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Analytical derivation of the optimum triplicator

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Abstract

The analytical derivation of the phase profile of a diffractive optical element that produces three equi-intense replicas of an input beam with the maximum efficiency is presented. Such derivation, based on a functional minimization procedure, leads to a closed form for the phase profile and to an efficiency value slightly lower than the predicted theoretical upper bound. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

Beam multipliers, i.e., devices that divide an input beam into a certain number N of output beams with equal power, are among the most popular diffractive optical elements (or DOEs for short). It is well known that, generally speaking, they are almost untractable in analytical terms. For this reason, specialized numerical algorithms have been devised for designing such elements [1]. As far as the diffraction efficiencies are concerned, expressions of the upper bounds have been established by Wyrowski for either amplitude or phase DOEs [2,3], and in Ref. [4] they are used to derive upper bounds on the diffraction efficiency of beam multipliers (for N from 2 to 25) generated using phase-only filters.

Although the knowledge of an upper bound can be helpful in estimating the applicability of a DOE in actual situations, sometimes it could be important to know what is the maximum value of the efficiency that can be reached by a real diffractive phase element.

The answer is quite simple for the cases $N=1$ and $N=2$. In the first one, a unitary efficiency is obtained with a blazed grating, while in the second one the maximum efficiency, that is $8/\pi^2$, is reached by a $(0,\pi)$ binary Ronchi phase grating, as can easily verified. To our knowledge, this are the only cases in which the phase profile corresponding to the maximum efficiency can be computed analytically.

Even for $N=3$, the phase profile corresponding to the optimum multiplier has not been derived. On the other hand, a divide-by-three device, or *triplicator*, has also application interest (see, for example, Ref. [5]). One may wonder whether for such a simple case a precise maximum efficiency (not an upper bound) can be evaluated and what sort of continuous phase profile would ensure such a maximum. In the past, some work has been made about it. For example a triplicator with efficiency of the order of 0.92 was obtained by slightly modificating a sinusoidal phase profile [6]. In Refs. [7] and [8] a theoretical efficiency of 0.926 was found by using optimization criteria in order to have at the same time high values of the efficiency and high uniformity of the intensity of the diffracted orders.

In this paper we prove that an optimum triplicator exists, whose phase transmittance we give in a simple analytic form. As is expected, its efficiency, which, we stress, is the maximum obtainable, is lower than the upper bound given in Ref. [4], i.e., 0.938. Furthermore, we prove that the obtained phase profile is the only one able to produce such an efficiency.

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2. Theoretical analysis

We intend to evaluate the transmission function of the DOE that produces three equi-intense replicas of an incident field and assures the maximum diffraction efficiency. We shall refer to a phase grating with unitary period. We assume that the geometrical features of the DOE are large enough, with respect to the wavelength, to allow a scalar treatment of the problem.

If we denote by $\phi(x)$ the phase profile, the transmission function of the grating is $\tau(x)=\exp[i\phi(x)]$. By virtue of its periodicity, $\tau(x)$ can be expanded into its Fourier series, i.e.,

$$\tau(x)=\sum_{n=-\infty}^{\infty}\tau_n\exp(i2\pi nx), \tag{1}$$

where

$$\tau_n=\int_{-1/2}^{1/2}\tau(x)\exp(-i2\pi nx)\mathrm{d}x. \tag{2}$$

Our aim is to find the function, say $\bar{\phi}(x)$, such that the powers carried by the orders 0, $+1$, and -1 are equal to each other, i.e.,

$$|\tau_{-1}|=|\tau_0|=|\tau_1|, \tag{3}$$

and the efficiency, defined as

$$\eta=|\tau_{-1}|^2+|\tau_0|^2+|\tau_1|^2, \tag{4}$$

assumes its highest value. This form of the efficiency derives from the fact that, for a phase-only grating, the following relation holds:

$$\sum_{n=-\infty}^{\infty}|\tau_n|^2=1. \tag{5}$$

Furthermore, the partial reflection of the incident beam on the DOE surface has been neglected.

The three chosen orders are the central ones but this does not represent a restriction. Indeed, any different choice of contiguous orders can be reconducted to this one, by adding to $\bar{\phi}(x)$ a suitable linear function of x.

First of all, we note that while the phase of the zero order does not depend on a spatial shift of the grating along the x-direction, this is not the case for τ_1 and τ_{-1} [see Eq. (2)]. Since we are interested in the cases in which $|\tau_1|=|\tau_{-1}|$, the origin of the x-axis can be set in such a way that the equality

$$\tau_{-1}=-\tau_1 \tag{6}$$

is satisfied. In this way, we reduce by one the number of degrees of freedom at our disposal and the only independent coefficients remain τ_0 and τ_1.

Now, let us write the phase function $\phi(x)$ in terms of its even and odd parts, as follows:

$$\phi(x)=\phi_e(x)+\phi_o(x). \tag{7}$$

Inserting Eq. (7) into Eq. (2), and taking Eq. (6) into account, the following expressions are obtained for the amplitudes of the orders 0 and $+1$:

$$\tau_0=2\int_0^{1/2}\exp[i\phi_e(x)]\cos[\phi_o(x)]\mathrm{d}x,$$

$$\tau_1=2\int_0^{1/2}\exp[i\phi_e(x)]\sin[\phi_o(x)]\sin(2\pi x)\,\mathrm{d}x. \tag{8}$$

Let us now introduce the functional

$$\mathscr{J}(\phi)=|\tau_0|+a|\tau_1|, \tag{9}$$

with a a positive constant. We shall determine the phase profile which maximizes $\mathscr{J}$ for an arbitrary value of a. Then condition (3) will determine the value of a. Under condition (3), maximizing $\mathscr{J}$ leads to the maximum efficiency of the triplicator.

Before going on, we note that, in order to maximize $\mathscr{J}(\phi)$, the even part of ϕ, i.e. ϕ_e, can be set to zero. This can be shown by starting from the following inequalities [easily derived from Eq. (8)]:

$$|\tau_0|\le 2\int_0^{1/2}\left|\cos[\phi_o(x)]\right|\mathrm{d}x,$$

$$|\tau_1|\le 2\int_0^{1/2}\left|\sin[\phi_o(x)]\right|\sin(2\pi x)\,\mathrm{d}x. \tag{10}$$

In Appendix A it is proved that the equality signs in Eq. (10) hold if $\sin(\phi_o)$ and $\cos(\phi_o)$ are positive in the interval $(0,1/2)$ and, at the same time, ϕ_e assumes an (arbitrarily chosen) constant value. In particular, we set $\phi_e=0$ and Eq. (10) becomes

$$|\tau_0|=2\int_0^{1/2}\cos[\phi(x)]\mathrm{d}x,$$

$$|\tau_1|=2\int_0^{1/2}\sin[\phi(x)]\sin(2\pi x)\,\mathrm{d}x, \tag{11}$$

where the equality $\phi=\phi_o$ has been used. Note that, comparing Eq. (11) with Eq. (8) written for $\phi_e=0$, the coefficients τ_0 and τ_1 turn out to be positive quantities. Hence, from Eq. (6), τ_{-1} is negative. This agrees with the results obtained in Ref. [4] for a continuous-phase grating for $N=3$.

Inserting Eq. (11) into Eq. (9), the following expression is obtained for the functional $\mathscr{J}(\phi)$:

$$\mathscr{J}(\phi)=2\int_0^{1/2}\{\cos[\phi(x)]+a\sin[\phi(x)]\sin(2\pi x)\}\mathrm{d}x. \tag{12}$$

According to the variational calculus, the first variation of $\mathscr{J}$ must be 0 around the optimum phase, $\bar{\phi}(x)$. This means that letting

$$\phi(x)=\bar{\phi}(x)+\varepsilon(x), \tag{13}$$

where $\varepsilon(x)$ is an arbitrary small perturbation, the corresponding first variation of $\mathscr{J}(\phi)$, that is

$$\begin{aligned}\delta\mathscr{J} &= \mathscr{J}(\bar\phi+\varepsilon)-\mathscr{J}(\bar\phi)\\ &\approx 2\int_0^{1/2}\varepsilon(x)\left\{a\cos\left[\bar\phi(x)\right]\sin(2\pi x)\right.\\ &\qquad\left.-\sin\left[\bar\phi(x)\right]\right\}\mathrm{d}x. \qquad (14)\end{aligned}$$

must vanish for every $\varepsilon(x)$. Therefore, $\bar\phi$ must be a solution of the following equation:

$$a\cos\left[\bar\phi(x)\right]\sin(2\pi x)-\sin\left[\bar\phi(x)\right]=0, \qquad (15)$$

whence

$$\bar\phi(x;a)=\tan^{-1}\left[a\sin(2\pi x)\right], \qquad (16)$$

where the explicit dependence on a has been shown in the argument of $\bar\phi$.

It is worth stressing that the function in Eq. (16) maximizes $\mathscr{J}$ for any choice of the parameter a and that we have not yet imposed the condition that $|\tau_0|$ and $|\tau_1|$ have to be equal to each other. Actually, as remarked above, it is just this condition that fixes the value of a.

Inserting Eq. (16) into Eq. (11) we can express the expression of the absolute values of the coefficients τ_0 and τ_1 that maximize $\mathscr{J}$ in terms of elliptic integrals, as follows:

$$|\tau_0| = 2\int_0^{1/2}\frac{\mathrm{d}x}{\sqrt{1+a^2\sin^2(2\pi x)}} = \frac{2}{\pi}K(-a^2),$$

$$\begin{aligned}|\tau_1| &= 2a\int_0^{1/2}\frac{\sin^2(2\pi x)}{\sqrt{1+a^2\sin^2(2\pi x)}}\,\mathrm{d}x\\ &= \frac{2}{\pi a}\left[E(-a^2)-K(-a^2)\right], \qquad (17)\end{aligned}$$

where K and E denote the complete elliptic integrals of first and second kind [9], respectively. The dependence of $|\tau_0|$ (solid) and $|\tau_1|$ (dotted) on a is shown in Fig. 1.

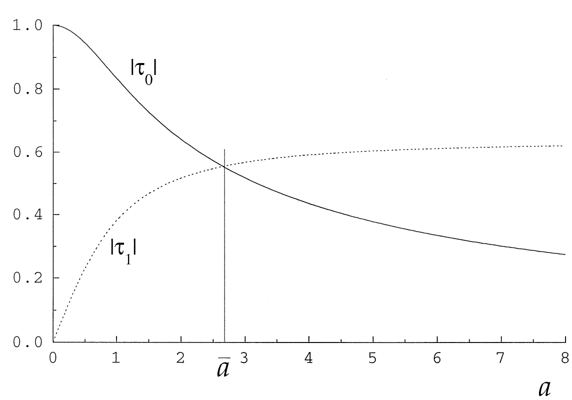

Fig. 1. Behavior of the modula of the zero (solid) and the first (dotted) diffraction order for a triplicator as a function of the a parameter [see Eq. (17)].

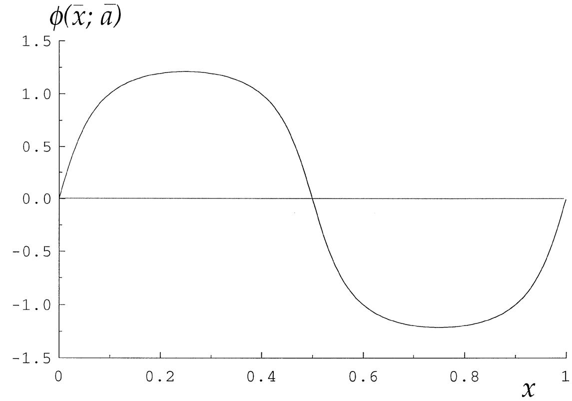

Fig. 2. Phase profile corresponding to the optimum triplicator, as a function of x, for an unitary grating period.

Equating to each other the two expressions in Eq. (17), the following value of a, say $\bar a$, is obtained:

$$\bar a = 2.65718\ldots. \qquad (18)$$

In conclusion, the phase profile of the grating producing a triplication of an incident light beam with the highest efficiency is given by the function in Eq. (16), where the value of a has to be taken as $\bar a$, given in Eq. (18). Its efficiency can be evaluated by means of Eqs. (4), (6) and (17), and turns out to be

$$\eta = 0.92556\ldots. \qquad (19)$$

Fig. 2 shows the phase profile corresponding to the optimum triplicator, for a unitary grating period. It should be noted that both the phase profile and the efficiency value are similar to those obtained by means of numerical algorithms in Refs. [6] and [7].

Before ending this section, we want to show how, by choosing suitable values of a, it is also possible to obtain from Eq. (16) the phase distribution of the optimum *monoplicator* and *duplicator*. Indeed, when $a=0$ or $a\to\infty$, maximizing $\mathscr{J}$ is tantamount to maximizing $|\tau_0|$ or $|\tau_1|$ (and then also $|\tau_{-1}|$), respectively. From Eqs. (16) and (11) we have, indeed,

$$\lim_{a\to 0}\bar\phi(x;a)=0 \Rightarrow \begin{cases}|\tau_0|=1\\ |\tau_1|=0\end{cases}, \qquad (20)$$

and

$$\lim_{a\to\infty}\bar\phi(x;a)=\frac{\pi}{2}\,\mathrm{sgn}\left[\sin(2\pi x)\right] \Rightarrow \begin{cases}|\tau_0|=0\\ |\tau_1|=2/\pi\end{cases}. \qquad (21)$$

From the first equation we deduce that the grating which maximizes the diffraction efficiency of order 0 has a constant phase profile and produces only one order with unitary efficiency. This is a trivial result. On the other hand, the second equation shows that the optimum duplicator is a $(0,\pi)$ phase Ronchi grating, with efficiency equal to $8/\pi^2$, which coincides with the value given in Ref. [4] for the Wyrowski upper bound.

3. Conclusions

The analytical derivation of the optimum triplicator has been presented. It leads to a closed-form expression for the phase profile ensuring the maximum conceivable efficiency for a DOE that produces three replicas of equal power of an input beam. This maximum efficiency is shown to be $\eta = 0.92556\ldots$, slightly smaller than the upper bound given by Wyrowski.

The mathematical technique used in this paper can be extended to find efficiency and phase profile of optimum divide-by-N DOEs, but in such cases numerical methods have to be used for maximization of the involved functionals [7].

Appendix A

Here, we want to prove that, given the following integrals:

$$I_1 = \int_0^{1/2} \exp[i\phi_e(x)]\cos[\phi_o(x)]\,\mathrm{d}x,$$

$$I_2 = \int_0^{1/2} |\cos[\phi_o(x)]|\,\mathrm{d}x, \tag{A.1}$$

$$I_3 = \int_0^{1/2} \exp[i\phi_e(x)]\sin[\phi_o(x)]\sin(2\pi x)\,\mathrm{d}x,$$

$$I_4 = \int_0^{1/2} |\sin[\phi_o(x)]|\sin(2\pi x)\,\mathrm{d}x, \tag{A.2}$$

the equalities

$$|I_1| = I_2 \tag{A.3}$$

and

$$|I_3| = I_4, \tag{A.4}$$

are verified only for particular choices of the function ϕ_e.

We begin from the case of Eqs. (A.1) and (A.3). The interval $(0,1/2)$ can be divided into N sub-intervals of width $\delta = 1/2N$. Then, if the integrands in Eq. (A.1) are continuous functions, as we suppose, we can write

$$I_1 = \delta \sum_{k=1}^{N} \exp[i\phi_e(\xi_k)]\cos[\phi_o(\xi_k)], \tag{A.5}$$

$$I_2 = \delta \sum_{k=1}^{N} |\cos[\phi_o(\eta_k)]|, \tag{A.6}$$

where ξ_k and η_k are points suitably chosen inside the kth sub-interval.

For simplicity, we set

$$\cos[\phi_o(\xi_k)] = C_k, \quad \cos[\phi_o(\eta_k)] = D_k. \tag{A.7}$$

By evaluating the squared modulus of I_1 and I_2, the following results are obtained:

$$|I_1|^2 = \delta^2\left[\sum_k |C_k|^2 + 2\sum_{h,k>h} |C_h||C_k|\gamma_{hk}\right], \tag{A.8}$$

$$|I_2|^2 = \delta^2\left[\sum_k |D_k|^2 + 2\sum_{h,k>h} |D_h||D_k|\right], \tag{A.9}$$

where

$$\gamma_{hk} = \cos[\phi_e(\xi_h) + \alpha(\xi_h) - \phi_e(\xi_k) - \alpha(\xi_k)]. \tag{A.10}$$

Here, $\alpha(\xi_k)$ is the phase of C_k and assumes values 0 or π, depending on the sign of C_k.

Let us note that, in the limit $N \to \infty$, ξ_k tends to η_k, so that $C_k \to D_k$. Then, in this limit, the equality (A.3) is verified only if [see Eqs. (A.8) and (A.9)]

$$\gamma_{hk} = 1, \quad \forall (h,k). \tag{A.11}$$

This means that, in terms of the function ϕ_e, the following condition must be met:

$$\phi_e(x) + \alpha(x) = \Phi, \tag{A.12}$$

with Φ constant. With such choice for ϕ_e, it can be easily verified that Eq. (A.3) holds. In particular, if $\cos[\phi_o(x)] \geq 0, \quad \forall x \in (0,1/2)$, Eq. (A.12) reduces to $\phi_e(x) = \Phi$.

When similar arguments are applied to Eq. (A.4), we find that $\phi_e = \Phi'$, with Φ' constant, when $\sin(\phi_o) \geq 0$ and $\phi_e = \Phi' - \pi$ otherwise. So, if there exist intervals in which $\sin(\phi_o)$ and $\cos(\phi_o)$ have opposite sign, it is not possible to find a function ϕ_e so that both conditions (A.3) and (A.4) are simultaneously satisfied.

On the contrary, if a phase $\bar{\phi}(x)$ is found as a solution of our optimization problem, for which both $\sin(\phi_o)$ and $\cos(\phi_o)$ have the same sign in the interval $(0,1/2)$, then $\phi_e(x)$ must be constant, in order that both Eq. (A.3) and Eq. (A.4) hold. In particular, this is the case for the solution given in Eq. (16), for which $\sin(\phi_o)$ and $\cos(\phi_o)$ are both positive.

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