Maximizing Young's fringe visibility through reversible optical transformations

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Received November 15, 2006; accepted November 28, 2006;
posted December 8, 2006 (Doc. ID 77141); published February 15, 2007

When a Young's interferometer is fed by an electromagnetic beam, fringes of low, or even zero, visibility do not necessarily indicate lack of correlation between two typical field components at the pinholes. The passage of light that emerges from one of the pinholes through a nonabsorbing anisotropic optical element may enhance the visibility. We inquire about the maximum visibility that can be attained through such a reversible transformation (polarizers being excluded). We find that such a quantity can be evaluated in closed form. Its value is directly related to the Ky Fan 1-norm of the correlation matrix of the illuminating beam.

OCIS codes: 030.1640, 260.5430.

In recent years, there has been considerable interest in the coherence properties of electromagnetic beams.[1–12] In addition, a couple of pioneering papers about coherence of an electromagnetic field in the space–time domain were published long ago by Karczewski.[13,14] In particular, the visibility of fringes seen at the output of a Young's interferometer when the field correlation matrix of the illuminating beam is taken into account has been investigated. An example, used in Refs. 7 and 15 and sometimes referred to when discussing quantum erasers,[16] is of help. Suppose that the beam is linearly polarized along the x-axis at one pinhole and linearly polarized along the y-axis at the other. Further, suppose that such fields have the same optical intensity and are perfectly correlated. No fringes are seen, because light has orthogonal states of polarization at the pinholes. Yet, on inserting a $\pi/2$ rotator at one of the pinholes, fringes with unit visibility will appear. The same result can be obtained by covering both holes with a linear polarizer set at an angle $\pi/4$ between the x- and the y-axes. In both cases the original state of the field is modified by the presence of the anisotropic element. There is, however, a basic difference between them. In fact, in the first case the original state can be restored simply by using a $-\pi/2$ rotator, whereas in the second case certain field components are lost in an unrecoverable way. In other words, the first type of element (rotator) causes a reversible transformation, while the second one (polarizer) induces an irreversible transformation. As far as intrinsic properties of the illuminating field are concerned, reversible transformations seem to be more significant than irreversible ones. To clarify this point, let us consider a further example. Suppose that the fields at the pinholes have completely correlated x-components and completely uncorrelated y-components. Then a linear polarizer aligned to the x-axis produces perfect fringes, while a polarizer aligned to the y-axis gives no fringes at all. In both alignments, some important information about the fields has been destroyed.

In this paper, we concentrate on reversible transformations and determine the value of visibility that can be achieved by inserting at one pinhole a general anisotropic nonabsorbing element and acting on its parameters until a maximum of the visibility is reached. As we shall see, the value of the maximum visibility is easily evaluated in closed form. Furthermore, such a value does not change if a second anisotropic element is inserted at the other pinhole.

Let us recall that, in the space–time domain, we can account for the complete set of correlation functions at two typical points $\mathbf{r}_1$ and $\mathbf{r}_2$ by using the beam coherence-polarization (BCP) matrix,[2,3]

$$\hat{J}(\mathbf{r}_1,\mathbf{r}_2)=\begin{bmatrix} J_{xx}(\mathbf{r}_1,\mathbf{r}_2) & J_{xy}(\mathbf{r}_1,\mathbf{r}_2) \\ J_{yx}(\mathbf{r}_1,\mathbf{r}_2) & J_{yy}(\mathbf{r}_1,\mathbf{r}_2) \end{bmatrix}, \tag{1}$$

whose elements $J_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$, $(\alpha,\beta=x,y)$ give the cross-correlation between the α- and β-components of the electric field at points $\mathbf{r}_1$ and $\mathbf{r}_2$ for zero time delay. It was noted in Refs. 2 and 3 that the visibility of fringes seen at the output of a Young's interferometer depends only on the trace of the matrix (when $\mathbf{r}_1$ and $\mathbf{r}_2$ specify the positions of the pinholes). To synthesize this result, an *equivalent* mutual intensity at the pinholes was defined as

$$J_{\mathrm{eq}}(\mathbf{r}_1,\mathbf{r}_2)=\mathrm{Tr}\{\hat{J}(\mathbf{r}_1,\mathbf{r}_2)\}, \tag{2}$$

where Tr stands for trace. The meaning of this quantity is as follows. If no anisotropic elements are used, then the visibility predicted by the vectorial theory is the same as that of the ordinary scalar theory,[17] provided that the usual mutual intensity is replaced by J_{eq}. More explicitly, the visibility of the fringes, say $\mathcal{V}$, in the interference pattern is given by[2,3,17]

$$\mathcal{V} = \frac{2|J_{\mathrm{eq}}(\mathbf{r}_1,\mathbf{r}_2)|}{I(\mathbf{r}_1) + I(\mathbf{r}_2)}, \tag{3}$$

where $I(\mathbf{r}_j) = \mathrm{Tr}\{\hat{J}(\mathbf{r}_j,\mathbf{r}_j)\}$, $j=1,2$.

From now on, we shall be interested only in the elements of the BCP matrix for which $\mathbf{r}_1$ and $\mathbf{r}_2$ are position vectors of the two pinholes of a Young's interferometer. These elements form a 2×2, not necessarily Hermitian, matrix, that will be denoted in the following by $\hat{J}_{12}$.

To examine the changes suffered by the visibility when a nonabsorbing anisotropic element is inserted into the path of light emerging from one of the pinholes, we first note that the denominator of Eq. (3) remains unchanged. Therefore the change of $\mathcal{V}$ is determined by that of J_{eq}. Next, let us recall[18] that a nonabsorbing anisotropic element can always be specified by a suitable 2×2 *unitary* matrix, i.e., an element of the $U(2)$ group, say $\hat{M}$. The unitary character implies

$$\hat{M}\hat{M}^{\dagger} = \hat{M}^{\dagger}\hat{M} = \hat{I}, \tag{4}$$

where the daggers denote Hermitian conjugation and $\hat{I}$ stands for the identity matrix. When such an element is located at one of the pinholes, matrix $\hat{J}_{12}$ goes onto $\hat{J}'_{12} = \hat{J}_{12}\hat{M}$.[3] The modulus of $\mathrm{Tr}\{\hat{J}'_{12}\}$ will determine the new fringe visibility at the output of the Young's interferometer.

In particular, for maximizing the fringe visibility, it is sufficient to maximize the quantity $|\mathrm{Tr}\{\hat{J}_{12}\hat{M}\}|$, when $\hat{M}$ spans the $U(2)$ group. Taking into account Eq. (2), we shall denote such a maximum by $|J_{\mathrm{eq}}|_{\mathrm{Max}}$.

The above maximization problem is well known in the context of matrix analysis, and its solution is provided by the so-called Ky-Fan 1-norm.[19] More precisely, consider the singular value decomposition of the matrix $\hat{J}_{12}$,[18] according to which we can always write

$$\hat{J}_{12} = \hat{U}\hat{S}\hat{V}^{\dagger}, \tag{5}$$

where $\hat{S}$ is a 2×2 diagonal matrix whose elements, say σ_1 and σ_2, are the (nonnegative) singular values, while $\hat{U}$ and $\hat{V}$ are suitable unitary matrices whose columns, say, $\mathbf{u}_i$ and $\mathbf{v}_i$ $(i=1,2)$, respectively, are related to σ_i by the following coupled linear equations:

$$\hat{J}_{12}\mathbf{v}_i = \sigma_i\mathbf{u}_i, \qquad \hat{J}^{\dagger}_{12}\mathbf{u}_i = \sigma_i\mathbf{v}_i. \tag{6}$$

In particular, the squares of the σ_i coincide with the eigenvalues of the (nonnegative definitive) operators $\hat{J}^{\dagger}_{12}\hat{J}_{12}$ and $\hat{J}_{12}\hat{J}^{\dagger}_{12}$.

The Ky-Fan n-norm of the matrix $\hat{J}_{12}$, say, $N_n(\hat{J}_{12})$, is defined as[19]

$$N_n(\hat{J}_{12}) = (\sigma_1^n + \sigma_2^n)^{1/n}. \tag{7}$$

In particular, it is possible to prove[19] that $N_1(\hat{J}_{12})$ has the following variational characterization:

$$N_1(\hat{J}_{12}) = \sigma_1 + \sigma_2 = \max\{|\mathrm{Tr}\{\hat{J}_{12}\hat{M}\}| : \hat{M} \in U(2)\}, \tag{8}$$

which directly provides the solution to our problem. Moreover, the maximum value is attained by choosing as $\hat{M}$ the matrix $\hat{\mathcal{M}} = \hat{V}\hat{U}^{\dagger}$. In fact, on using Eqs. (4) and (5), such a choice leads to

$$\hat{J}_{12}\hat{\mathcal{M}} = \hat{U}\hat{S}\hat{V}^{\dagger}\hat{V}\hat{U}^{\dagger} = \hat{U}\hat{S}\hat{U}^{\dagger}, \tag{9}$$

so that

$$\mathrm{Tr}\{\hat{J}_{12}\hat{\mathcal{M}}\} = \mathrm{Tr}\{\hat{U}\hat{S}\hat{U}^{\dagger}\} = \mathrm{Tr}\{\hat{S}\} = \sigma_1 + \sigma_2, \tag{10}$$

where use has been made of the fact that the matrices $\hat{U}\hat{S}\hat{U}^{\dagger}$ and $\hat{S}$ are connected by a similarity transformation so that they have the same trace.

This proves that, for any given BCP matrix of the field at the pinholes, the equivalent mutual intensity obtained by the insertion of a suitable nonabsorbing anisotropic element at one of the pinholes has a maximum modulus given by the positive number

$$|J_{\mathrm{eq}}|_{\mathrm{Max}} = N_1(\hat{J}_{12}) = \sigma_1 + \sigma_2, \tag{11}$$

which is invariant under $U(2)$ transformations. We further note that the anisotropic element that leads to $|J_{\mathrm{eq}}|_{\mathrm{Max}}$ is determined up to a phase factor. As seen above, in fact, such an element is specified by the Jones matrix $\hat{\mathcal{M}} = \hat{V}\hat{U}^{\dagger}$. However, applying to $\hat{J}_{12}$ a matrix of the form $\hat{\mathcal{M}}\exp(i\varphi)$, with arbitrary real φ, would also transform $\hat{J}_{12}$ into a matrix whose trace has a modulus equal to $|J_{\mathrm{eq}}|_{\mathrm{Max}}$. Finally, we remark that the same results would have been obtained with a further anisotropic element inserted at the second pinhole.

It should be recalled that a somewhat similar approach has been recently used by Réfrégier and co-workers,[10,15,20] who proposed new definitions of degrees of coherence as the singular values of a suitable matrix, obtained by $\hat{J}$. It has also been shown that such degrees of coherence can be related to the visibility of the fringes produced in a Young's interferometer equipped with both polarizers and nonabsorbing anisotropic elements. Such a setup, however, introduces irreversible transformations on the impinging field. This remark also applies to the scheme proposed in a paper by Mujat *et al.*[21] relating to the Fresnel–Arago laws, where polarizers and rotators were inserted at the pinholes.

It is easy to show that the Ky Fan 1-norm of $\hat{J}_{12}$ can be expressed in closed form so that $|J_{\mathrm{eq}}|_{\mathrm{Max}}$ can be evaluated without explicitly calculating the singular values. In fact, as already said, σ_1^2 and σ_2^2 are the eigenvalues of $\hat{J}_{12}\hat{J}^{\dagger}_{12}$. Furthermore, the determinant of the matrix $\hat{S}$ appearing in Eq. (5) is equal to the modulus of the determinant of $\hat{J}_{12}$. As a consequence, we have

$$|J_{\mathrm{eq}}|_{\mathrm{Max}} = [\mathrm{Tr}\{\hat{J}_{12}\hat{J}^{\dagger}_{12}\} + 2|\mathrm{Det}\{\hat{J}_{12}\}|]^{1/2}, \tag{12}$$

where Det stands for the determinant.

Equation (12) has a further virtue. It shows that

$|J_{\mathrm{eq}}|_{\mathrm{Max}}$ can be related to the so-called *electromagnetic degree of coherence*,[7] which gives account of the correlations between *all* the pairs of components of the electric field at the two pinholes of a Young's interferometer so that it is not directly associated with the visibility of the interference pattern.[22–24] In such an approach, the equivalent mutual intensity in Eq. (2) is replaced by a quantity, say, $J_{\mathcal{E}}$, defined as the Frobenius norm of $\hat{J}_{12}$, i.e.,[19]

$$J_{\mathcal{E}} = \sqrt{\mathrm{Tr}\{\hat{J}_{12}\hat{J}_{12}^{\dagger}\}}, \tag{13}$$

which, from Eq. (7), is seen to coincide with the Ky Fan 2-norm of matrix $\hat{J}_{12}$. On comparing Eqs. (12) and (13), the following relation is obtained:

$$|J_{\mathrm{eq}}|_{\mathrm{Max}} = [J_{\mathcal{E}}^2 + 2|\mathrm{Det}\{\hat{J}_{12}\}|]^{1/2}, \tag{14}$$

from which, in particular, it follows that $|J_{\mathrm{eq}}|_{\mathrm{Max}} \geqslant J_{\mathcal{E}}$. As can be easily verified, the equality sign holds if and only if $\hat{J}_{12}=\mathbf{a}_1\mathbf{a}_2^{\dagger}$, with $\mathbf{a}_1$ and $\mathbf{a}_2$ being two suitable vectors.

In scalar coherence theory,[17] the phrase *degree of coherence* was uniquely related to fringe visibility. The same type of relation was suggested by Wolf for the vectorial treatment,[6] making reference to a Young's interferometer in which the incoming field is not modified by the insertion of any anisotropic element. In the electromagnetic case, however, we have seen that the same phrase (possibly supplemented by some adjective) is used with different meanings. In other words, the terms *coherence* and *correlation* are not universally used as synonyms. On the other hand, there seems to be no doubt about the meaning of the word *visibility*. In the present paper, we looked for the maximum value that the visibility may reach through the use of nonabsorbing anisotropic elements. In this way, the field undergoes only a reversible transformation. Accordingly, the quantity $|J_{\mathrm{eq}}|_{\mathrm{Max}}$ specified by Eq. (12) can be thought of as an intrinsic feature of the electromagnetic field at the pinholes.

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