could be used to image on $G$ the field emerging from $\Phi$, or vice versa. It is important to stress, however, that for the synthesis of significant classes of ESM sources, namely, those for which the argument of $P_{xy}$ is independent of the spatial coordinate $u$, the element $\Phi$ consists of a conventional retardation plate that is used to introduce a spatially uniform phase delay between the two polarization components. In such cases, the synthesis of the source is performed using only conventional optical elements and amplitude transparencies. An example of such sources will be presented in the following section.

As far as the second term on the right-hand side of Eq. (14) is concerned, it is not difficult to realize that its synthesis can be achieved by using another laser beam, $\ell_-$, polarized at $45^\circ$, feeding the same interferometer from the other face of the PBS, as depicted in Fig. 2(b). The lens $L$ images the field emerging from a transparency, located across $\Pi_L$ and characterized by the field transmission function $\tau = A / \mu_x$. The role of the PBS is somewhat reversed, since the reflected component (polarized along $y$) passes through the lower arm and is imaged onto the transparency $\eta_x$, while the transmitted one (polarized along $x$) passes through the upper arm of the interferometer and is modulated by $\eta_y$. As for the previous case, the two components are recombined by the BS and imaged onto the plane $\Pi_i$, by the lens $L$, but now they are in phase opposition because, within the interferometer, the $y$ component of the electric field undergoes more reflections than in the previous case (see Fig. 2). Again, the phase difference $\phi$ between the orthogonal components of the field is given to the mode through the element $\Phi$.

If the interferometer is fed by $\ell_+$ and $\ell_-$ at the same time, a spatially incoherent source with locally varying polarization matrix $\hat{P}$ is produced across the plane $\Pi_i$. Such source can be used as the input of the optical system in Fig. 1 to produce ESM sources with assigned coherence–polarization properties. In the next section, results obtained here will be applied to the case of EGSM sources.

5. EGSM SOURCES

The most general EGSM source is characterized by the following BCP matrix [15]:

$$J_{\alpha\beta}(r_1, r_2) = I_{\alpha\beta} \exp\left(-\frac{r_1^2}{4\sigma_\alpha^2}\right) \exp\left(-\frac{r_2^2}{4\sigma_\beta^2}\right) \exp\left[-\frac{(r_2 - r_1)^2}{2\delta_{\alpha\beta}}\right],$$

where $\sigma_\alpha, \delta_{\alpha\beta} (\alpha, \beta = x, y)$ are real and positive parameters. Furthermore, $I_{\alpha\alpha} > 0$, while $I_{xy} = I_{\alpha\beta}$ can be complex.

The number of independent real parameters is nine, but the values of such parameters can be chosen at will only within some specific ranges, which are specified by imposing the nonnegative definiteness of the BCP matrix [17,37,41]. In fact, the condition in Eq. (13), ensuring nonnegativeness of the BCP matrix, in this case reads...
for any \( u \). In particular, since the functions on the left- and right-hand sides of Eq. (19) decrease monotonically, on considering such inequality for the limiting cases \( u \to 0 \) and \( u \to \infty \), the following fork inequality is derived:
from the results presented in the previous section, it follows that the EGSM source in Eq. (18) can be generated as the output of the optical system in Fig. 1, starting from a primary, spatially incoherent source characterized by a position-dependent polarization matrix whose elements are given by

$$P_{\alpha\beta}(u) = \frac{2\pi I_{\alpha\beta} \delta^{2}_{\alpha\beta}}{\lambda^2 \beta^2} \exp \left( -\frac{2\pi^2 \delta^{2}_{\alpha\beta} u^2}{\lambda^2 \beta^2} \right),$$

with a transmission mask at the plane $$\Pi$$, chosen as

$$t_{o}(r) = \exp \left( -\frac{r^2}{4\sigma_o^2} \right).$$

In this case, the phase of the upper off-diagonal term of the polarization matrix coincides with the argument of the complex number $$I_{xy}$$, and therefore it is independent of the spatial coordinate. Therefore, the phase modulator present in the experimental setup of Section 4 reduces to a spatially uniform retardation plate. The synthesis of any EGSM can then be achieved using only amplitude transparencies and conventional optical elements. In the following, for simplicity, we assume $$I_{xy}$$ to be real, so that we can get rid of the above retardation plate.

Numerical examples, shown in Fig. 3, refer to three different choices of the parameters corresponding to three typical cases of EGSM sources. Spatial coordinates and distances have been normalized in such a way that $$\sqrt{I_f} = 1$$, i.e., $$u/\sqrt{I_f} \rightarrow u$$ and $$\delta_{\alpha\beta} \rightarrow \delta_{\alpha\beta} (\alpha, \beta=x,y)$$. While the first case [Fig. 3(a)] corresponds to a rather generic situation, in the second one [Fig. 3(b)] there is no correlation between the orthogonal components of the field ($$I_{xy}=0$$), and in the third one [Fig. 3(c)] the orthogonal components have the same amplitude ($$I_{xx}=I_{yy}$$) and the same width of the degree of coherence ($$\delta_{xx}=\delta_{yy}$$).

The last example, in particular, corresponds the case studied in [34], where an EGSM source was synthesized by superimposing, through a MZI, two spatially incoherent sources with suitable intensity profiles, linearly polarized along x and y, respectively. Such profiles represented the sum and the difference, respectively, of two Gaussian functions having different widths and peak values. The EGSM was then produced after free propagation from the resulting spatially incoherent source, through the vCZ theorem. The final form of the BCP matrix of the source was then obtained by carrying out a rotation of $$\pi/4$$ of the polarization direction of the field.

It is worth showing how the setup presented here encompasses, as a particular case, that proposed and implemented in [34]. In fact, on choosing $$\delta_{xx}=\delta_{yy}$$ and $$I_{xx}=I_{yy}$$, the diagonal elements of $$P$$ turn out to be coincident, i.e., $$P_{xx}=P_{yy}$$. Therefore, from Eqs. (15) and (17), we have

$$\eta = 1,$$