Realizability condition for electromagnetic Schell-model sources

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The set of functions that appear in the correlation matrix of an electromagnetic source must satisfy the constraint of nonnegative definiteness. Here we derive a necessary and sufficient condition for nonnegativeness for the class of electromagnetic Schell-model sources. This result also suggests a possible synthesis procedure for this type of source. As an illustration, two specific examples of electromagnetic Schell-model sources are discussed. © 2008 Optical Society of America

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1. INTRODUCTION

Spatial correlation functions must satisfy the well known nonnegative definiteness constraint [1,2]. Obviously, violation of this constraint entails that the corresponding source simply cannot exist. A condition ensuring that the definiteness constraint is satisfied will be loosely referred to as a realizability condition. In this paper, we shall consider realizability conditions for Schell-model (SM, for short) sources.

SM sources were introduced more than 40 years ago to describe scalar partially coherent light sources having shift-invariant degree of spatial coherence [3]. Since then, they have played a role of utmost importance in the context of the theory of coherence [1,4], because the wave field radiated by several natural sources turns out to be of the SM type. The most celebrated example is of course given by the light radiated by stars, which, when observed from the Earth, presents a shift-invariant degree of spatial coherence, well described by a besinc function (the definition of besinc will be recalled shortly). This, in turn, allowed the angular diameters of very distant astronomic objects to be measured via the Michelson stellar interferometer [5] and synthetic-aperture techniques to be developed [6,7]. A significant feature of scalar SM sources is that they can always be produced starting from a spatially incoherent light source and applying the van Cittert–Zernike theorem [5] for the propagated field [8–17]. The reason for this stems from the fact that the necessary and sufficient condition for a scalar SM source to be physically realizabile is that the Fourier transform of its degree of coherence be a nonnegative function [11].

In the present paper, we address the realizability problem for SM sources in the more general framework of electromagnetic stochastic wave fields. Sources of this kind were considered not long ago [18–21], but finding realizability conditions for them turned out to be more demanding than for scalar sources, as witnessed, for instance, by [19,21]. In particular, we shall derive the necessary and sufficient condition that must be fulfilled by the shift-invariant parts of the correlation matrix characterizing a typical electromagnetic Schell-model source. We will show that such a condition has a simple physical interpretation in terms of the generalized van Cittert–Zernike theorem [18], which, in turn, suggests a possible procedure for synthesizing electromagnetic SM sources. As an example, we shall apply the necessary and sufficient condition to the particular case of an electromagnetic Gaussian Schell-model (EGSM) source and make a comparison with the results recently found by Roychowdhury and Korotkova in [21]. A second example, mainly involved in the study of the light emitted by disk-shaped sources, such as stars, will also be considered.

2. PRELIMINARIES

A. Scalar Schell-Model Sources

The spatial coherence features of partially coherent scalar sources are described in terms of the mutual intensity, in the space–time domain, or of the cross-spectral density, in the space–frequency domain [1,4,5,22]. In the present paper, we shall adopt the first choice. On the other hand, the extension to the space–frequency approach is straightforward.

A partially coherent scalar source is said to be of the SM type [1,4] if its mutual intensity at two typical points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ can be written in the following form:

$$J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = s(\boldsymbol{\rho}_1)s(\boldsymbol{\rho}_2)j(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2), \tag{1}$$

where s is the square root of the optical intensity, namely,

$$s(\boldsymbol{\rho}) = \sqrt{J(\boldsymbol{\rho},\boldsymbol{\rho})}, \quad (2)$$

while $j(\cdot)$ denotes the (shift-invariant) degree of spatial coherence.

It is well known that, in general, the form of j cannot be chosen at will, but rather it is subjected to restrictions aimed at ensuring that the corresponding $J(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$ is a genuine mutual intensity. As a matter of fact, the (obvious) condition $|j|\leq 1$ is not enough to ensure that J be a nonnegative definite kernel, a mandatory requirement for an actual spatial correlation function [5]. In the scalar case, a necessary and sufficient condition warranting the above nonnegativeness feature is provided by Bochner's theorem [23], according to which the function J specifies a nonnegative definite kernel if and only if the Fourier transform of its shift-invariant part is a nonnegative function [11].

B. Electromagnetic Schell-Model Sources

A quasi-monochromatic electromagnetic source will be described by means of the beam-coherence–polarization (BCP) matrix [24,25], whose elements—say, $J_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)$—are given by

$$J_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \langle E^*_\alpha(\boldsymbol{\rho}_1,t)E_\beta(\boldsymbol{\rho}_2,t)\rangle; \quad (\alpha,\beta = x,y), \quad (3)$$

where $E_\alpha(\boldsymbol{\rho},t)$ $(\alpha=x,y)$ is the analytic signal associated with the electric field component along the α axis of a quasi-monochromatic wave at position $\boldsymbol{\rho}$ and time t. The angular brackets denote time averages. An electromagnetic source will be called of the SM type if the elements of the corresponding BCP matrix have the following form [4]:

$$J_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = s_\alpha(\boldsymbol{\rho}_1)s_\beta(\boldsymbol{\rho}_2)j_{\alpha\beta}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2), \quad (\alpha,\beta = x,y), \quad (4)$$

with

$$s_\alpha(\boldsymbol{\rho}) = \sqrt{J_{\alpha\alpha}(\boldsymbol{\rho},\boldsymbol{\rho})}; \quad (\alpha = x,y). \quad (5)$$

Of course, since the diagonal elements of the BCP matrix represent scalar correlation functions, s_α $(\alpha=x,y)$ is a nonnegative function. In addition, since both j_{xx} and j_{yy} possess the same properties as a scalar degree of spatial coherence, their Fourier transforms have to be nonnegative.

It should be noted how the choice of the functional form of the off-diagonal elements of the BCP matrix suffers from constraints additional to those of the nonnegative definiteness of the whole matrix. This clearly appears in Eq. (4). In fact, while the choice of the shift-varying part of the diagonal matrix elements, represented by the nonnegative functions s_α, is arbitrary, it necessarily fixes the functional form of the shift-varying part of the off-diagonal elements $J_{\alpha\beta}$ as well. Due to such constraint, the shift-invariant character of all the normalized degrees of correlation $j_{\alpha\beta}$ is ensured, so that Eq. (4) provides the natural extension of the Schell model to the vectorial realm.

3. REALIZABILITY CONDITION

The present section is aimed at deriving a necessary and sufficient condition for nonnegativeness of the BCP matrix of any electromagnetic SM source.

It is easily seen that, for sources of the type specified by Eq. (4), the nonnegativeness condition [1] can be expressed by requiring that the quadratic form

$$Q = \sum_{\alpha=x,y}\sum_{\beta=x,y}\int\int j_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)f^*_\alpha(\boldsymbol{\rho}_1)f_\beta(\boldsymbol{\rho}_2)\mathrm{d}^2\rho_1\mathrm{d}^2\rho_2, \quad (6)$$

where f_x and f_y are arbitrary (well-behaving) functions, be nonnegative for any choice of f_x and f_y. Note that only the normalized parts, $j_{\alpha\beta}$, of the BCP matrix elements are involved in this condition. On expressing the functions $j_{\alpha\beta}$ through their Fourier transform (to be denoted by a tilde), viz.,

$$j_{\alpha\beta}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2) = \int \tilde{j}_{\alpha\beta}(\boldsymbol{\eta})\exp[2\pi i\boldsymbol{\eta}\cdot(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)]\mathrm{d}^2\eta, \quad (7)$$

and on inserting such representation into Eq. (6), we obtain

$$Q = \sum_{\alpha=x,y}\sum_{\beta=x,y}\int \tilde{f}^*_\alpha(\boldsymbol{\eta})\tilde{f}_\beta(\boldsymbol{\eta})\tilde{j}_{\alpha\beta}(\boldsymbol{\eta})\mathrm{d}^2\eta. \quad (8)$$

Furthermore, on taking into account the relation

$$\tilde{j}_{yx}(\boldsymbol{\eta}) = \tilde{j}^*_{xy}(\boldsymbol{\eta}), \quad (9)$$

which follows from the fact that $j_{xy}(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)=j^*_{yx}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)$ [25], Eq. (8) can be written as

$$Q = \int [|\tilde{f}_x(\boldsymbol{\eta})|^2\tilde{j}_{xx}(\boldsymbol{\eta}) + |\tilde{f}_y(\boldsymbol{\eta})|^2\tilde{j}_{yy}(\boldsymbol{\eta}) + 2\,\mathrm{Re}\{\tilde{f}^*_x(\boldsymbol{\eta})\tilde{f}_y(\boldsymbol{\eta})\tilde{j}_{xy}(\boldsymbol{\eta})\}]\mathrm{d}^2\eta, \quad (10)$$

where, as already noted, $\tilde{j}_{xx}$ and $\tilde{j}_{yy}$ are nonnegative functions.

It is not difficult to see that, in order for Q to be nonnegative, it is sufficient that the following condition be met:

$$|\tilde{j}_{xy}(\boldsymbol{\eta})| \leq \sqrt{\tilde{j}_{xx}(\boldsymbol{\eta})\tilde{j}_{yy}(\boldsymbol{\eta})}, \quad (11)$$

for any $\boldsymbol{\eta}$.

To prove that Eq. (11) is also necessary, we shall argue by contradiction. Suppose that (i) Q is nonnegative for any choice of $\tilde{f}_x$ and $\tilde{f}_y$; (ii) the condition in Eq. (11) is not satisfied for $\boldsymbol{\eta}$ belonging to a certain domain $\mathcal{D}$. From a general result of matrix analysis [26], it follows that the integrand in Eq. (10) is nonnegative at the point $\boldsymbol{\eta}$ if and only if the inequality in Eq. (11) is fulfilled at $\boldsymbol{\eta}$. Because of assumption (ii), it will always be possible, at any point in $\mathcal{D}$, to choose $\tilde{f}_x(\boldsymbol{\eta})$ and $\tilde{f}_y(\boldsymbol{\eta})$ in such a way that the corresponding value of the integrand is negative. Consequently, on letting $\tilde{f}_x(\boldsymbol{\eta})$and $\tilde{f}_y(\boldsymbol{\eta})$ to be identically vanishing for $\boldsymbol{\eta}\notin\mathcal{D}$, we would obtain $Q<0$, and this contradicts hypothesis (i). Accordingly, we conclude that Eq. (11) is a necessary and sufficient condition to warrant the nonnegativeness of Q.

Equation (11) is the main result of this paper. Note that it can be seen as the generalization of Bochner's theorem for the case of tensorial operators.

An interesting consequence of Eq. (11) is that the value of the correlation between the orthogonal components of the field at a typical point across a SM source is not limited by the Schwarz inequality only. This can be seen on letting

$$B_{\alpha\beta} = j_{\alpha\beta}(0), \tag{12}$$

with $B_{xx}=B_{yy}=1$ and B_{xy} $(=B^*_{yx})$ a complex number representing the normalized correlation between the x and the y component of the field at the same point. From the Schwarz inequality it directly follows that [4]

$$|B_{xy}| \le \sqrt{B_{xx}B_{yy}} = 1, \tag{13}$$

but a possibly stronger limitation comes from Eq. (11). In fact, on taking into account that

$$B_{xy} = j_{xy}(0) = \int \tilde{j}_{xy}(\boldsymbol{\eta})\mathrm{d}^2\eta, \tag{14}$$

from Eq. (11) we have at once

$$|B_{xy}| = \left|\int \tilde{j}_{xy}(\boldsymbol{\eta})\mathrm{d}^2\eta\right| \le \int |\tilde{j}_{xy}(\boldsymbol{\eta})|\mathrm{d}^2\eta \le \int \sqrt{\tilde{j}_{xx}(\boldsymbol{\eta})\tilde{j}_{yy}(\boldsymbol{\eta})}\mathrm{d}^2\eta. \tag{15}$$

This constraint for the correlation matrix is generally more tightening than the Schwarz inequality. In order to see this, let us take into account the following inequality:

$$\sqrt{\tilde{j}_{xx}(\boldsymbol{\eta})\tilde{j}_{yy}(\boldsymbol{\eta})} \le \frac{\tilde{j}_{xx}(\boldsymbol{\eta}) + \tilde{j}_{yy}(\boldsymbol{\eta})}{2}, \tag{16}$$

where the equality sign holds if and only if $\tilde{j}_{xx}(\boldsymbol{\eta})=\tilde{j}_{yy}(\boldsymbol{\eta})$. From Eqs. (15) and (16) we then have

$$|B_{xy}| \le \int \sqrt{\tilde{j}_{xx}(\boldsymbol{\eta})\tilde{j}_{yy}(\boldsymbol{\eta})}\mathrm{d}^2\eta \le \int \frac{\tilde{j}_{xx}(\boldsymbol{\eta}) + \tilde{j}_{yy}(\boldsymbol{\eta})}{2}\mathrm{d}^2\eta = \frac{B_{xx}+B_{yy}}{2} = 1. \tag{17}$$

Therefore, unless $\tilde{j}_{xx}(\boldsymbol{\eta})$ is identically equal to $\tilde{j}_{yy}(\boldsymbol{\eta})$, the upper bound in Eq. (15) is strictly smaller than one.

We can look at such a bound from a different point of view by taking the square of both members of Eq. (11) and integrating them on the whole $\boldsymbol{\eta}$ plane, thus obtaining

$$\int |\tilde{j}_{xy}(\boldsymbol{\eta})|^2\mathrm{d}^2\eta \le \int \tilde{j}_{xx}(\boldsymbol{\eta})\tilde{j}_{yy}(\boldsymbol{\eta})\mathrm{d}^2\eta. \tag{18}$$

Next, we come back to the $\boldsymbol{\rho}$ plane, where, thanks to the Parseval theorem, Eq. (18) reads

$$\int |j_{xy}(\boldsymbol{\rho})|^2\mathrm{d}^2\rho \le \int j^*_{xx}(\boldsymbol{\rho})j_{yy}(\boldsymbol{\rho})\mathrm{d}^2\rho. \tag{19}$$

Finally, on applying the Schwarz inequality to the right-hand side of Eq. (19), the following equation is derived:

$$\int |j_{xy}(\boldsymbol{\rho})|^2\mathrm{d}^2\rho \le \sqrt{\int |j_{xx}(\boldsymbol{\rho})|^2\mathrm{d}^2\rho \int |j_{yy}(\boldsymbol{\rho})|^2\mathrm{d}^2\rho}. \tag{20}$$

Equation (20) is interesting on its own, since it gives a possible interpretation of the inequality in Eq. (11). Paralleling a definition of coherence time proposed long ago by Mandel [27], the integral $\int|j_{\alpha\beta}(\boldsymbol{\rho})|^2\mathrm{d}^2\rho$ can be seen as an estimate of the "correlation area" across the source plane, that is, the area within which the correlation between the α and β components of the electric field is appreciable. Within such intepretation scheme, Eq. (20) states that the cross-polarization correlation area is upper bounded by the geometric mean of the co-polarization correlation areas.

As a final remark, we note that our present results can be slightly generalized by considering sources whose BCP matrix elements have the form

$$J_{\alpha\beta}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = s_\alpha(\boldsymbol{\rho}_1)s_\beta(\boldsymbol{\rho}_2)\exp\{\mathrm{i}[\phi_\beta(\boldsymbol{\rho}_2) - \phi_\alpha(\boldsymbol{\rho}_1)]\} \times j_{\alpha\beta}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2), \quad (\alpha,\beta = x,y), \tag{21}$$

where ϕ_α is an arbitrary phase distribution. Structures of this type appear, for example, in the BCP matrix elements pertaining to the field propagated a distance z away from the plane of a spatially incoherent source, in which case the phase distribution ϕ_α has the expression [18]

$$\phi_\alpha(\boldsymbol{\rho}) = \frac{\pi}{\lambda z}\rho^2, \tag{22}$$

where λ denotes the mean wavelength of the field. It is easily verified that our derivation of the necessary and sufficient realizability condition could be repeated giving to the BCP matrix elements the form in Eq. (21) and that the final result would still be expressed by Eq. (11).

4. EXAMPLES

A. Electromagnetic Gaussian Schell-Model Sources

Electromagnetic Gaussian Schell-model (EGSM) sources have been introduced as the natural extension to the vector case of scalar Gaussian Schell-model sources [19,28,29]. They have attracted the attention of several researchers, from both the theoretical and experimental standpoints, as can be seen from the recent literature [30–43]. A general EGSM source is characterized by a BCP matrix of the form given in Eq. (4), with [4]

$$s_\alpha(\boldsymbol{\rho}) = A_\alpha \exp\left(-\frac{\rho^2}{4\sigma_\alpha^2}\right),$$

$$j_{\alpha\beta}(\boldsymbol{\rho}) = B_{\alpha\beta} \exp\left(-\frac{\rho^2}{2\delta_{\alpha\beta}^2}\right), \tag{23}$$

where σ_α, $\delta_{\alpha\beta}$ $(=\delta_{\beta\alpha})$, and A_α $(\alpha,\beta=x,y)$ are real and positive parameters. The number of independent real parameters is nine. Using Eq. (11), it is not difficult to show that the condition

$$|B_{xy}|\delta_{xy}^2 \exp(-2\pi^2\delta_{xy}^2\eta^2) \leq \delta_{xx}\delta_{yy} \exp[-\pi^2(\delta_{xx}^2+\delta_{yy}^2)\eta^2] \tag{24}$$

has to be met for any η. Since the functions on the left-hand side and right-hand side of Eq. (24) decrease monotonically with η, on considering the inequality for the limiting cases $\eta=0$ and $\eta\to\infty$, the following "fork" inequality is found:

$$\sqrt{\frac{\delta_{xx}^2+\delta_{yy}^2}{2}} \leq \delta_{xy} \leq \sqrt{\frac{\delta_{xx}\delta_{yy}}{|B_{xy}|}}. \tag{25}$$

It is worthwhile to compare such an inequality with that obtained in [21], which reads

$$\max\{\delta_{xx},\delta_{yy}\} \leq \delta_{xy} \leq \frac{\min\{\delta_{xx},\delta_{yy}\}}{\sqrt{|B_{xy}|}}. \tag{26}$$

In particular, since

$$\sqrt{\frac{\delta_{xx}^2+\delta_{yy}^2}{2}} \leq \max\{\delta_{xx},\delta_{yy}\}, \quad \sqrt{\delta_{xx}\delta_{yy}} \geq \min\{\delta_{xx},\delta_{yy}\}, \tag{27}$$

we see that the interval specified by Eq. (26) fits into the, possibly larger, interval required by Eq. (25).

Furthermore, it should be noted that, in order for the inequality in Eq. (25) to be satisfied, it must be

$$\sqrt{\frac{\delta_{xx}^2+\delta_{yy}^2}{2}} \leq \sqrt{\frac{\delta_{xx}\delta_{yy}}{|B_{xy}|}}, \tag{28}$$

and therefore the modulus of B_{xy} has to fulfill the following condition:

$$|B_{xy}| \leq \frac{2}{\delta_{xx}/\delta_{yy}+\delta_{yy}/\delta_{xx}}. \tag{29}$$

Equation (29) gives none other than the upper bound derived in Section 3, as can be verified by using Eq. (15). As far as we know, such limitation on the value of $|B_{xy}|$ has not been noted before.

B. Electromagnetic Besinc Schell-Model Sources

Another model used to characterize a wide class of sources, within the scalar framework, is based on the use of a shift-invariant degree of spatial coherence described by a so-called "besinc" function. In the present section we propose an extension of such a model to the case of electromagnetic sources. Consider a BCP matrix, whose elements have the form of Eq. (3) with

$$j_{\alpha\beta}(\boldsymbol{\rho}) = B_{\alpha\beta}\,\mathrm{besinc}(\rho/\delta_{\alpha\beta}), \tag{30}$$

where $\delta_{\alpha\beta}$ $(=\delta_{\beta\alpha}$ with $\alpha,\beta=x,y)$ are real and positive parameters, and the besinc function is defined as

$$\mathrm{besinc}(t) = \frac{2J_1(\pi t)}{\pi t}, \tag{31}$$

$J_1(t)$ being the Bessel function of the first kind and order one [44]. On substituting from Eq. (30) into Eq. (11), and on taking the following Fourier transform into account,

$$\mathrm{besinc}\left(\frac{\rho}{\delta_{\alpha\beta}}\right) \xrightarrow{\mathrm{FT}} \frac{4\delta_{\alpha\beta}^2}{\pi}\,\mathrm{circ}(2\delta_{\alpha\beta}\eta), \tag{32}$$

with $\mathrm{circ}(u)$ being the characteristic function of the circle $u\leq 1$, the realizability condition is expressed by the inequality

$$\delta_{xx}\delta_{yy}\,\mathrm{circ}(2\delta_M\eta) \geq |B_{xy}|\delta_{xy}^2\,\mathrm{circ}(2\delta_{xy}\eta), \tag{33}$$

where $\delta_M=\max\{\delta_{xx},\delta_{yy}\}$. Equation (33) must be valid for any value of η. In particular, on taking the spatial extents of the circ functions into account and considering the value $\eta=0$, the following inequality is easily derived:

$$\max\{\delta_{xx},\delta_{yy}\} \leq \delta_{xy} \leq \sqrt{\frac{\delta_{xx}\delta_{yy}}{|B_{xy}|}}. \tag{34}$$

Furthermore, using the same arguments leading to Eq. (29), one can show that

$$|B_{xy}| \leq \min\left\{\frac{\delta_{xx}}{\delta_{yy}},\frac{\delta_{yy}}{\delta_{xx}}\right\}. \tag{35}$$

5. CONNECTION WITH THE VAN CITTERT–ZERNIKE THEOREM

The analysis presented in the previous sections finds an intuitive explanation in terms of the extension of the van Cittert–Zernike theorem to electromagnetic sources [18]. According to the latter theorem, the shift-invariant parts of the BCP matrix can be synthesized starting from a spatially incoherent source, whose polarization characteristics are in general point dependent, and considering the propagated radiation. The polarization matrix [4] across the incoherent source is specified by $\tilde{j}_{\alpha\beta}$. Within such an interpretation scheme, we see that Eq. (11) simply imposes that, at any point of the incoherent source, the polarization matrix be nonnegative definite, a well-known requirement for this type of matrix. Once the shift-invariant parts of the BCP matrix have been synthesized, the factors remaining in Eq. (21) can be introduced by means of suitable complex filters.

The problem of how, in practice, a spatially incoherent source, having a prescribed point-dependent polarization matrix, can be realized still represents, in the general case, an open issue, as witnessed by the attempts proposed in the past for the particular case of the EGSM sources [20,45]. In other cases, however, the above interpretation leads to simple synthesis schemes. As an example, we consider an electromagnetic source of the besinc type (see Subsection 4.B), with $\delta_{xx}=\delta_{yy}$. Equation (34) now gives

$$\delta_{xx} \leq \delta_{xy} \leq \frac{\delta_{xx}}{\sqrt{|B_{xy}|}}, \tag{36}$$

where, from Eq. (35), $|B_{xy}|\leq 1$. Let us further choose δ_{xy} equal to its upper bound. Taking Eqs. (30) and (32) into account, the polarization matrix of the incoherent source is easily seen to be proportional to

$$\begin{bmatrix} \mathrm{circ}(2\delta_{xx}\eta) & \mathrm{circ}(2\delta_{xy}\eta) \\ \mathrm{circ}(2\delta_{xy}\eta) & \mathrm{circ}(2\delta_{xx}\eta) \end{bmatrix}, \tag{37}$$

where the spatial frequency is proportional, through a suitable factor, to the coordinate across the source plane.

To synthesize the incoherent source specified by Eq. (37), we first note that, since $\delta_{xx} < \delta_{xy}$, for $\eta < 1/(2\delta_{xy})$ the state of polarization is linear at $\pi/4$. Conversely, for $1/(2\delta_{xy}) < \eta < 1/(2\delta_{xx})$ the field is completely unpolarized. In other words, the incoherent source is constituted by a disk radiating linearly polarized light, surrounded by an unpolarized annular region with the same intensity.

6. CONCLUSIONS

Devising genuine spatial correlation matrices for general electromagnetic sources is still an open problem. Here, we solved such a problem for the whole class of SM sources by deriving a necessary and sufficient condition for satisfying the nonnegative definiteness constraint. We also showed that any electromagnetic SM source can in principle be synthesized starting from a suitably polarized spatially incoherent source. These results should be useful for both modeling and synthesizing electromagnetic sources.

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