IOP PUBLISHING JOURNAL OF OPTICS A: PURE AND APPLIED OPTICS

J. Opt. A: Pure Appl. Opt. **11** (2009) 085706 (7pp) doi:10.1088/1464-4258/11/8/085706

On genuine cross-spectral density matrices

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Received 9 March 2009, accepted for publication 6 May 2009
Published 28 May 2009
Online at stacks.iop.org/JOptA/11/085706

Abstract

For an electromagnetic stochastic beam, the choice of the mathematical structure of the cross-spectral density matrix is limited by the constraint of non-negative definiteness. We present a sufficient condition for building these matrices in such a way that this constraint is automatically satisfied. This allows us to put into evidence that electromagnetic beams can exhibit very peculiar correlation properties, some of which would not be encountered in scalar treatments. These results are illustrated by means of a number of examples.

Keywords: coherence, polarization, partially coherent electromagnetic sources

1. Introduction

It is well known that correlation functions for optical fields cannot be chosen at will because of the non-negative definiteness constraints. Let us recall such a constraint for the scalar case. We denote by $W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ the cross-spectral density (CSD) ([1], section 4.1) at two typical points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ of a planar source, omitting, for brevity, the dependence on the temporal frequency. Next, we consider the quadratic form

$$Q = \int \int f^*(\boldsymbol{\rho}_1) f(\boldsymbol{\rho}_2) W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2, \qquad (1)$$

where $f(\boldsymbol{\rho})$ is an arbitrary function, the integrals are extended across the source plane, and the asterisk denotes the complex conjugate. Then, the non-negative definiteness constraint means that, for any choice of f, the inequality

$$Q \geqslant 0, \qquad (2)$$

has to be satisfied by any genuine CSD. To ascertain whether this inequality is met for a given form of W is not trivial, because of the arbitrariness in the choice of f. In fact, it was shown that even plausible forms of the CSD can violate the above constraint [2]. A notable exception is constituted by Schell-model sources ([1], section 5.3). For such sources, the normalized form of W, namely the spectral degree of coherence ([1], section 4.2), is shift-invariant. It turns out that the non-negative definiteness for scalar Schell-model sources is ensured if the spectral degree of coherence has a non-negative Fourier transform [3]. In the general case, however, things are not that easy. A sufficiency condition for satisfying the constraint in equation (2) has recently been presented in [2]. More explicitly, it was found that W is a *bona fide* CSD if it can be expressed as follows

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int p(\boldsymbol{v}) H^*(\boldsymbol{\rho}_1, \boldsymbol{v}) H(\boldsymbol{\rho}_2, \boldsymbol{v}) \mathrm{d}^2 v, \qquad (3)$$

where $p(\boldsymbol{v})$ is an arbitrary non-negative weight function and $H(\boldsymbol{\rho}, \boldsymbol{v})$ is an arbitrary kernel. A particular case of equation (3) is obtained when $H(\boldsymbol{\rho}, \boldsymbol{v}) = V(\boldsymbol{\rho} - \boldsymbol{v})$, where V can be thought of as a coherent field distribution. In this case, the source associated with equation (3) is a superposition of mutually shifted and uncorrelated replicas of a given field. This type of superposition was first used in [4] and [5], and lately developed in [6, 7]. In what follows, for convenience, we shall loosely refer to equation (3) as the superposition rule.

It is worthwhile to recall that any genuine CSD is endowed, according to the definition of the CSD, with the Hermiticity property, i.e.,

$$W(\boldsymbol{\rho}_2, \boldsymbol{\rho}_1) = W^*(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2). \qquad (4)$$

In physical terms, this means that the interchange of points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ leads to a phase reversal, while leaving the modulus of the CSD unchanged. In other words, the two points are essentially equivalent.

In the present paper, we shall first extend the condition in equation (3) to the electromagnetic case, so that we will establish a superposition rule for CSD matrices. Then, we will discuss some peculiar features of the resulting constructed CSD matrices. In particular, we shall underline that the two points can play quite different roles. A number of examples will help to epitomize these issues. The significant case of electromagnetic Gaussian beams with partial coherence and partial polarization, generally quoted as electromagnetic Gaussian Schell-model beams [8–12], will also be touched upon. We shall see how the recipe that we are going to derive permits a safe construction of such beams, which are specified by a high number of parameters, without fear of violating the non-negative definiteness constraint.

The following notations will be used in the paper

$$\mathrm{rect}(t) = \begin{cases} 1, & |t| \leqslant 1/2 \\ 0, & |t| > 1/2; \end{cases}$$
$$\mathrm{step}(t) = \begin{cases} 1, & t \geqslant 0 \\ 0, & t < 0; \end{cases} \qquad \mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}. \tag{5}$$

2. Non-negative definiteness condition for cross-spectral density matrices

Let us denote by $\widehat{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ the CSD matrix ([1], section 9.1) at points $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2$ in the source plane. More explicitly, we let

$$\widehat{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \begin{bmatrix} W_{xx}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) & W_{xy}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \\ W_{yx}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) & W_{yy}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \end{bmatrix}. \tag{6}$$

The matrix elements are given by

$$W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \langle E_\alpha^*(\boldsymbol{\rho}_1) E_\beta(\boldsymbol{\rho}_2)\rangle; \qquad (\alpha = x, y; \beta = x, y), \tag{7}$$

where $E_\alpha(\boldsymbol{\rho})$ is the fluctuating electric field component along the α-axis at point $\boldsymbol{\rho}$ and the angular brackets denote an ensemble average. Even in this case, the frequency dependence is not made explicit. From now on, we shall use the symbols α and β as indexes, each of which can be either x or y.

In order to recall the non-negative definiteness condition, we define the quadratic form

$$Q = \sum_\alpha \sum_\beta \int\int f_\alpha^*(\boldsymbol{\rho}_1) f_\beta(\boldsymbol{\rho}_2) W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2, \tag{8}$$

where $f_x(\cdot)$ and $f_y(\cdot)$ are two arbitrary (well-behaving) functions. The non-negative definiteness condition is still expressed by equation (2), i.e., for any choice of f_x and f_y, Q must be non-negative ([13], section 6.6.1).

In order to extend the superposition rule specified by equation (3), we give the following form to the elements of the CSD matrix

$$W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int p_{\alpha\beta}(\boldsymbol{v}) H_\alpha^*(\boldsymbol{\rho}_1, \boldsymbol{v}) H_\beta(\boldsymbol{\rho}_2, \boldsymbol{v})\mathrm{d}^2 v, \tag{9}$$

where $p_{\alpha\beta}(\boldsymbol{v})$ are the elements of the following weight matrix

$$\widehat{p}(\boldsymbol{v}) = \begin{bmatrix} p_{xx}(\boldsymbol{v}) & p_{xy}(\boldsymbol{v}) \\ p_{xy}^*(\boldsymbol{v}) & p_{yy}(\boldsymbol{v}) \end{bmatrix}, \tag{10}$$

while $H_x(\boldsymbol{\rho}, \boldsymbol{v})$ and $H_y(\boldsymbol{\rho}, \boldsymbol{v})$ are two arbitrary kernels. In order to see the features of the $p_{\alpha\beta}$ elements, we first note that W_{xx} and W_{yy} have the same nature as scalar CSDs. In fact, if we let the beam pass through a linear polarizer aligned to the α-axis, the outcoming field is fully specified by the scalar CSD $W_{\alpha\alpha}$. Accordingly, the diagonal elements of the $\widehat{p}$ matrix must satisfy the conditions

$$p_{\alpha\alpha}(\boldsymbol{v}) \geqslant 0. \tag{11}$$

Now, let us insert the expressions from equation (9) into (8). Interchanging the order of integrations, this gives

$$Q = \sum_\alpha \sum_\beta \int p_{\alpha\beta}(\boldsymbol{v}) h_\alpha^*(\boldsymbol{v}) h_\beta(\boldsymbol{v})\mathrm{d}^2 v, \tag{12}$$

where

$$h_\alpha(\boldsymbol{v}) = \int f_\alpha(\boldsymbol{\rho}) H_\alpha(\boldsymbol{\rho}, \boldsymbol{v})\mathrm{d}^2\rho. \tag{13}$$

Taking inequalities in equation (11) into account, it is easily seen that Q will be non-negative provided that the further requirement

$$p_{xx}(\boldsymbol{v}) p_{yy}(\boldsymbol{v}) - |p_{xy}(\boldsymbol{v})|^2 \geqslant 0, \tag{14}$$

is satisfied for any $\boldsymbol{v}$. The pair of equations (11) and (14) imply that, for any $\boldsymbol{v}$, the matrix $\widehat{p}$ is non-negative definite and thus the quantity Q given by equation (12) is non-negative. To gain a deeper insight into the present problem, another way to derive these results is summarized in the appendix. In conclusion, constructing $W_{\alpha\beta}$ according to equation (9), which will be called the generalized superposition rule, ensures that the resulting CSD matrix satisfies the constraint of non-negative definiteness. The arbitrariness of the choice of $\widehat{p}$ (subject to equations (11) and (14)) and H_α will allow us to devise a wealth of genuine CSD matrices.

3. The kernels

Similarly to the scalar case [2], equation (9) can be intuitively read as describing the CSD matrix at the output of an optical system endowed with impulse responses H_x and H_y with respect to the two field components and fed by a spatially incoherent source, whose local polarization matrix is proportional to $\widehat{p}(\boldsymbol{v})$. Differences between H_x and H_y can be due to some kind of anisotropic behaviour of the optical system. For example, the system could be based on an interferometer in which orthogonal states of polarization are sent along different arms, where they encounter distinct optical elements. As far as the mathematical features of the CSD matrix are concerned, however, H_x and H_y simply represent kernels of linear transformations (with respect to $\boldsymbol{v}$). We then have plenty of choices for such kernels and each choice is likely to lead to distinct classes of CSD matrices. Fresnel, Fourier, Laplace, Hankel, and Mellin are examples of transforms that immediately come to mind. While it would be hopeless to give even a partial account of all of the cases that imagination might suggest, some significant features can be explored by focusing attention on a single type of integral transform.

A simple and significant class of CSD matrices is obtained by giving H_α a Fourier-like structure. More explicitly, we let

$$H_\alpha(\boldsymbol{\rho}, \boldsymbol{v}) = F_\alpha(\boldsymbol{\rho}) \exp[-2\pi i \boldsymbol{g}_\alpha(\boldsymbol{\rho}) \cdot \boldsymbol{v}], \tag{15}$$

where $\boldsymbol{g}_\alpha$ are arbitrary vectorial real functions and F_α are possible complex profile functions. On inserting from equation (15) into (9) we obtain

$$W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = F_\alpha^*(\boldsymbol{\rho}_1) F_\beta(\boldsymbol{\rho}_2) \tilde{p}_{\alpha\beta}[\boldsymbol{g}_\beta(\boldsymbol{\rho}_2) - \boldsymbol{g}_\alpha(\boldsymbol{\rho}_1)], \tag{16}$$

where the tilde denotes Fourier transformation. Since $\boldsymbol{g}_\alpha$ is arbitrary, it is clear that the elements $W_{\alpha\beta}$ can have rather sophisticated structures. Let us begin with the simplest case, namely

$$\boldsymbol{g}_\alpha(\boldsymbol{\rho}) = a\boldsymbol{\rho}, \tag{17}$$

where a is a constant. Then, equation (16) becomes

$$W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = F_\alpha^*(\boldsymbol{\rho}_1) F_\beta(\boldsymbol{\rho}_2) \tilde{p}_{\alpha\beta}[a(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)], \tag{18}$$

so that the source is of the vectorial Schell-model type ([1], section 9.4). This particular case has already been treated in [14].

In principle, we could choose $\boldsymbol{g}_\alpha$ in equation (15) as bizarre as we like and in any case we would obtain a valid CSD matrix. A simple choice, however, will suffice to show some novelty elements. Let

$$\boldsymbol{g}_\alpha(\boldsymbol{\rho}) = a_\alpha \boldsymbol{\rho}, \tag{19}$$

where a_α are constants. Now, equation (16) becomes

$$W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = F_\alpha^*(\boldsymbol{\rho}_1) F_\beta(\boldsymbol{\rho}_2) \tilde{p}_{\alpha\beta}(a_\beta \boldsymbol{\rho}_2 - a_\alpha \boldsymbol{\rho}_1), \tag{20}$$

or, more explicitly

$$\begin{aligned} W_{xx}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) &= F_x^*(\boldsymbol{\rho}_1) F_x(\boldsymbol{\rho}_2) \tilde{p}_{xx}[a_x(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)], \\ W_{yy}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) &= F_y^*(\boldsymbol{\rho}_1) F_y(\boldsymbol{\rho}_2) \tilde{p}_{yy}[a_y(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)], \\ W_{xy}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) &= F_x^*(\boldsymbol{\rho}_1) F_y(\boldsymbol{\rho}_2) \tilde{p}_{xy}(a_y \boldsymbol{\rho}_2 - a_x \boldsymbol{\rho}_1). \end{aligned} \tag{21}$$

It is seen that, if $a_x \neq a_y$, the Fourier transforms appearing in the diagonal elements have different scale factors, but in any case they depend on $\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$ only. In other words, W_{xx} and W_{yy} are the same as scalar CSDs of the Schell-model type. Something different happens for W_{xy}. In fact, since $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ can be multiplied by different factors, we see that adding a term, say $\delta\boldsymbol{\rho}$, once to $\boldsymbol{\rho}_1$ and another time to $\boldsymbol{\rho}_2$, may produce different effects. As a limiting example, let $a_x \neq 0$ and $a_y = 0$. Then, W_{xy} becomes independent of $\boldsymbol{\rho}_2$. This means that while $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ play essentially the same role as far as the diagonal elements are concerned, things can go differently for the off-diagonal elements. A comment is in order. As we already noted, in scalar theory of coherence, we are accustomed to think that, up to a phase reversal, points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ can be interchanged. This continues to be true for diagonal elements. In a sense, the two points cannot be distinguished from one another. Apparently, a different behaviour can be exhibited by off-diagonal elements. A spontaneous question is: How can we distinguish point $\boldsymbol{\rho}_1$ from point $\boldsymbol{\rho}_2$? The answer is found at once if we consider how W_{xy} is to be measured [15]. Essentially, points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ can be distinguished from one another because the first is covered with a linear polarizer aligned to the x-axis whereas the second has a similar polarizer aligned to the y-axis. (One of the polarizers is then followed by a $\pi/2$ rotator, so that interference can take place.) Although the different roles that $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ can play in off-diagonal elements could be seen in other known CSD matrices, the phenomenon seems to be particularly evident in the present example.

Another possibility is worth mentioning. Let $a_x = -a_y$. Then the Fourier transform appearing in W_{xy} depends on $\boldsymbol{\rho}_2 + \boldsymbol{\rho}_1$ instead of being shift-invariant as it occurs for the diagonal elements. Finally, another interesting case is obtained when the vector $\boldsymbol{g}_\alpha(\boldsymbol{\rho})$ that appears in equation (15) is orthogonal to $\boldsymbol{\rho}$ and proportional to its length. As we shall see in the examples, this may lead to twist phenomena.

The above remarks may give a faint idea of how rich the structure of a CSD matrix can be. On the other hand, one would hardly write complicated forms for the matrix elements without possessing a guarantee that they have physical sense. The sufficient condition established in section 2 affords a safe guide for investigating new structures of the CSD matrix.

A note about the physical dimensions of the variable $\boldsymbol{v}$ should be made. Such dimensions depend on the form of the kernels. For example, if the kernels represent a displaced version of some basic coherent contribution (see section 1), $\boldsymbol{v}$ has the same dimensions as $\boldsymbol{\rho}$, whereas, for a Fourier kernel $\boldsymbol{v}$ may have dimensions of a spatial frequency.

4. Elementary examples

Some simple examples can help to illustrate the results of the previous sections. We shall begin with a pair of cases in which the quantities of interest depend on one transverse coordinate only. Denoting by (ξ, η) and (u, w) the Cartesian coordinates associated with $\boldsymbol{\rho}$ and $\boldsymbol{v}$, respectively, we are going to assume η and w to be immaterial. For our first example, we shall assume that H_x and H_y have, apart from an arbitrary common profile factor F, the form of simple Fourier exponentials with opposite sign in the exponent, namely

$$\begin{aligned} H_x(\xi, u) &= F(\xi) \exp(-2\pi i u \xi); \\ H_y(\xi, u) &= F(\xi) \exp(2\pi i u \xi). \end{aligned} \tag{22}$$

Furthermore, we suppose all of the elements of the matrix $\widehat{p}$ to equal

$$p_{\alpha\beta}(u) = S_0 \,\mathrm{rect}(u/a), \tag{23}$$

where S_0 and a are positive constants. Then, using equation (9), we easily find

$$W_{\alpha\alpha}(\xi_1, \xi_2) = S_0 F^*(\xi_1) F(\xi_2) \mathrm{sinc}[a(\xi_2 - \xi_1)]. \tag{24}$$

We then see that the diagonal elements of the CSD matrix possess a Schell-model structure. It is not so for the off-diagonal elements. In fact, the following expression is easily derived for W_{xy}

$$W_{xy}(\xi_1, \xi_2) = S_0 F^*(\xi_1) F(\xi_2) \mathrm{sinc}[a(\xi_2 + \xi_1)]. \tag{25}$$

There is perfect correlation between the x- and the y-component whenever we let $\xi_1 = -\xi_2$. It is worthwhile to write the local polarization matrix, which turns out to be

$$\widehat{W}(\xi, \xi) = S_0 |F(\xi)|^2 \begin{bmatrix} 1 & \mathrm{sinc}(2a\xi) \\ \mathrm{sinc}(2a\xi) & 1 \end{bmatrix}. \tag{26}$$

Since the diagonal elements are identical, the (spectral) degree of polarization P ([1], section 9.2) depends essentially on the off-diagonal elements. Using equation (26) we find

$$P(\xi) = |\mathrm{sinc}(2a\xi)|, \tag{27}$$

so that the field is polarized only for a certain extent around the origin of the ξ-axis. Let us further recall the recently introduced [16, 17] degree of cross-polarization, say P_c, defined as

$$P_c(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \sqrt{1 - \frac{4 \det \widehat{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)}{[\mathrm{tr}\, \widehat{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)]^2}}, \tag{28}$$

which reduces to the ordinary (spectral) degree of polarization P when $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2$. If we apply this definition to our present example, we find

$$P_c(\xi_1, \xi_2) = \left| \frac{\mathrm{sinc}[a(\xi_2 + \xi_1)]}{\mathrm{sinc}[a(\xi_2 - \xi_1)]} \right|. \tag{29}$$

In particular, letting $\xi_2 = -\xi_1$, we obtain

$$P_c(\xi_1, -\xi_1) = \frac{1}{|\mathrm{sinc}(2a\xi_1)|}, \tag{30}$$

which is always greater than or equal to one. It is known, in fact, that the degree of cross-polarization has no upper bound.

We can wonder how this type of source could be physically realized. To answer this question, we note that, disregarding the profile function F, for each value of u ($|u| < a/2$) we have a pair of plane waves, as indicated by equation (22). One of them is linearly polarized along the x-axis, while the other is similarly polarized along the y-axis. They are perfectly correlated since, according to equation (23), $p_{xy} = S_0$, like p_{xx} and p_{yy}. Furthermore, the two waves have opposite components of the wavevector along the ξ-axis. The various pairs are then superposed without correlation. It should not be difficult to devise an experimental apparatus for obtaining this type of superposition by using an interferometric arrangement similar to the one suggested for scalar specular sources [18].

We shall now discuss another example by considering the following structures for the kernels

$$\begin{aligned} H_x(\xi, u) &= F_x(\xi) \exp[-\tau(u - \xi)] \mathrm{step}(u - \xi); \\ H_y(\xi, u) &= F_y(\xi) \exp[\tau(u - \xi)] \mathrm{step}(\xi - u), \end{aligned} \tag{31}$$

where τ is a positive constant. They could be realized as coherent impulse responses in optical systems with suitable complex pupil functions ([19], section 8.3). We further assume the elements $p_{\alpha\beta}$ to be identically equal to $2\tau S_0$. Then, the diagonal elements of the CSD matrix turn out to be

$$W_{\alpha\alpha}(\xi_1, \xi_2) = S_0 F_\alpha^*(\xi_1) F_\alpha(\xi_2) \exp[-\tau|\xi_2 - \xi_1|]. \tag{32}$$

Taking the van Cittert–Zernike theorem into account ([1], section 3.2; [20]), we understand that they have the same form as the scalar CSD of a secondary source put at a suitable distance from a primary, spatially incoherent source, whose intensity profile has a Lorentzian shape. This secondary source is covered by a filter with transmission function $F_\alpha(\xi)$, thus realizing a Schell-model source. On the other hand, the off-diagonal elements are

$$\begin{aligned} W_{xy}(\xi_1, \xi_2) &= 2S_0 \tau F_x^*(\xi_1) F_y(\xi_2) (\xi_2 - \xi_1) \\ &\quad \times \exp[-\tau(\xi_2 - \xi_1)] \mathrm{step}(\xi_2 - \xi_1). \end{aligned} \tag{33}$$

Here, something peculiar occurs. Because of the presence of the step function, a correlation between the x-component of the electric field at point ξ_1 and the y-component at point ξ_2 may exist only if point ξ_2 is on the right of point ξ_1. We may notice that in the present example the kernel has a Laplace-like structure.

5. The Gaussian case

Here, we want to discuss a class of cases in which the elements of the weight matrix have a Gaussian shape. Furthermore, the kernels also include a Gaussian term. This class encompasses the so-called electromagnetic Gaussian Schell-model beams ([1], section 9.4; [8–12]), whose CSD matrix has the elements

$$\begin{aligned} W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) &= B_{\alpha\beta} C_\alpha C_\beta \\ &\quad \times \exp\left[-\frac{\rho_1^2}{4\sigma_\alpha^2} - \frac{\rho_2^2}{4\sigma_\beta^2} - \frac{(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1)^2}{2\delta_{\alpha\beta}^2}\right], \end{aligned} \tag{34}$$

where $B_{\alpha\alpha} = 1$, while B_{xy} which is a complex number, satisfies the inequality $|B_{xy}| \leqslant 1$.[4] All the other new symbols denote positive quantities. It is seen that the matrix elements are characterized by the remarkable number of 9 parameters (the modulus and the phase of B_{xy}, C_x, C_y, σ_x, σ_y, δ_{xx}, δ_{yy}, δ_{xy}). If these parameters are chosen at will it may well happen that the resulting beam is physically senseless. Finding conditions for satisfying the non-negative definiteness requirement turned out to be non-trivial [8, 11]. Now, we want to examine this subject on the grounds of our generalized superposition rule. To extend the previous treatment, anisotropic features will be introduced. We begin by giving the following expressions for the elements of $\widehat{p}$

$$p_{\alpha\beta}(\boldsymbol{v}) = A_{\alpha\beta} \exp(-\gamma_{\alpha\beta} u^2 - \varepsilon_{\alpha\beta} w^2). \tag{35}$$

Here, A_{xx} and A_{yy} are non-negative constants, whereas $A_{xy} = A_{yx}^*$ may be complex. Furthermore, all of the $\gamma_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ are non-negative. It is seen that equation (11) is met. As

[4] A more stringent upper bound has been established in [14].

for equation (14), it is easy to see that it is satisfied under the following conditions

$$|A_{xy}|^2 \leqslant A_{xx}A_{yy},$$
$$\gamma_{xy} \geqslant \frac{\gamma_{xx}+\gamma_{yy}}{2}; \qquad \varepsilon_{xy} \geqslant \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2}. \tag{36}$$

Let us now consider the kernels. We assume them to be

$$H_\alpha(\boldsymbol{\rho},\boldsymbol{v}) = F_\alpha(\boldsymbol{\rho}) \times \exp[-b_\alpha(\xi-u)^2 - c_\alpha(\eta-w)^2 - 2\pi\mathrm{i}\,\boldsymbol{g}_\alpha(\boldsymbol{\rho})\cdot\boldsymbol{v}], \tag{37}$$

where b_α and c_α are non-negative constants. It is worthwhile to reckon the number of parameters that may be necessary to specify the CSD matrix. In equation (35), there are 10 parameters (A_{xx}, A_{yy}, the modulus and the phase of A_{xy}, γ_{xx}, γ_{yy}, γ_{xy}, ε_{xx}, ε_{yy}, ε_{xy}), subject to the constraints in equations (36). Then, there are the 4 parameters b_α and c_α. Further parameters will be present in the expressions for F_α and $\boldsymbol{g}_\alpha$. Even if only one parameter could be enough for each of them, we would have reached a total of 18 quantities. It is also to be observed that the elliptical structures associated with equations (35) and (37) have their axes aligned to those of the reference frame. In the most general case, such elliptical structures could be rotated with respect to one another. We could align the reference axes to those of one of the ellipses, but we would need two angles to specify the other two. In conclusion, the number of parameters appearing in the CSD matrix could easily exceed the value of 20. Choosing in a random way so many quantities without incurring some violation of the non-negative definiteness condition would be unlikely. The virtue of the generalized superposition rule is that such choice can be done without problems by only satisfying equations (36).

A discussion of the general case would be very long and is outside the scope of this paper. We shall limit ourselves to the simple isotropic case

$$\gamma_{\alpha\beta} = \varepsilon_{\alpha\beta} = 0; \qquad b_x = b_y = c_x = c_y = b. \tag{38}$$

Then, on evaluating equation (9) we find

$$W_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \frac{\pi A_{\alpha\beta}F_\alpha^*(\boldsymbol{\rho}_1)F_\beta(\boldsymbol{\rho}_2)}{2b}\exp\left[-\frac{b(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2}{2}\right] \times \exp\left\{\frac{-\pi^2}{2b}[\boldsymbol{g}_\beta(\boldsymbol{\rho}_2)-\boldsymbol{g}_\alpha(\boldsymbol{\rho}_1)]^2 - \pi\mathrm{i}(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)\cdot[\boldsymbol{g}_\beta(\boldsymbol{\rho}_2)-\boldsymbol{g}_\alpha(\boldsymbol{\rho}_1)]\right\}. \tag{39}$$

We shall first consider the case

$$\boldsymbol{g}_x(\boldsymbol{\rho}) = \boldsymbol{g}_y(\boldsymbol{\rho}) = a\boldsymbol{\rho}, \tag{40}$$

which leads to the following matrix elements

$$W_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \frac{\pi A_{\alpha\beta}F_\alpha^*(\boldsymbol{\rho}_1)F_\beta(\boldsymbol{\rho}_2)}{2b}\exp\left[-\frac{b(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2}{2}\right] \times \exp\left\{\frac{-\pi^2a^2}{2b}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2 - \pi\mathrm{i}a(\boldsymbol{\rho}_2^2-\boldsymbol{\rho}_1^2)\right\}. \tag{41}$$

Now, let $a = 0$ and choose F_α of Gaussian shape. Then, equation (41) acquires the same mathematical form as equation (34) and we have the electromagnetic Gaussian Schell-model beam. In addition to the shift-invariant terms, the matrix elements contain an exponential with an imaginary exponent. If $a < 0$, terms of this type appear when a Gaussian Schell-model beam has propagated along some distance from the plane where the intensity distribution has its minimum variance ([1], section 9.4; [21]). This plane is conventionally called the waist plane. Accordingly, when $a < 0$, equation (41) can be thought of as pertaining to a Gaussian Schell-model beam whose waist plane lies in the half-space $z < 0$. On the other hand, if $a > 0$, the opposite sign in the imaginary exponent gives rise to a beam that is converging to its waist plane, located in the half-space $z > 0$.

As a second choice, we let

$$\boldsymbol{g}_x(\boldsymbol{\rho}) = a\boldsymbol{\rho}; \qquad \boldsymbol{g}_y(\boldsymbol{\rho}) = -a\boldsymbol{\rho}. \tag{42}$$

The resulting diagonal elements of the CSD matrix are

$$W_{xx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \frac{\pi A_{xx}F_x^*(\boldsymbol{\rho}_1)F_x(\boldsymbol{\rho}_2)}{2b}\exp\left[-\frac{b(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2}{2}\right] \times \exp\left\{\frac{-\pi^2a^2}{2b}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2 - \pi\mathrm{i}a(\boldsymbol{\rho}_2^2-\boldsymbol{\rho}_1^2)\right\}. \tag{43}$$

$$W_{yy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \frac{\pi A_{yy}F_y^*(\boldsymbol{\rho}_1)F_y(\boldsymbol{\rho}_2)}{2b}\exp\left[-\frac{b(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2}{2}\right] \times \exp\left\{\frac{-\pi^2a^2}{2b}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2 + \pi\mathrm{i}a(\boldsymbol{\rho}_2^2-\boldsymbol{\rho}_1^2)\right\}. \tag{44}$$

In order to appreciate the meaning of equations (43) and (44), let us suppose again F_x and F_y to be Gaussian shaped. Each of these two elements has the form pertaining to a scalar Gaussian Schell-model beam ([1], section 5.3). However, because of the opposite sign of the imaginary terms within the exponents, when $a > 0$, W_{xx} describes a beam whose waist is in the half-space $z > 0$, while W_{yy} refers to a beam having its waist in the half-space $z < 0$.

For the off-diagonal elements, we easily find

$$W_{xy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \frac{\pi A_{xy}F_x^*(\boldsymbol{\rho}_1)F_y(\boldsymbol{\rho}_2)}{2b}\exp\left[-\frac{b(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2}{2}\right] \times \exp\left\{\frac{-\pi^2a^2}{2b}(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)^2 + \pi\mathrm{i}a(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)^2\right\}. \tag{45}$$

This no longer has a Schell-model structure because of the presence of the term $(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)^2$ in the second exponential function. Therefore, the generalized superposition rule can generate new types of electromagnetic Gaussian beams.

The third choice we make for $\boldsymbol{g}_\alpha$ is

$$\boldsymbol{g}_x(\boldsymbol{\rho}) = \boldsymbol{g}_y(\boldsymbol{\rho}) = a\boldsymbol{\nu}, \tag{46}$$

where $\boldsymbol{\nu}$ has the same length as $\boldsymbol{\rho}$, but is orthogonal to it, i.e., $|\boldsymbol{\nu}| = |\boldsymbol{\rho}|$ and $\boldsymbol{\nu}\cdot\boldsymbol{\rho} = 0$, as briefly mentioned in section 3. Simple calculations lead to the following expression for the elements of the CSD matrix.

$$W_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \frac{\pi A_{\alpha\beta}F_\alpha^*(\boldsymbol{\rho}_1)F_\beta(\boldsymbol{\rho}_2)}{2b}\exp\left[-\frac{b(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2}{2}\right] \times \exp\left\{\frac{-\pi^2a^2}{2b}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)^2 - 2\pi\mathrm{i}a|\boldsymbol{\rho}_1\times\boldsymbol{\rho}_2|\right\}. \tag{47}$$

Here, we do have Schell-model correlation functions, but with a phase twist [22–24]. It is of interest to note that such twist does not require the functions F_α to be necessarily Gaussian. Thus, the present example generalizes the model introduced in [22].

6. Propagation effects

In the paraxial approximation, the field propagated at a distance z from the source plane possesses a CSD matrix whose elements, say $W_{z\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2)$, are given by

$$W_{z\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int\int W_{\alpha\beta}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) K_z^*(\boldsymbol{\rho}_1, \boldsymbol{r}_1) \times K_z(\boldsymbol{\rho}_2, \boldsymbol{r}_2) \mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2, \tag{48}$$

where the free propagation kernel K_z is given by ([1], section 9.4)

$$K_z(\boldsymbol{\rho}, \boldsymbol{r}) = -\frac{\mathrm{i}e^{\mathrm{i}kz}}{\lambda z} \exp\left[\frac{\mathrm{i}k}{2z}(\boldsymbol{\rho} - \boldsymbol{r})^2\right]. \tag{49}$$

Here, $k = 2\pi/\lambda$, with λ being the wavelength. On inserting equations (48) and (49) into equation (9), we easily obtain

$$W_{z\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int p_{\alpha\beta}(\boldsymbol{v}) H_{z\alpha}^*(\boldsymbol{r}_1, \boldsymbol{v}) H_{z\beta}(\boldsymbol{r}_2, \boldsymbol{v}) \mathrm{d}^2 v, \tag{50}$$

where we let

$$H_{z\alpha}(\boldsymbol{r}, \boldsymbol{v}) = \int H_\alpha(\boldsymbol{\rho}, \boldsymbol{v}) K_z(\boldsymbol{\rho}, \boldsymbol{r}) \mathrm{d}^2\rho. \tag{51}$$

It is seen that the structure of equation (50) is identical to that of equation (9) except that H_α is replaced by $H_{z\alpha}$, which plays the role of a *propagated kernel*. While this could be expected when H_α represents a coherent field distribution (see the comments following equation (3)), it is slightly less obvious when H_α is the kernel of a general integral transform.

7. Conclusions

In this paper, we saw the extension to the electromagnetic case of a sufficient condition for non-negative definiteness of a scalar CSD [2]. This affords a simple recipe to devise genuine CSD matrices. Its application leads us to put into evidence that general CSD matrices can be endowed with new and unfamiliar properties, as we showed through several examples.

Appendix

In this appendix, we will derive the non-negative definiteness condition for CSD matrices in a somewhat different way.

As a natural generalization of the (scalar) superposition rule given by equation (3), we introduce the expression

$$\widehat{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int \widehat{H}^\dagger(\boldsymbol{\rho}_1, \boldsymbol{v}) \widehat{p}(\boldsymbol{v}) \widehat{H}(\boldsymbol{\rho}_2, \boldsymbol{v}) \mathrm{d}^2 v, \tag{A.1}$$

where the dagger denotes the Hermitian adjoint, $\widehat{p}(\boldsymbol{v})$ is given by equation (10), and

$$\widehat{H}(\boldsymbol{\rho}, \boldsymbol{v}) = \begin{bmatrix} H_x(\boldsymbol{\rho}, \boldsymbol{v}) & 0 \\ 0 & H_y(\boldsymbol{\rho}, \boldsymbol{v}) \end{bmatrix}, \tag{A.2}$$

with $H_x(\boldsymbol{\rho}, \boldsymbol{v})$ and $H_y(\boldsymbol{\rho}, \boldsymbol{v})$ being two arbitrary kernels. It is to be noted that equation (A.1) is essentially the same as equation (9). The quadratic form for the non-negative definiteness condition given by equation (8) may be written, using a matrix formulation, in the form

$$Q = \int\int \widehat{f}^\dagger(\boldsymbol{\rho}_1) \widehat{W}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \widehat{f}(\boldsymbol{\rho}_2) \mathrm{d}^2\rho_1 \mathrm{d}^2\rho_2, \tag{A.3}$$

where

$$\widehat{f}(\boldsymbol{\rho}) = \begin{bmatrix} f_x(\boldsymbol{\rho}) \\ f_y(\boldsymbol{\rho}) \end{bmatrix}, \tag{A.4}$$

with $f_x(\boldsymbol{\rho})$ and $f_y(\boldsymbol{\rho})$ being two arbitrary (well-behaving) functions.

On substituting from equation (A.1) into (A.3) and interchanging the order of integrations, we obtain for Q the expression

$$Q = \int \widehat{h}^\dagger(\boldsymbol{v}) \widehat{p}(\boldsymbol{v}) \widehat{h}(\boldsymbol{v}) \mathrm{d}^2 v, \tag{A.5}$$

where the 2×1 $\widehat{h}(\boldsymbol{v})$ matrix is given by

$$\widehat{h}(\boldsymbol{v}) = \int \widehat{H}(\boldsymbol{\rho}, \boldsymbol{v}) \widehat{f}(\boldsymbol{\rho}) \mathrm{d}^2\rho. \tag{A.6}$$

Obviously equations (A.5) and (A.6) are essentially the same as equations (12) and (13), respectively.

The quantity Q given by equation (A.5) is non-negative, i.e.,

$$Q \geqslant 0, \tag{A.7}$$

if the Hermitian matrix $\widehat{p}(\boldsymbol{v})$ is non-negative definite. It is widely known that the Hermitian matrix $\widehat{p}(\boldsymbol{v})$ is non-negative definite if and only if all of the principal minors of the matrix are non-negative, i.e.,

$$\begin{gathered} p_{xx}(\boldsymbol{v}) \geqslant 0, \qquad p_{yy}(\boldsymbol{v}) \geqslant 0, \\ p_{xx}(\boldsymbol{v}) p_{yy}(\boldsymbol{v}) - |p_{xy}(\boldsymbol{v})|^2 \geqslant 0, \end{gathered} \tag{A.8}$$

for any $\boldsymbol{v}$. Accordingly, the quantity Q given by equation (A.5) and thus by equation (A.3) is non-negative, provided that equations (A.8) is satisfied for any $\boldsymbol{v}$.

To conclude, the CSD matrix constructed in accordance with the recipe given by equation (A.1) is necessarily non-negative definite, namely, it is physically realizable, as far as the condition in equations (A.8) for $\widehat{p}(\boldsymbol{v})$ is satisfied.

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