Partially correlated thin annular sources: the scalar case

Franco Gori,[1,*] Massimo Santarsiero,[1] Riccardo Borghi,[2] and Chun-Fang Li[3,4]

[1]*Dipartimento di Fisica, Università degli Studi "Roma Tre," and Consorzio Nationale Interuniversitario per le Scienze Fisiche della Materia, via della Vasca Navale, 84, 00146 Rome, Italy*
[2]*Dipartimento di Elettronica Applicata, Università degli Studi "Roma Tre," and Consorzio Nationale Interuniversitario per le Scienze Fisiche della Materia, via della Vasca Navale, 84, 00146 Rome, Italy*
[3]*Department of Physics, Shanghai University, Shanghai 200444, China*
[4]*State Key Laboratory of Transient Optics and Photonics, Xi'an Institute of Optics and Precision Mechanics of Chinese Academy of Sciences, Xi'an 710119, China*
**Corresponding author: gori@uniroma3.it*

Received July 3, 2008; accepted August 26, 2008;
posted September 15, 2008 (Doc. ID 98346); published October 23, 2008

Thin annular sources, either coherent or completely incoherent from the spatial standpoint, have played a significant role in the synthesis of diffraction-free and J_0-correlated fields, respectively. Here, we consider thin annular sources with partial correlation. A scalar description is developed under the assumption that the correlation function between two points depends on their angular distance only. We show that for any such source the modal expansion can easily be found. Further, we examine how the correlation properties of the radiated fields change on free propagation. We also give a number of examples and present possible synthesis schemes.

OCIS codes: 030.1640, 260.5430.

1. INTRODUCTION

Thin annular apertures have been used to synthesize two important classes of optical beams. In fact, when illuminated with spatially coherent light, a thin annular aperture can be used to produce diffraction-free beams [1,2], whereas under spatially incoherent illumination, it gives rise to J_0-correlated fields [3,4].

In the present paper we investigate the properties of sources constituted by thin annular apertures illuminated by partially coherent light. We will develop a scalar treatment, while the extension to the electromagnetic case will be considered elsewhere. We shall refer to the case in which the correlation functions between two points depend on their angular distance only, i.e., when such functions are *shift-invariant* in the angular sense. We shall show that the pertinent modal expansions [5,6] can be found by elementary means. Further, we shall see how to evaluate the correlation functions of the fields propagated away from such sources.

Since the sources of our interest are essentially one-dimensional, a comparison can be made with the case of a rectilinear geometry. In such case, shift-invariant correlation functions correspond to *homogeneous* sources [7]. As is well known, such sources are considered to be rather unphysical, because they should have an infinite extent. As a matter of fact, this is why *quasi-homogeneous* sources were introduced [8]. Furthermore, shift-invariant correlation functions do not belong to the class of Hilbert–Schmidt kernels, so that the theory of modal expansion cannot be applied. Of course, such limitations do not apply to the case of angular shift-invariance.

In a sense, our sources are obtained by wrapping a rectilinear source around a circle, something reminiscent of the Born–von Karman boundary conditions for an ideal crystal. This leads to sources whose correlation functions are angularly periodic. Then, the Fourier series can be used as the basic analytical tool. In fact, it is this tool that allows us to determine the modal expansions as well as the expressions for the propagated fields. From the experimental point of view, we shall see how these sources could be synthesized.

The following definitions of functions will be used throughout the paper:

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases},$$

$$\mathrm{tri}(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & |x| > 1 \end{cases},$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x},$$

$$\mathrm{dir}_N(x) = \frac{1}{2N+1}\frac{\sin[(2N+1)x/2]}{\sin(x/2)}. \tag{1}$$

The names of the first three functions are rather familiar. As for the latter, it arises from this function's association with the name of Dirichlet.

2. MODAL ANALYSIS

We shall describe the scalar spatial coherence properties using the mutual intensity [9], say $J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z)$, between two points specified by the vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ across the plane z=constant. The mutual intensity across the source plane is defined as

$$J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)=\langle V(\boldsymbol{\rho}_1,0,t)V^*(\boldsymbol{\rho}_2,0,t)\rangle, \tag{2}$$

where $V(\boldsymbol{\rho},z,t)$ is the scalar analytic signal describing the quasi-monochromatic field at point $(\boldsymbol{\rho},z)$ and time t. By assuming the process to be stationary and ergodic, the angle brackets can be thought of as denoting a time average. It should be noted that the analysis we are going to develop could be based on the use of the cross-spectral density [5,6]. In the synthesis schemes, however, we will have to deal with temporal averages, so the mutual intensity seems to be a more suitable tool.

Let us now recall that the mutual intensity has to be a nonnegative definite kernel [5,10]. This means that the quadratic quantity,

$$Q=\iint J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)g^*(\boldsymbol{\rho}_1)g(\boldsymbol{\rho}_2)\mathrm{d}^2\rho_1\mathrm{d}^2\rho_2, \tag{3}$$

has to satisfy the condition

$$Q\geqslant 0 \tag{4}$$

for any choice of the (well-behaving) function $g(\boldsymbol{\rho})$. We further recall that by modal expansion we essentially mean the Mercer's series [5]

$$J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)=\sum_n \Lambda_n\Phi_n(\boldsymbol{\rho}_1)\Phi_n^*(\boldsymbol{\rho}_2), \tag{5}$$

where Λ_n and Φ_n denote eigenvalues and eigenfunctions, respectively, of the homogeneous Fredholm integral equation

$$\int J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)\Phi(\boldsymbol{\rho}_2)\mathrm{d}^2\rho_2=\Lambda\Phi(\boldsymbol{\rho}_1), \tag{6}$$

the integral being extended to the source plane. In Eq. (5), a typical eigenfunction (as well as the corresponding eigenvalue) is assumed to be specified by a single index n, although for a planar two-dimensional source, two indices are generally used.

We now refer to sources in the form of an infinitely thin annulus. Using polar coordinates, we let $\boldsymbol{\rho}_j=(\rho_j,\varphi_j)$, $(j=1,2)$, and describe the annulus by means of radial delta functions. Accordingly, we write an angularly shift-invariant mutual intensity as

$$J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)=K\delta(\rho_1-a)\delta(\rho_2-a)J_a(\varphi_1-\varphi_2), \tag{7}$$

where K is a positive constant and a is the radius of the annulus. It will be apparent later on (see Section 4 on propagation) that K has dimensions of an area. The correlation properties along the annulus are accounted for by the the shift-invariant function $J_a(\varphi_1-\varphi_2)$, whose dimensions are those of a mutual intensity. It is with this function that we shall be mainly concerned. We shall loosely refer to it as the mutual intensity although, strictly speaking, the latter is given by Eq. (7).

Because of the basic relation

$$J(\boldsymbol{\rho}_2,\boldsymbol{\rho}_1,0)=J^*(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0), \tag{8}$$

we see that J_a is Hermitian. Note that the optical intensity along the annulus is proportional to $J_a(0)$ and, hence, is uniform. We shall further suppose J_a to possess a Fourier series expansion as

$$J_a(\varphi_1-\varphi_2)=\sum_{n=-\infty}^{\infty}\gamma_n\exp[in(\varphi_1-\varphi_2)]. \tag{9}$$

Since the function $J_a(\cdot)$ is Hermitian, the coefficients γ_n are real. Furthermore, in order to represent a bona fide angular, shift-invariant correlation function, J_a must also be nonnegative definite. On invoking the Bochner theorem [11], and thanks to the periodicity of J_a, it turns out that the necessary and sufficient condition for this is that all Fourier coefficients γ_n be nonnegative.

We now see that Eq. (9) can be read as a modal expansion. The orthonormal eigenfunctions are $\exp(in\varphi)/\sqrt{2\pi}$, and the associated (nonnegative) eigenvalues are $2\pi\gamma_n$. This is due to the angular shift-invariance of the correlation function (see also [12]). Let $J_R(\varphi_{12})$ be the restriction of the periodic function $J_a(\varphi_{12})$ to the interval $(-\pi,\pi)$, where, for brevity, we let $\varphi_{12}=\varphi_1-\varphi_2$. Then, the γ_n coefficients, which, for a Hermitian function, have the well-known expression

$$\gamma_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}J_a(\varphi_{12})\exp(-in\varphi_{12})\mathrm{d}\varphi_{12}$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Re\{J_a(\varphi_{12})\exp(-in\varphi_{12})\}\mathrm{d}\varphi_{12}, \tag{10}$$

coincide, up to a proportionality factor, with the samples of the one-dimensional Fourier transform (FT from now on) of J_R, taken at a distance $1/(2\pi)$ from one another. In fact, we have

$$\tilde{J}_R\left(\frac{n}{2\pi}\right)=\int_{-\pi}^{\pi}J_a(\varphi_{12})\exp(-in\varphi_{12})\mathrm{d}\varphi_{12}=2\pi\gamma_n, \tag{11}$$

where the tilde denotes FT.

This remark furnishes a sufficient criterion to decide whether a given form of J_a corresponds to a genuine mutual intensity. It is so whenever the FT of the truncated function J_R is everywhere nonnegative. For instance, the function

$$J_R(\varphi_{12})=I_0\,\mathrm{tri}\left(\frac{\varphi_{12}}{\varepsilon}\right), \tag{12}$$

where I_0 is a positive constant and $0<\epsilon<\pi$, can be surely taken as a bona fide mutual intensity because its FT is nowhere negative. On the other hand, the deceptively reasonable choice

$$J_R(\varphi_{12})=I_0\,\mathrm{rect}\left(\frac{\varphi_{12}}{2\varepsilon}\right) \tag{13}$$

is actually senseless, because some of the eigenvalues would necessarily be negative. In fact, in this case, the Fourier coefficients γ_n turn out to be proportional to

$\mathrm{sinc}(n\epsilon/\pi)$ and it is not possible to choose a value of $\epsilon(<\pi)$ such that all of them are positive.

3. EXAMPLES

The simplest example is that of a delta-correlated source. On propagation this gives rise to J_0-correlated partially coherent fields [3], which have been studied in several papers [4,13–22]. Although a deltalike correlation function does not belong to the class of Hilbert–Schmidt kernels, a modal expansion can be found [23]. Here, we simply note that in such limiting case all the eigenvalues become equal to one another, as can be seen from Eq. (10) by formally replacing J_a by a delta function. Therefore, there is only one eigenvalue with infinite degeneracy.

As a second example we assume that in the interval $(-\pi,\pi)$, J_a is given by

$$J_R(\varphi_{12}) = I_0\,\mathrm{rect}\left(\frac{\varphi_{12}}{2\pi}\right)\sum_{m=-\infty}^{\infty}\mathrm{tri}[(\varphi_{12}-2\pi m)/\varepsilon], \quad (14)$$

where $0<\epsilon\leq 2\pi$ and I_0 is a positive constant. In words, the periodic function J_a is constituted by a series of isosceles triangles centered at $0,\pm 2\pi,\pm 4\pi,\ldots$ and having basis length 2ϵ. The triangles do not overlap if $\epsilon\leq\pi$. On increasing ϵ beyond π the triangles overlap more and more, until for $\epsilon=2\pi$ the function J_a becomes flat and the coherent limit is reached (see Fig. 1). On the other hand, the incoherent limit is approached for $\epsilon\ll 2\pi$.

The eigenvalues $2\pi\gamma_n$ can be evaluated by inserting Eq. (14) into Eq. (10); they turn out to be given by

$$2\pi\gamma_0 = I_0\varepsilon, \quad (15)$$

$$2\pi\gamma_n = \frac{4I_0}{n^2\varepsilon}\sin^2\left(\frac{n\varepsilon}{2}\right) \qquad (n\neq 0). \quad (16)$$

Let us first consider the limiting coherent case $\epsilon=2\pi$. As expected, all the eigenvalues vanish except that of index zero. On decreasing ϵ, more and more eigenvalues become significant. Eventually, they tend to coalesce into a single, degenerate eigenvalue when ϵ tends to zero (although all of them tend to zero).

For another simple example, we suppose J_a to have the form

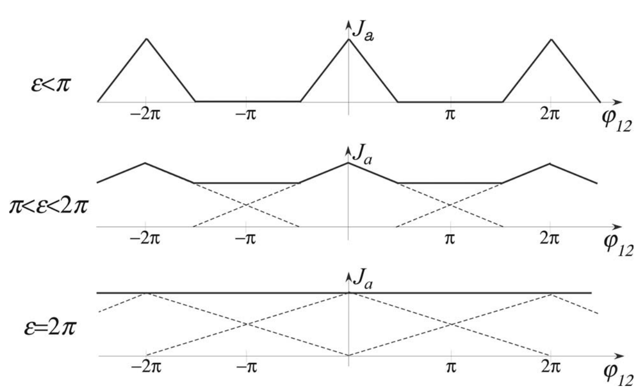

Fig. 1. Plot of Eq. (14) for different values of ϵ.

$$J_a(\varphi_{12}) = I_0\frac{(1-q)^2}{1+q^2-2q\cos\varphi_{12}}, \quad (17)$$

where $0<q<1$. This is the well-known Airy function [9]. The modal expansion is found at once because the Fourier series

$$\frac{1-q^2}{1+q^2-2q\cos\varphi_{12}} = \sum_{n=-\infty}^{\infty} q^{|n|}\exp(in\varphi_{12}) \quad (18)$$

is known to hold, so that

$$2\pi\gamma_n = I_0\frac{1-q}{1+q}q^{|n|}. \quad (19)$$

Accordingly, the eigenvalues decrease in a geometric way as functions of the modulus of their index. For each of them there is a twofold degeneracy, except for $n=0$.

In the previous examples, there is an infinite number of modes (except in the limiting coherent cases, when $\epsilon=2\pi$ or $q\to 0$). Of course, this is not a necessary feature of the modal expansions. Consider the case in which the mutual intensity J_a is given by

$$J_a(\varphi_{12}) = I_0\,\mathrm{dir}_N(\varphi_{12}). \quad (20)$$

Thanks to the equality

$$\mathrm{dir}_N(\varphi_{12}) = \frac{1}{2N+1}\sum_{n=-N}^{N}\exp(in\varphi_{12}), \quad (21)$$

we see that there is only one eigenvalue with degeneracy $2N+1$.

Similarly, letting

$$J_a(\varphi_{12}) = I_0\,\mathrm{dir}_N^2(\varphi_{12}) \quad (22)$$

and recalling the equality

$$\mathrm{dir}_N^2(\varphi_{12}) = \frac{1}{(2N+1)^2}\sum_{n=-2N}^{2N}(2N+1-|n|)\exp(in\varphi_{12}), \quad (23)$$

we find a set of eigenvalues that decrease linearly with the modulus of their index and are endowed with twofold degeneracy (except the highest one).

It is worthwhile to note that in all our examples the mutual intensity can be thought of as the superposition of equally spaced replicas of a single function having a possibly infinite support. Indeed, the Airy function is a superposition of Lorentzian curves, the dir_N function is a superposition of sinc curves, and the dir_N^2 of sinc^2 curves. All this stems from the celebrated Poisson's formula [24], which establishes a simple relation between a single function, say $f(x)$, and an infinite set of replicas of $f(x)$ spaced at a distance X from one another. Such formula reads

$$\sum_{n=-\infty}^{\infty} f(x+nX) = \sum_{n=-\infty}^{\infty}\tilde{f}\left(\frac{n}{X}\right)\exp(2\pi inx/X). \quad (24)$$

Essentially, Poisson's formula states that if we sample the FT of a function, then the inverse FT of the set of samples, taken as weighted delta functions, is the superposition of equally spaced replicas of the original function.

Now, among all possible functions, few have had the same importance in coherence theory as the Gaussian. Therefore, it is interesting to investigate a periodic mutual intensity made of a superposition of Gaussian curves. Replacing x by φ_{12} and letting $X=2\pi$ in Eq. (24) we can write

$$\sqrt{\frac{\mu}{\pi}}\sum_{n=-\infty}^{\infty}\exp[-\mu(\varphi_{12}+2\pi n)^2]=\sum_{n=-\infty}^{\infty}\exp\left(-\frac{n^2}{4\mu}+in\varphi_{12}\right), \tag{25}$$

where μ is a positive constant. The right-hand side is nothing but one of the ϑ-functions of Jacobi. More precisely, a superposition of Gaussian curves leads to a mutual intensity of the form

$$J_a(\varphi_{12})=I_0\vartheta_3\left[\frac{\varphi_{12}}{2},\exp\left(-\frac{1}{4\mu}\right)\right], \tag{26}$$

where the $\vartheta_3(x,q)$ function is defined through the equation (24)

$$\vartheta_3(x,q)=\sum_{n=-\infty}^{\infty}q^{n^2}\exp(2inx), \tag{27}$$

where $0<q<1$. Clearly, in this case the eigenvalues are proportional to q^{n^2}. As far as we know, this is one of the few examples in which this venerable function occurs in optics.

As a last example, we take a mutual intensity of the form

$$J_a(\varphi_{12})=I_0J_0\left(2q\sin\frac{\varphi_{12}}{2}\right), \tag{28}$$

where J_n stands for the nth Bessel function of the first kind, and $q>0$ [25]. As we shall see in Section 5, this case is tightly connected to the synthesis of shift-invariant annular sources. The coefficients γ_n are obtained from formula 6.681.6 of [24] and, up to a common proportionality factor, read as

$$\gamma_n=J_n^2(q). \tag{29}$$

Even for this example, use could be made of the Poisson formula [Eq. (24)]. In this case, however, the function $f(\varphi)$ has not a simple form and should be obtained numerically.

4. PROPAGATION

In this section we will study how the spatial coherence properties of the field radiated by annular sources change upon free propagation. We shall limit ourselves to paraxial approximation.

Let us recall the propagation formula for the mutual intensity from the source plane $z=0$ to a typical observation plane $z=\text{const.}$ [9]:

$$J(\mathbf{r}_1,\mathbf{r}_2,z)=\frac{1}{\lambda^2z^2}\iint J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,0)\times\exp\left\{\frac{ik}{2z}[(\mathbf{r}_1-\boldsymbol{\rho}_1)^2-(\mathbf{r}_2-\boldsymbol{\rho}_2)^2]\right\}\mathrm{d}^2\rho_1\mathrm{d}^2\rho_2, \tag{30}$$

where $k=2\pi/\lambda$, λ being the mean wavelength. When Eq. (7) is inserted into Eq. (30) the following result is obtained:

$$J(\mathbf{r}_1,\mathbf{r}_2,z)=\frac{Ka^2}{\lambda^2z^2}\exp\left[\frac{ik}{2z}(r_1^2-r_2^2)\right]\iint J_a(\varphi_1-\varphi_2)\times\exp\left\{\frac{-ika}{z}[r_1\cos(\varphi_1-\vartheta_1)-r_2\cos(\varphi_2-\vartheta_2)]\right\}\mathrm{d}\varphi_1\mathrm{d}\varphi_2, \tag{31}$$

where r_j, ϑ_j, $(j=1,2)$, are polar coordinates in the observation plane. It is seen that, for dimensional consistency, K must have dimensions of squared length. The origin of this lies in the use of the delta functions in Eq. (7).

As a next step, the Fourier series in Eq. (9) is used. This gives rise to

$$J(\mathbf{r}_1,\mathbf{r}_2,z)=\frac{Ka^2}{\lambda^2z^2}\exp\left[\frac{ik}{2z}(r_1^2-r_2^2)\right]\sum_{n=-\infty}^{\infty}\gamma_n\times\int\exp\left[in\varphi_1-\frac{ikar_1}{z}\cos(\varphi_1-\vartheta_1)\right]\mathrm{d}\varphi_1\times\int\exp\left[-in\varphi_2+\frac{ikar_2}{z}\cos(\varphi_2-\vartheta_2)\right]\mathrm{d}\varphi_2. \tag{32}$$

Next, we use a change of variables of the form $\varphi_j=\alpha_j+\vartheta_j-\pi/2$, $(j=1,2)$, and we recall the integral representation for the Bessel function $J_n(u)$ of the first kind and order n [24], i.e.,

$$J_n(u)=\frac{1}{2\pi}\int_0^{2\pi}\exp[i(n\epsilon-u\sin\epsilon)]\mathrm{d}\epsilon. \tag{33}$$

Then, we arrive at

$$J(\mathbf{r}_1,\mathbf{r}_2,z)=\frac{Kk^2a^2}{z^2}\exp\left[\frac{ik}{2z}(r_1^2-r_2^2)\right]\times\sum_{n=-\infty}^{\infty}\gamma_nJ_n\left(\frac{kar_1}{z}\right)J_n\left(\frac{kar_2}{z}\right)\exp[in(\vartheta_1-\vartheta_2)]. \tag{34}$$

The series on the right-hand side represents, within the paraxial approximation, the modal expansion of the propagated field. Up to a constant factor, the modes, say $\Phi_n(\mathbf{r},z)$, are given by

$$\Phi_n(\mathbf{r},z) = \exp\left(\frac{ik}{2z}r^2\right)J_n\left(\frac{kar}{z}\right)\exp(in\vartheta), \qquad (35)$$

while the eigenvalues are proportional to the γ_n coefficients.

In particular, the optical intensity across the observation plane is

$$I(\mathbf{r},z) \equiv J(\mathbf{r},\mathbf{r},z) = \frac{Kk^2a^2}{z^2}\sum_{n=-\infty}^{\infty}\gamma_n J_n^2\left(\frac{kar}{z}\right). \qquad (36)$$

Therefore, the transverse intensity pattern is shape-invariant within the paraxial region [26]. Upon propagation, it is simply attenuated as $1/z^2$ and enlarged by a factor proportional to z. It will be noted that in the coherent limit, in which only the γ_0 coefficient survives, the optical intensity is a J_0^2 structure [27], whereas it becomes uniform in the opposite limit, in which the γ's tend to be equal to one another. This ensues from the well-known formula [24]

$$\sum_{n=-\infty}^{\infty} J_n^2(x) = 1, \qquad (37)$$

for any value of x.

In the general case, the structure of the transverse intensity pattern will be governed by the distribution of the γ_n coefficients.

5. SYNTHESIS

In this section, we discuss two possible experimental procedures for synthesizing scalar, shift-invariant correlation functions along the annulus. In the first approach, we start from a circularly symmetric, spatially incoherent planar source and let the radiation emerging from it impinge on a thin annular aperture. By virtue of the van Cittert–Zernike (vCZ for short) theorem [9], the mutual intensity between two points across a plane parallel to the source depends on their Euclidean distance only [28], so that the mutual intensity of the radiation emerging from the annular mask depends only on the difference of their angular coordinates.

In the second technique, a rotating transparency is put in front of the annular aperture, and the latter is illuminated by a spatially coherent, uniform, and equiphase light field, such as a plane wave impinging orthogonally. Provided that the response times of the instruments used to detect the radiation emerging from the transparency are not shorter than the rotation period, the mutual intensity of the field after the transparency turns out to be exactly of the shift-invariant type discussed here.

A. Use of Primary Incoherent Sources

We start by considering a primary incoherent planar source with intensity distribution described by the radial function $I_S(\rho)$, with $\boldsymbol{\rho}$ being the position vector across the source. Suppose an infinitely thin annulus to be placed on a plane at a distance D from the source. On applying the vCZ theorem and taking the radial symmetry of the intensity distribution into account, the correlation function $J_a(\varphi_{12})$ of the radiation after the annulus, disregarding proportionality factors, turns out to be

$$J_a(\varphi_{12}) \propto \int_0^{\infty} I_S(\rho)J_0\left(2\alpha\rho\sin\frac{\varphi_{12}}{2}\right)\rho\mathrm{d}\rho, \qquad (38)$$

where $\alpha = ka/D$. On taking Eqs. (28) and (29) into account, after some algebra the γ_n coefficients are found to be given by

$$\gamma_n \propto \int_0^{\infty} I_S(\rho)J_n^2(\alpha\rho)\rho\mathrm{d}\rho. \qquad (39)$$

The latter are strictly positive and, therefore, give rise to a nonnegative definite correlation function. Furthermore, $\gamma_{-n} = \gamma_n$.

There are some cases in which the coefficients in Eq. (39) can be expressed in closed-form terms. For instance, if the primary incoherent source is itself a thin annulus with radius R, its intensity distribution $I_S(\rho)$ is proportional to $\delta(\rho - R)$. Therefore, the mutual intensity of the radiation after the annular mask is, from Eq. (38), proportional to $J_0[2\alpha R\sin(\varphi_{12}/2)]$, and the coefficients turn out to be

$$\gamma_n = J_n^2(\alpha R). \qquad (40)$$

Here and in the following examples, the coefficients are normalized to their sum. Equation (40) exactly corresponds to the example presented at the end of Section 3.

Different primary sources with radial intensity distributions can be obtained by simply superimposing annular sources with different radii and intensities, according to Eq. (38). For instance, if we consider a disk of radius R incoherently and uniformly illuminated, then the following coefficients are found:

$$\gamma_n = J_n^2(\alpha R) - J_{n+1}(\alpha R)J_{n-1}(\alpha R). \qquad (41)$$

For a Gaussian-shaped primary source having width σ_I, we have $I_S(\rho) = \exp(-\rho^2/\sigma_I^2)$ and

$$\gamma_n = \exp\left(-\frac{\alpha^2\sigma_I^2}{2}\right)\mathcal{I}_n\left(\frac{\alpha^2\sigma_I^2}{2}\right), \qquad (42)$$

with $\mathcal{I}_n$ being the modified Bessel function of first kind and order n.

Figure 2 shows the behaviors of the γ_n coefficients obtained from Eq. (40) (squares), Eq. (41) (open circles), and Eq. (42) (solid circles) as functions of the index n for $\alpha R = \alpha\sigma_I = 10$. These plots clearly show that the form of J_a can strongly affect the behavior of the eigenvalues. The latter play a key role in the structure of the propagated field, as we know from Eq. (34).

B. Use of Rotating Transparencies

Suppose that a plane wave illuminates orthogonally an opaque mask (M) in which an annular aperture is pierced (see Fig. 3). The emerging radiation then passes through a rotating transparency (τ). We assume that the transmission function of the transparency depends on the angular coordinate only and we denote it by $\tau(\varphi)$. Therefore, the disturbance at a typical point φ and time t has the form

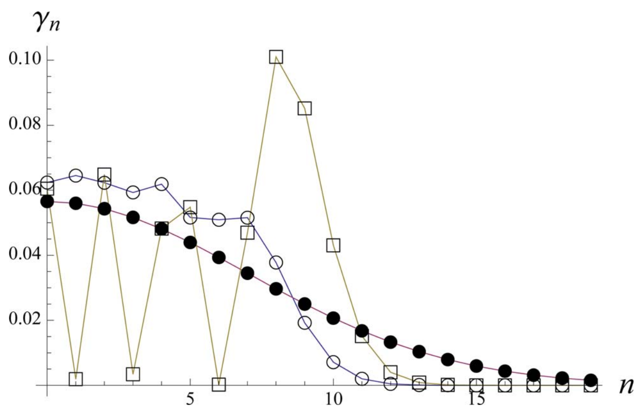

Fig. 2. (Color online) Coefficients obtained from Eq. (40) (squares), Eq. (41) (open circles), and Eq. (42) (solid circles) as functions of the index n for $\alpha R = \alpha\sigma_I = 10$.

$$V(\varphi, t) = A_i(t)\tau(\varphi - \omega t), \tag{43}$$

where $A_i(t)$ is the complex amplitude of the illuminating wave and ω is the angular velocity of the rotating transparency. The time dependence of A_i accounts for possible fluctuations in amplitude and phase of the incident plane wave.

The mutual intensity of the emerging field can be evaluated through a temporal average over one period. More precisely, we have

$$\begin{aligned} J(\varphi_1, \varphi_2) &= \frac{1}{T}\int_0^T V(\varphi_1, t)V^*(\varphi_2, t)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T |A_i(t)|^2 \tau(\varphi_1 - \omega t)\tau^*(\varphi_2 - \omega t)\mathrm{d}t \\ &= \frac{I_i}{T}\int_0^T \tau(\varphi_1 - \omega t)\tau^*(\varphi_2 - \omega t)\mathrm{d}t, \end{aligned} \tag{44}$$

where T is the rotation period. In the last passage it has been assumed that the intensity I_i of the plane wave is nearly constant over times of the order of the period and can be drawn out of the integral [29]. A simple change of variable shows that this gives rise to the shift-invariant mutual intensity

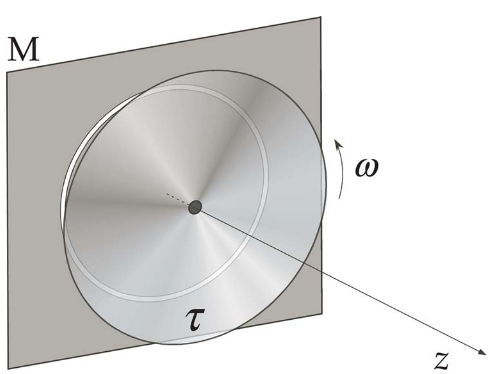

Fig. 3. (Color online) Rotating transparency coherently illuminated by an annular field distribution.

$$J(\varphi_1, \varphi_2) = J_a(\varphi_{12}) = I_i C_r(\varphi_{12}), \tag{45}$$

where C_τ denotes the autocorrelation of the transmission function; that is

$$C_r(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \tau^*(\xi)\tau(\xi + \varphi)\mathrm{d}\xi. \tag{46}$$

It is apparent that this is a genuine mutual intensity, thanks to the physical interpretation of the setup. In mathematical terms, the nonnegativity of the kernel specified by Eq. (45) is a consequence of its superposition nature [10,30].

Let us give a simple example by taking

$$\tau(\varphi) = \mathrm{rect}\left(\frac{\varphi}{\epsilon}\right). \tag{47}$$

On evaluating C_τ and using Eq. (45), we find a periodic function whose restriction to the interval $(-\pi, \pi)$ coincides with that given by Eq. (14).

As we shall see, all the mutual intensities seen in Section 3, as well as any other angularly shift-invariant mutual intensity, could be synthesized using the above approach, by means of suitable transparencies.

Let us consider a periodic function $\tau(\varphi)$ admitting the Fourier expansion

$$\tau(\varphi) = \sum_{m=-\infty}^{\infty} \tau_m \exp(im\varphi). \tag{48}$$

On inserting from Eq. (48) into Eq. (46) we obtain at once

$$C_\tau(\varphi) = \sum_{m=-\infty}^{\infty} |\tau_m|^2 \exp(im\varphi). \tag{49}$$

For synthesizing an arbitrary mutual intensity we could proceed as follows. Expand J_a in a Fourier series. This will furnish the $|\tau_m|^2$. The square roots of these terms give a set of Fourier coefficients that specify a transparency whose autocorrelation gives the required mutual intensity [31]. Let us apply such a procedure to some of the examples presented in Section 3. In the case of the Airy function, Eq. (17), we can take as τ_m just the square roots of the coefficients γ_m, with zero phases. This gives, apart from proportionality factors, $\tau_m = (\sqrt{q})^{|m|}$, so that the angular dependence of the transparency τ can be chosen itself as an Airy function, but with parameter given by $\sqrt{q}$.

The same arguments hold for the fourth example in Section 3, namely, the function dir_N. In fact, by virtue of Eq. (21), the coefficients γ_m equal $1/(2N+1)$ for any $|m| \leqslant N$ and vanish otherwise. Therefore, coefficients τ_m can also be chosen as equal to one another for $|m| \leqslant N$ and vanishing for $|m| > N$, so that the form of the function $\tau(\varphi)$ turns out to be the same as that of its autocorrelation, i.e., $\mathrm{dir}_N(\varphi)$. Note, instead, that the autocorrelation of dir_N^2 is no longer a dir_N^2 function, as can be seen from Eq. (23). In the case of a mutual intensity proportional to a ϑ_3 function [Eq. (25)], taking the square root of the Fourier coefficients corresponds to doubling the pertinent μ parameter, but the shape of the transmission function keeps the same analytical form.

6. CONCLUSIONS

Diffraction-free and J_0-correlated fields were introduced more than twenty years ago. Yet the interest in their properties and applications is far from subsiding. Diffraction-free and J_0-correlated fields epitomize the type of radiation emitted by a thin annular source under coherent and incoherent illumination, respectively. In the present paper, we showed that these fields constitute limiting cases of a more general class of annular sources with partial correlation. Under the hypothesis of angularly shift-invariant correlation functions, we have seen that, for any such source, the modal expansion is easily determined, thus adding a whole new class to the relatively small set of sources for which the modal analysis can be performed in closed-form terms. This, in turn, allows us to study in a simple and unified manner the correlation properties of the fields propagating away from such sources. A wise choice of the form of the mutual intensity along the annulus allows us to favor certain modes with respect to the others, thus influencing the structure of the field propagated away from the source. We have also seen that feasible experimental synthesis schemes exist, thus enabling laboratory checks of theoretical predictions. In the present paper, we limited ourselves to a scalar treatment. The extension to the electromagnetic case, which will be discussed in a subsequent paper, will reveal further features of partially correlated thin annular sources.

ACKNOWLEDGMENT

This work was performed while C.-F. Li was a guest of Roma Tre University.

REFERENCES AND NOTES

1. J. Durnin, J. J. Miceli, Jr., and J. H. Eberly, "Diffraction-free beams," Phys. Rev. Lett. **58**, 1499–1501 (1987).
2. G. Indebetouw, "Nondiffracting optical fields: some remarks on their analysis and synthesis," J. Opt. Soc. Am. A **6**, 150–152 (1989).
3. F. Gori, G. Guattari, and C. Padovani, "Modal expansion of J_0-correlated sources," Opt. Commun. **64**, 311–316 (1987).
4. J. Turunen, A. Vasara, and A. T. Friberg, "Propagation invariance and self-imaging in variable-coherence optics," J. Opt. Soc. Am. A **8**, 282–289 (1991).
5. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U. Press, 1995).
6. E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light* (Cambridge U. Press, 2007).
7. E. Wolf and W. H. Carter, "Angular distribution of radiant intensity from sources of different degrees of spatial coherence," Opt. Commun. **13**, 205–209 (1975).
8. W. H. Carter and E. Wolf, "Coherence and radiometry with quasihomogeneous planar sources," J. Opt. Soc. Am. **67**, 785–796 (1977).
9. M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge U. Press, 1999).
10. F. Gori and M. Santarsiero, "Devising genuine spatial correlation functions," Opt. Lett. **32**, 353–355 (2007).
11. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Blackie and Sons, 1956).
12. A. Burvall, P. Martinsson, and A. T. Friberg, "Communication modes in large-aperture approximation," Opt. Lett. **32**, 611–613 (2007).
13. M. W. Kowarz and G. S. Agarwal, "Bessel-beam representation for partially coherent fields," J. Opt. Soc. Am. A **12**, 1324–1330 (1995).
14. C. Palma and G. Cincotti, "Imaging of J_0-correlated Bessel-Gauss beams," IEEE J. Quantum Electron. **33**, 1032–1040 (1997).
15. R. Borghi, "Superposition scheme for J_0-correlated partially coherent sources," IEEE J. Quantum Electron. **35**, 849–856 (1999).
16. S. A. Ponomarenko, "A class of partially coherent beams carrying optical vortices," J. Opt. Soc. Am. A **18**, 150–156 (2001).
17. G. Gbur and T. D. Visser, "Can spatial coherence effects produce a local minimum of intensity at focus?" Opt. Lett. **28**, 1627–1629 (2003).
18. J. Garnier, J.-P. Ayanides, and O. Morice, "Propagation of partially coherent light with the Maxwell–Debye equation," J. Opt. Soc. Am. A **20**, 1409–1417 (2003).
19. L. Wang and B. Lü, "Propagation and focal shift of J_0-correlated Schell-model beams," Optik (Stuttgart) **117**, 167–172 (2006).
20. L. Rao, X. Zheng, Z. Wang, and P. Yei, "Generation of optical bottle beams through focusing J_0-correlated Schell-model vortex beams," Opt. Commun. **281**, 1358–1365 (2007).
21. S. A. Ponomarenko, W. Huang, and M. Cada, "Dark and antidark diffraction-free beams," Opt. Lett. **32**, 2508–2510 (2007).
22. T. van Dijk, G. Gbur, and T. D. Visser, "Shaping the focal intensity distribution using spatial coherence," J. Opt. Soc. Am. A **25**, 575–581 (2008).
23. F. Gori, M. Santarsiero, and R. Borghi, "Modal expansion for J_0-correlated electromagnetic sources," Opt. Lett. **33**, 1857–1859 (2008).
24. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed., A. Jeffrey and D. Zwillinger, eds. (Academic, 2000).
25. Following the standard usage, we denote by J both the mutual intensity and the Bessel functions of the first kind. No confusion should arise since Bessel functions have a numerical index.
26. R. Borghi, M. Santarsiero, and R. Simon, "Shape invariance and a universal form for the Gouy phase," J. Opt. Soc. Am. A **21**, 572–579 (2004).
27. Actually, a different coherent limit in which the mutual intensity has the form $J_a(\varphi_{12})=I_0\exp(im\varphi_{12})$ with integer m could be considered. This corresponds to illuminating the annulus with a coherent field carrying a vortex of order m. In that case, the intensity of the propagated field would have a J_m^2 structure.
28. Quadratic phase factors can be compensated for through the use of a suitable lens.
29. Stability performances allowing such an approximation are often ensured by commercial lasers.
30. P. Vahimaa and J. Turunen, "Finite-elementary-source model for partially coherent radiation," Opt. Express **14**, 1376–1381 (2006).
31. In principle, each term could be multiplied by an arbitrary phase factor.