Shape-Invariance Error for Axially Symmetric Light Beams

Silvia Vicalvi, Riccardo Borghi, Massimo Santarsiero, and Franco Gori

***Abstract*—A significant aspect of the propagation of coherent light beams is that the shape of the transverse field distribution changes. In this paper, the concepts of *shape-invariance error* and *shape-invariance range* are used to characterize such effects in a quantitative way. Applications of the theoretical analysis to some simple but significant cases are presented.**

***Index Terms*—Beams, laser beam distortion, laser beams, laser modes, optical propagation.**

I. Introduction

IT IS WELL KNOWN that a Hermite–Gauss or a Laguerre–Gauss beam [1] is shape-invariant. Starting from the waist plane, the beam undergoes a transverse scale magnification upon propagation. In addition, its wavefront passes from planar to spherical and its initial phase changes. Therefore, it can be said that, up to a transverse scale factor, a quadratic phase factor, and an overall phase change, the shape of the beam cross section remains everywhere the same as at the waist plane. More general beams are not endowed with such a property and their transverse amplitude and phase distributions change in a more complicated way in the course of propagation. Nevertheless, except for pathological cases to be discussed later, a typical beam exhibits an approximate shape invariance for a certain range of distances along the mean axis of propagation. Possessing quantitative information about shape changes (i.e., modifications of the transverse amplitude and phase distributions) of a beam through propagation can be useful, especially whenever a field with assigned intensity and phase profiles is requested for practical applications. Moreover, it could permit a better evaluation of the tolerances in experimental apparatus and give a useful tool in designing optical instruments. For instance, a new definition of depth of focus, depending on the incident beam, could be used in the case of focusing systems.

In this paper we investigate the shape-invariance properties of a light beam. For the sake of simplicity, we limit ourselves to the case of axial symmetry. Using such concepts as the embedded Gaussian beam [2], [3] and the effective radius of curvature [4], we first discuss how the departure from shape invariance can be quantified through a *shape-invariance error* (SIE) (Section II). This quantity allows us to define a *shape-invariance range* (SIR) up to any desired degree of accuracy. The problem remains of how these quantities can be evaluated for a typical beam. We tackle this problem by using series expansions in Laguerre–Gauss (LG) beams. Once the expansion coefficients are known, the SIE can be evaluated at any cross section of the beam (Sections III and IV). Furthermore, the use of LG expansions gives a clear insight into the physical phenomenon that leads to losing shape invariance. In order to illustrate the main results, we work out some simple examples (Section V).

II. The Shape-Invariance Error

Let us consider a general axially symmetric beam, specified by the field distribution $V_0(r)$ at the transverse plane $z = 0$. Under the paraxial approximation, the propagated field at a distance z from the starting plane will be given by the Fresnel formula, that is [1],

$$V_z(r) = \frac{-i}{\lambda z} \exp\left[i\left(kz + \frac{k}{2z}r^2\right)\right] \int_0^\infty V_0(\rho) J_0\left(\frac{k\rho r}{z}\right) \cdot \exp\left(i\frac{k}{2z}\rho^2\right)\rho\, d\rho \quad (1)$$

where λ is the wavelength, $k = 2\pi/\lambda$, and J_0 is the Bessel function of the first kind and zeroth order.

Obviously, the structure of the propagated field depends on the analytical expression of the field at the starting plane, and closed forms for $V_z(r)$ are obtained in very few cases. Nonetheless, some of the properties of the propagated field can be predicted without explicitly evaluating the integral in (1) and hold for any beam propagating under the paraxial approximation. For instance, if a "width" of the beam is defined at a given transverse plane as the square root of the variance of the corresponding intensity distribution, namely

$$\sigma_z^2 = \frac{\int_0^\infty r^3 |V_z(r)|^2\, dr}{\int_0^\infty r |V_z(r)|^2\, dr} \quad (2)$$

it is known [2] that the following propagation law holds:

$$\sigma_z^2 = \sigma_{z_0}^2 \left[1 + \left(\frac{M^2\lambda}{2\pi\sigma_{z_0}^2}\right)^2 (z - z_0)^2\right] \quad (3)$$

where z_0 is the longitudinal coordinate of the waist plane, which is defined as the plane where the width reaches its

Manuscript received January 21, 1998; revised July 28, 1998.

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Publisher Item Identifier S 0018-9197(98)08085-3.

minimum value σ_{z_0}, and M^2 is the well-known factor that gives account of the divergence attitude of the beam, once the value of σ_{z_0} is fixed. In terms of the second-order moments of the transverse intensity distribution, it turns out to be [2]

$$M^2 = 2\pi\sigma_{z_0}\sigma_\infty \tag{4}$$

where σ_∞ is the width of the Fourier transform of the field across the waist.

Another useful quantity that can be defined for any paraxial beam is the average curvature radius, that is, the curvature radius of the spherical wave that matches the real wavefront of the beam as best as it can. In formulas, it is defined as [4]

$$R_z = -k\sigma_z^2\left[2\pi\int_0^\infty \left(r\,\frac{\partial\phi_z(r)}{\partial r}\right)|V_z(r)|^2 r\,dr\right]^{-1} \tag{5}$$

where ϕ_z is the phase of the transverse field distribution at the plane z. Even in this case the propagation law has a simple expression, i.e., [4]

$$R_z = (z-z_0) + \left(\frac{2\pi\sigma_{z_0}^2}{M^2\lambda}\right)^2 \frac{1}{(z-z_0)} \tag{6}$$

which is valid for any beam propagating under the paraxial approximation.

Incidentally, note that width and average curvature radius of a general beam exactly follow the same propagation laws as the analogous quantities of the fundamental Gaussian beam having spot-size at the waist given by

$$w_{z_0} = \sqrt{\frac{2\sigma_{z_0}^2}{M^2}} \tag{7}$$

which is known as the *embedded* Gaussian beam [2], [3].

In the following, for simplicity, we shall limit ourselves to the case in which the plane $z = 0$ coincides with the waist plane of the beam, and then $z_0 = 0$. The results obtained in this way will be generalized in Section IV to the case $z_0 \neq 0$.

Coming back to the purpose of this work, we want to quantify the modifications undergone by the shape of the transverse profile of a beam on propagation. Then, the effect produced on the field by the transverse scale magnification should be weeded out, as well as the spherical bending of the wavefront. To this end, we define, at any plane $z > 0$, a reference field in the following way. Take the propagated field and scale it down by the scale factor pertaining to the embedded Gaussian beam. In order to preserve the total energy of the beam, such scale contraction must be accompanied by a proportional increase of the field amplitude. Further, eliminate the spherical curvature using again the knowledge of the embedded beam. The result is the reference field at the plane $z =$ constant, that will be denoted by V_z^R.

A further quantity that could be present in the reference field, but should be removed, is a uniform phase factor. In fact, although we are interested in a comparison between starting and reference fields (and not between their intensities), a phase factor common to all points in the transverse plane is not very significant from a physical point of view. For example, for the case of a LG beam, for which a strict shape invariance is expected, the reference field for any z is equal to the field at the waist, up to an overall phase factor, arising from the propagation term $\exp(ikz)$ and from the presence of the Gouy phase [1]. So, it is clear that in this case such phases should be subtracted from the reference field to obtain a strict shape invariance.

In the case of a general axially symmetric beam, the procedure is analogous, but it is complicated by the fact that the determination of the uniform phase to be subtracted is not that simple. Thus, what we are going to do is to determine a sort of *average* phase, which will be subtracted from the phase of the reference field. In this way, the comparison between the fields takes into account only phase modulations around this average value, in addition to amplitude variations.

In order to evaluate the average phase of the reference field, we proceed as follows. First, we define the quantity

$$Q(\delta_z) = \sqrt{\frac{2\pi\int_0^\infty |V_0(r) - V_z^R(r)\,\exp(-i\delta_z)|^2 r\,dr}{4\pi\int_0^\infty |V_0(r)|^2 r\,dr}} \tag{8}$$

which is proportional to the mean squared difference between the field at $z = 0$ and the reference field, multiplied by the phase term $\exp(-i\delta_z)$. As can be easily verified, such a quantity is normalized to the interval [0, 1]. Now, in view of the above remarks about the uniform phase term, the average phase of the reference field can be defined as that value of δ_z, say $\hat{\delta}_z$, that minimizes Q.

In order to derive an explicit expression of $\hat{\delta}_z$, we expand (8), assuming, for simplicity, that V_0 (and hence V_z^R) carries unit power, that is

$$2\pi\int_0^\infty |V_0(r)|^2 r\,dr = 2\pi\int_0^\infty |V_z^R(r)|^2 r\,dr = 1. \tag{9}$$

We then find

$$\begin{aligned} Q(\delta_z) &= \sqrt{1 - \mathrm{Re}\left\{\exp(i\delta_z)2\pi\int_0^\infty V_0(r)V_z^{R*}(r)r\,dr\right\}} \\ &= \sqrt{1 - \left|2\pi\int_0^\infty V_0(r)V_z^{R*}(r)r\,dr\right|\cos(\alpha_z+\delta_z)} \end{aligned} \tag{10}$$

where Re stands for the real part, the asterisk denotes the complex conjugate, and

$$\alpha_z = \arg\left\{2\pi\int_0^\infty V_0(r)V_z^{R*}(r)r\,dr\right\} \tag{11}$$

the symbol $\arg$ denoting the argument.

Since we look for the value of δ_z that minimizes Q, we have from (10)

$$\hat{\delta}_z = -\alpha_z = -\arg\left\{2\pi\int_0^\infty V_0(r)V_z^{R*}(r)r\,dr\right\}. \tag{12}$$

We note that, when V_0 and V_z^R differ only for a uniform phase factor (as in the case of a LG beam), (12) just returns such phase.

In conclusion, we introduce the **shape-invariance error** (SIE) at distance z from the waist as the mean squared difference between the field at the starting plane and the reference field, from whose phase the constant term given in (12) has been weeded out. More explicitly,

$$\varepsilon_z = Q(\hat{\delta}_z) = \sqrt{1 - \left| 2\pi \int_0^\infty V_0(r) V_z^{R*}(r) r\, dr \right|}. \quad (13)$$

In particular, $\varepsilon_z = 0$ means that the two fields are equal to each other up to a uniform phase term, while if $\varepsilon_z = 1$ they are mutually orthogonal. Finally, we define the **shape-invariance range** (SIR) of the field [5] as the distance from the waist plane, say ζ, such that ε_z remains less than some given quantity if $z \leq \zeta$. It is a parameter that can be useful in practice. For example, if a given field profile is required at a certain transverse plane, the SIR quantitatively expresses how far one can move from that plane, according to the maximum error dictated by the experimental requirements.

III. Modal Representation of the Beam

In order to obtain more useful expressions for the quantities introduced in the previous section, we expand the beam under consideration into a series of LG beams [1]. Since we are studying axially symmetric field distributions, only LG modes endowed with axial symmetry will be present in the expansion. Moreover, the waist plane of the component LG modes can be chosen as coincident with that of the beam under consideration (i.e., $z = 0$). This means that the field amplitude at the waist can be written as

$$V_0(r) = \sum_{n=0}^\infty c_n \psi_n(r;\, w_0) \quad (14)$$

where $\psi_n(r;\, w_0)$ are the LG functions at the waist, defined as [1]

$$\psi_n(r; w_0) = \sqrt{\frac{2}{\pi w_0^2}} L_n\left(\frac{2r^2}{w_0^2}\right) \exp\left(-\frac{r^2}{w_0^2}\right), \quad n = 0, 1, \cdots. \quad (15)$$

Here, L_n is the Laguerre polynomial of order n [6], w_0 is the spot size of the mode, and the expansion coefficients are given by

$$c_n = 2\pi \int_0^\infty V_0(r) \psi_n(r;\, w_0) r\, dr, \qquad n = 0,\, 1,\, \cdots \quad (16)$$

It is worth noting that, for any fixed field distribution, the value of the parameter w_0 can be chosen arbitrarily and that the coefficients c_n depend on the particular choice.

The expression of the propagated field in the paraxial regime can be deduced starting from the modal representation (14). Across a typical plane $z > 0$, we have

$$V_z(r) = \exp(ikz - i\Phi_z) \exp\left(\frac{ik}{2R_z} r^2\right) \cdot \sum_{n=0}^\infty c_n \exp(-i2n\Phi_z) \psi_n(r;\, w_z) \quad (17)$$

where w_z, R_z, and Φ_z are given by the well-known expressions [1]

$$w_z = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$
$$R_z = z\left[1 + \left(\frac{z_R}{z}\right)^2\right]$$
$$\Phi_z = \arctan\left(\frac{z}{z_R}\right) \quad (18)$$

where z_R denotes the Rayleigh distance, that is

$$z_R = \frac{\pi w_0^2}{\lambda}. \quad (19)$$

We see from (17) that the modifications suffered by the beam on propagation can be ascribed to the behavior of each component Gaussian beam. In particular, through propagation each mode undergoes:

- an overall dephasing, independent of r and equal for all the modes, taken into account by the factor $\exp(ikz - i\Phi_z)$;
- a spherical bending of the wavefront, given by $\exp(ikr^2/2R_z)$;
- a scaling of the transverse profile, due to the presence of the term w_z in the argument of ψ_n;
- an index-dependent dephasing specified by the term $\exp(-i2n\Phi_z)$.

Now, although the features of the propagated beam cannot depend on the specific choice of the spot size of the modes used to represent it, each of the above-mentioned phenomena is conditioned by this choice, because of the relations (18) and (19). Suppose, however, that a certain value of the spot size, say $\overline{w}_0$, can be found such that each of the modes spreads with the same rate as the total field and, at any fixed transverse plane, its curvature radius matches the average spherical curvature of the beam. In that case, changes of the shape of the field could be simply ascribed to the dephasing among the modes.

On the basis of the considerations contained in the previous section, we see that the value of the spot size ensuring that the above requirements are met is just the one pertaining to the embedded Gaussian beam, namely,

$$\overline{w}_0 = \sqrt{\frac{2\sigma_0^2}{M^2}}. \quad (20)$$

From now on, for any real beam we shall refer to the basis of LG beams having spot size $\overline{w}_0$ as its *natural* basis. By making use of the formula given in the previous section, we can now see that the expression assumed by the reference field is

$$V_z^R(r) = \sum_{n=0}^\infty c_n \exp(-i2n\Phi_z) \psi_n(r;\, \overline{w}_0). \quad (21)$$

According to the definition of the reference field, the transverse width of the modes equals the value at the waist for any z and the curvature term has been eliminated. As regards the phase term $\exp(ikz - i\Phi_z)$, which is constant at any transverse plane, it has been omitted because it would be in any case counterbalanced by $\exp(-i\hat{\delta}_z)$.

By inserting (14) and (21) into (11), the following result is obtained for α_z:

$$\begin{aligned}\alpha_z &= \arg\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n c_m^* \exp(2im\Phi_z) \right.\\ &\qquad \left. \cdot 2\pi \int_0^{\infty} \psi_n(r;\overline{w}_0)\psi_m(r;\overline{w}_0) r\, dr\right\}\\ &= \arg\left\{\sum_{n=0}^{\infty} |c_n|^2 \exp(2in\Phi_z)\right\} \end{aligned} \tag{22}$$

where the orthogonality among the modes has been taken into account.

By virtue of (12), the expression in (22) also provides, apart from a minus sign, an explicit expression of the compensating phase term $\hat{\delta}_z$. In order to give a simple physical interpretation of this term, it is worth referring to the phase factor $\exp(i\hat{\delta}_z)$, given by

$$\exp(i\hat{\delta}_z) = \frac{\displaystyle\sum_{n=0}^{\infty} |c_n|^2 \exp(-2in\Phi_z)}{\left|\displaystyle\sum_{n=0}^{\infty} |c_n|^2 \exp(-2in\Phi_z)\right|}. \tag{23}$$

This expression can be interpreted as follows: the uniform-phase term acquired by the reference field in (21) on propagation is just the average of the phase terms pertinent to the modes used for its representation, weighted with the squared moduli of the coefficients c_n. Then the average phase acquired by the field propagated along a distance z is obtained by adding to $kz - \Phi_z$, the phase derived from (23).

It can be easily proved that the present definition of $\hat{\delta}_z$ coincides with the phase introduced in [5], that is $-2\overline{n}\Phi_z$, when sufficiently small values of z (and then of ε_z) are considered.

Finally, on inserting (14) and (21) into (13) and making use again of the orthogonality properties of the LG functions, we obtain for the SIE

$$\varepsilon_z = \sqrt{1 - \left|\sum_{n=0}^{\infty} |c_n|^2 \exp(2in\Phi_z)\right|}. \tag{24}$$

This is one of the basic results of this work and shows how the changes in the transverse shape of an axially symmetric coherent beam can be traced back solely to the average mutual dephasing of the component modes (in the natural basis).

IV. Extension to the Case $z_0 \neq 0$

The results contained in the previous two sections were obtained in the case in which the starting plane coincides with the waist plane of the beam. Now this limitation will be removed and the generalization to the case of a field whose waist is not at $z = 0$ will be derived. On the basis of the analysis carried out in Section III, the beam must be expanded by means of LG modes, having waist planes coincident with that of the beam, that is, at $z = z_0$, and spot size at the waist given by (7), here reported for clarity:

$$\overline{w}_{z_0} = \sqrt{\frac{2\sigma_{z_0}^2}{M^2}}. \tag{25}$$

Then, the natural basis pertinent to the field is known if we are able to evaluate the coordinate of the waist plane, the M^2 factor, and the width σ_{z_0} of the beam at the waist, starting from the knowledge of the field distribution at $z = 0$. The solution of this problem, relative to the case of a one-dimensional partially coherent beam, is contained in [7]. By following the same lines outlined in that paper, the following results are obtained for the case of an axially symmetric coherent beam:

$$z_0 = \frac{2\pi}{\lambda} \frac{\mathrm{Im}\left\{\displaystyle\int_0^{\infty} V_0(r)\frac{\partial V_0^*(r)}{\partial r} r^2\, dr\right\}}{\displaystyle\int_0^{\infty}\left|\frac{\partial V_0(r)}{\partial r}\right|^2 r\, dr} \tag{26}$$

$$\begin{aligned} M^2 = \frac{2\pi}{N}\Bigg[&\int_0^{\infty} |V_0(r)|^2 r^3\, dr \int_0^{\infty}\left|\frac{\partial V_0(r)}{\partial r}\right|^2 r\, dr\\ &- \left(\mathrm{Im}\left\{\int_0^{\infty} V_0(r)\frac{\partial V_0^*(r)}{\partial r} r^2\, dr\right\}\right)^2\Bigg]^{1/2} \end{aligned} \tag{27}$$

$$\begin{aligned} \sigma_{z_0}^2 = \frac{2\pi}{N}\Bigg(&\int_0^{\infty} |V_0(r)|^2 r^3\, dr\\ &- \frac{\lambda}{2\pi}\mathrm{Im}\left\{\int_0^{\infty} V_0(r)\frac{\partial V_0^*(r)}{\partial r} r^2\, dr\right\}\Bigg) \end{aligned} \tag{28}$$

where N denotes the power of the beam, which is assumed equal to 1 [see (9)], and Im stands for the imaginary part.

Once z_0 and $\overline{w}_{z_0}$ have been evaluated by means of (25)–(28), we can write the field at $z = 0$ as a superposition of LG functions having spot size $\overline{w}_{z_0}$ at the waist plane, that is

$$\begin{aligned} V_0(r) = &\exp(-i\Phi_0)\exp\left(\frac{ik}{2R_0} r^2\right)\\ &\cdot \sum_{n=0}^{\infty} c_n \exp(-2in\Phi_0)\psi_n(r;\overline{w}_0) \end{aligned} \tag{29}$$

where $\overline{w}_0$, R_0, and Φ_0 are the parameters of the modes at $z = 0$, whose values can be obtained by using the propagation laws given in (18) and (19), provided that w_0 is replaced by $\overline{w}_{z_0}$ and z by $z - z_0$. The evaluation of the expansion coefficients c_n involves the solution of the following integrals:

$$\begin{aligned} c_n = &2\pi \exp[i(2n+1)\Phi_0]\int_0^{\infty} V_0(r)\psi_n(r;\overline{w}_0)\\ &\cdot \exp\left(-\frac{ik}{2R_0} r^2\right) r\, dr, \qquad n = 0, 1, \cdots \end{aligned} \tag{30}$$

From (29), it can be noted that, unlike the case treated in the previous section, the field V_0 has an average phase different from zero and a finite average radius of curvature, and this must be taken into account when the reference field at a general plane $z > 0$ has to be considered. Consequently, some points of the rule given in Section II for the derivation of V_z^R from the propagated field V_z must be changed. In particular, the transverse scale and the average radius of curvature of V_z

must be related to the corresponding quantities of V_0, so that the reference field will be given by

$$V_z^R(r) = \exp\left(\frac{ik}{2R_0}\, r^2\right) \sum_{n=0}^{\infty} c_n \exp(-2in\Phi_z)\psi_n(r;\, \overline{w}_0). \tag{31}$$

A remark about phase $\hat{\delta}_z$ is also needed. In fact, in this case, $\hat{\delta}_z$ accounts for the uniform phase acquired by V_z^R with respect to that already present in V_0. Its expression turns out to be

$$\hat{\delta}_z = -\arg\left\{\sum_{n=0}^{\infty} |c_n|^2 \exp[2in(\Phi_z - \Phi_0)]\right\} \tag{32}$$

as can be easily derived by following a procedure analogous to the one reported in Section III.

Finally, the SIE assumes the following form:

$$\varepsilon_z = \sqrt{1 - \left|\sum_{n=0}^{\infty} |c_n|^2 \exp[2in(\Phi_z - \Phi_0)]\right|}. \tag{33}$$

We see that, when the starting plane does not coincide with the waist plane, the SIE depends on the dephasing of each component mode with respect to its initial phase. In particular, since the phase Φ_z saturates to the value $\pi/2$ as $z \to \infty$, if z_0 is negative and its modulus is much greater than the Rayleigh distance, the SIE will be approximately equal to zero for any $z > 0$, and consequently the SIR will be infinite. Of course, this is what one expects because of the shape-invariance properties of diffracted fields in the Fraunhofer region [8].

V. Examples

A. A Single LG Beam

An immediate check of the correctness of the theory above exposed can be performed by considering a field given by a single LG mode of order n and spot size v_0, i.e.,

$$V_0(r) = \psi_n(r;\, v_0) \tag{34}$$

which is known to be shape-invariant throughout the space, for paraxial propagation. In this case, by inserting the expressions of σ_0^2 and M^2, given by

$$\sigma_0^2 = \frac{2n+1}{2}\, v_0^2 \tag{35}$$

$$M^2 = 2n+1 \tag{36}$$

into (20), we see that the value of $\overline{w}_0$ is just v_0. This means that only one term is present in the expansion of this field when its natural basis is used.

Moreover, the phase δ_z evaluated by means of (23) is trivially equal to $-2n\Phi_z$, so that $V_z^R \exp(-i\delta_z)$ is exactly the same as the field distribution at the plane $z = 0$. As a consequence, the shape-invariance error is zero for any value of z, as was expected, and then the SIR of a Gaussian beam is infinite.

B. Finite Superposition of LG Beams

A more significant example is given by the superposition of a finite number of LG modes with the same spot-size v_0. At the common waist plane of the beams, the field distribution can be written as

$$V_0(r) = \sum_{k=1}^{N} c_{n_k} \psi_{n_k}(r;\, v_0) \tag{37}$$

where N is the total number of the component modes and is greater or equal to 2.

Following analogous lines as those proposed by Martínez-Herrero and Mejías in [9] for the determination of the first- and second-order intensity moments of one-dimensional partially coherent beams, we obtain in the present case for σ_0^2 and M^2 in (38) and (39), shown at the bottom of the page, where $\delta_{n,m}$ is the Kronecker symbol.

It can be easily verified that v_0 does not coincide with the value of the natural spot size, given in (20), unless only nonadjacent modes are present in (37). In this case, M^2 and σ_0^2 have the expressions relative to a superposition of mutually uncorrelated modes [2] and the natural spot size turns out to be just v_0.

As a simple example, we consider the case $N = 2$ and write

$$V_0(r) = c_n\psi_n(r;\, v_0) + c_m\psi_m(r;\, v_0). \tag{40}$$

If $|n - m| \geq 2$, due to the above reason, the expression on the right-hand side of (40) is nothing but the natural modal expansion of the field V_0. Regarding the SIE, starting from (24), the following closed-form result is obtained:

$$\varepsilon_z = \sqrt{1 - \sqrt{1 - 4|c_n|^2|c_m|^2 \sin^2(|n-m|\Phi_z)}}. \tag{41}$$

The behavior of ε_z is shown in Fig. 1, for $|c_n|^2 = 0.3$, $|c_m|^2 = 0.7$, $|n - m| = 6$, $v_0 = 1$ mm, and $\lambda = 0.5$ μm. It can be seen that the error vanishes at certain distances from the waist. More precisely this happens at the distances z_ℓ,

$$\sigma_0^2 = \frac{v_0^2}{2} \sum_{k=1}^{N}\sum_{h=1}^{N} [|c_{n_k}|^2(2n_k+1)\delta_{n_k,\,n_h} - 2n_k \operatorname{Re}\{c_{n_k}^* c_{n_h}\}\delta_{n_k,\,n_h+1}] \tag{38}$$

$$M^2 = \sqrt{\left[\sum_{k=1}^{N} |c_{n_k}|^2(2n_k+1)\right]^2 - 4\left[\sum_{k,\,h=0}^{N} n_k \operatorname{Re}\{c_{n_k}^* c_{n_h}\}\delta_{n_k,\,n_h+1}\right]^2} \tag{39}$$

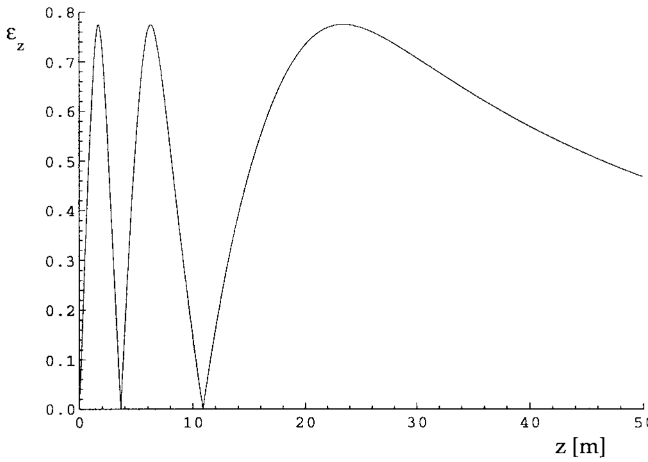


Fig. 1. Behavior of the SIE as a function of z for a superposition of two LG modes, with $|c_n|^2 = 0.3$, $|c_m|^2 = 0.7$, $|n-m| = 6$, $v_0 = 1$ mm, and $\lambda = 0.5$ μm.

satisfying the following equation:

$$|n-m|\Phi_{z_\ell} = \ell\pi \tag{42}$$

with integer ℓ. This behavior is easily understood by noting that, when $z = z_\ell$, the two propagated LG fields have the same phase difference as at the plane $z = 0$, so that the overall field distribution self-reproduces, apart from scale, uniform phase, and curvature factors.

We note that ε_z vanishes also in the limit $z \to \infty$ and this means that V_0 is self-reproducing under Fourier transformation, i.e., it is a self-Fourier function [10]–[15]. Such a behavior is observed in general, whenever (40) contains LG modes with the same parity, and then $|n-m|$ is even, as can be seen from (42). Indeed LG functions of even order are eigenfunctions of the Fourier kernel with eigenvalue 1, while those of odd order have eigenvalue -1. Therefore, any linear combination of (two or more) LG functions with even (odd) indexes gives a self-Fourier function and then an SIE vanishing in the far field. In addition, this kind of superposition always gives rise to beams whose natural spot size coincides with the spot size of the modes.

A similar analysis can be performed when $|n-m| = 1$ but, in such a case, since the condition (42) is never satisfied, the interpretation of the results is not so immediate. Moreover, since the natural spot size does not coincide with v_0, an infinite number of terms has to be considered in the natural expansion of V_0.

C. Flattened Gaussian Beams

As already said, the knowledge of how the transverse shape of a beam changes upon propagation is useful, for example, for practical applications. A significant example is that of beams showing a flattop profile at a certain transverse plane, say the waist plane. Beams of this kind are required, for instance, in optical data processing, laser welding and branding, and laser-stimulated etching. In these cases, it may be interesting to know how far from the waist plane the field distribution approximately maintains its shape. On the basis of our analysis, this is tantamount to evaluating the range of distances, for which the SIE keeps below an assigned value, chosen according to the experimental requirements.

Among the various mathematical models used to describe flattop profiles [16]–[18], we choose the so-called *flattened Gaussian beams* (FGB's) [17]. They proved to be particularly useful in the study of the paraxial propagation problem [19].

The field distribution of an FGB of order N at the waist plane is given by [19]

$$V_0^{(N)}(r) = A_0 \exp\left[-\frac{(N+1)r^2}{v_0^2}\right] \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{\sqrt{N+1}}{v_0}\,r\right)^{2n} \tag{43}$$

where A_0 is an amplitude factor, v_0 is a real quantity expressing the width of the beam, and N is an integer related to the flatness of the profile. For $N = 0$, it gives a Gaussian function, becomes flatter and flatter with increasing N, and tends to the function $\mathrm{circ}(r/v_0)$ in the limit $N \to \infty$.

Up to now, the study of the properties of a FGB through propagation has been performed by expanding the field by means of $N+1$ LG functions having spot size $v_0/\sqrt{N+1}$ [19]. Now, in order to study how the profile of a FGB changes upon propagation, we set about evaluating its natural modal expansion. Of course, in this case we expect that an infinite number of terms will contribute to the modal expansion, so that the SIE has to be evaluated numerically.

First of all, recalling the results obtained by Bagini *et al.* for σ_0^2 and M^2 [19, eqs. (35) and (36)], the value of $\overline{w}_0$ (20) turns out to be

$$\overline{w}_0 = v_0\left[\frac{2^N N!}{(N+1)}\right]^{1/2}\left[\frac{R_N}{(2N+1)!}\right]^{1/4} \tag{44}$$

where R_N can be expressed as follows:

$$R_N = \sum_{n=0}^{N}\sum_{m=0}^{N} \frac{n+m+1}{2^{n+m}}\binom{n+m}{n}. \tag{45}$$

By inserting (43) into (16) and using [20, formula 2.19.3.2] to solve the integral on the right-hand side, the following expression is obtained for the coefficients c_n:

$$c_n^{(N)} = A_0\sqrt{2\pi}\,\overline{w}_0\,\alpha_N \sum_{m=0}^{N} \frac{1}{(1+\alpha_N)^{m+1}} \cdot P_n^{(0,\,m-n)}\left(1-\frac{4\alpha_N}{1+\alpha_N}\right), \qquad n = 0, 1, \cdots \tag{46}$$

where $P_n^{(\alpha,\,\beta)}$ are the Jacobi polynomials [6] and the parameter α_N is defined as

$$\alpha_N = \frac{1}{\overline{w}_0^2}\,\frac{v_0^2}{(N+1)}. \tag{47}$$

On inserting the values of the coefficients $c_n^{(N)}$ into (24), it is possible to evaluate the SIE as a function of z. Results obtained for several values of N are shown in Fig. 2. We remark that, at any transverse plane $z = \mathrm{const}$, the SIE increases with increasing the order N. This is a consequence of the fact that, on increasing N, the profile more and more resembles a circ

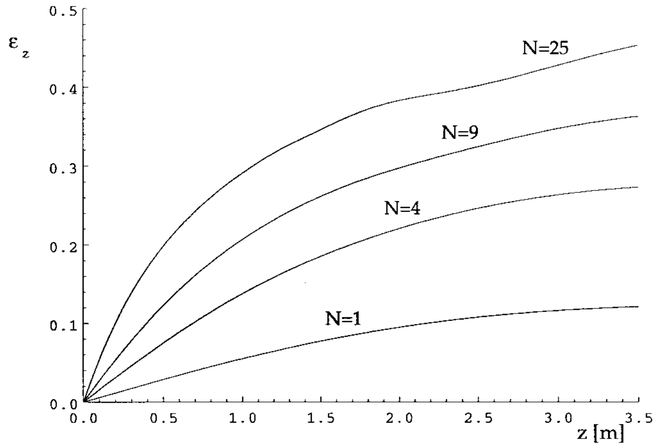


Fig. 2. SIE versus z for FGB's with $v_0 = 1$ mm, $\lambda = 0.5$ $\mu\mu$m, and different values of N.

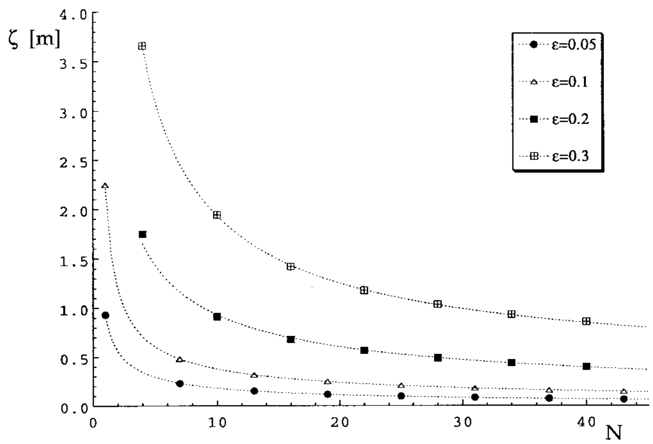
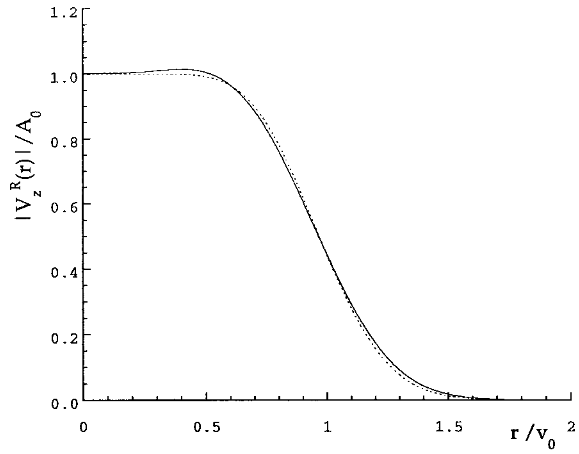
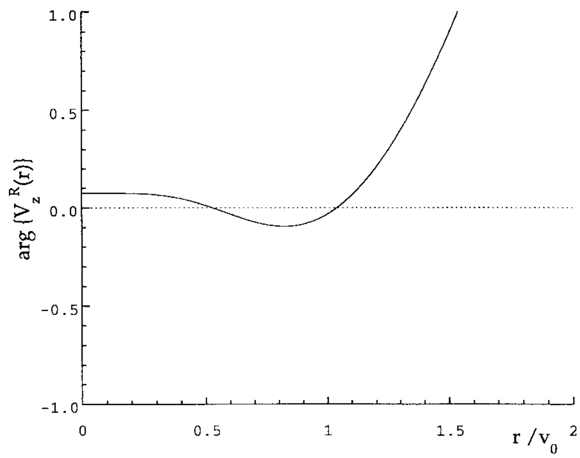


Fig. 3. SIR versus N for FGB's with $v_0 = 1$ mm and $\lambda = 0.5$ μm, for different values of ε.



(a)



(b)

Fig. 4. Behavior of (a) modulus and (b) phase (in radians) of the reference field V_z^R (solid curve) together with those of V_0 (dotted curve), versus r/v_o, for FGB's with $v_0 = 1$ mm, $\lambda = 0.5$ μm, $N = 4$, $z = 340$ mm, corresponding to a value of the SIE of 5%.

function so that, even for small values of z, significant changes of the shape are observed.

In Fig. 3, the SIR is drawn as a function of N for four different values of the error, say ε. For practical purposes, this figure may be used to estimate, for a given order of the beam, the maximum distance, for which the error remains lower than ε.

As an example, in Fig. 4, moduli [Fig. 4(a)] and phases [Fig. 4(b)] of V_z^R (solid curve) and V_0 (dashed curve) are plotted together for $N = 4$. The value of the distance z is chosen in such a way as to give rise to $\varepsilon = 5\%$. In Fig. 5(a) and (b), the same quantities are plotted for $N = 25$, keeping fixed the value of ε (5%). As expected, the higher the order of the beam, the smaller the value of z. It can be noted that, for the chosen value of ε, not only the amplitudes, but also the phase profiles of the reference field are very similar to those of the starting field, at least for those values of r for which the field is significantly different from zero.

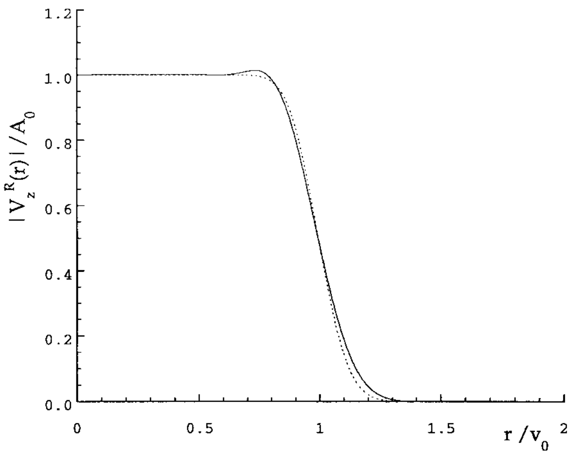
A separate comment is deserved for the case of the circ function (or whenever discontinuities are present in the starting field). It is known [21], indeed, that in such case M^2 diverges and, as a consequence, (20) furnishes a vanishing value for the natural spot size. This is, of course, a pathological case, to which formulas given in the previous sections cannot be applied. It should be considered a limiting case of the class of the FGB's for $N \to \infty$. For example, Fig. 3 gives a clear hint of the fact that, for any fixed value of ε, the SIR goes to zero as $N \to \infty$, and this agrees with the features of the field diffracted by a circular aperture under Fresnel approximation.

VI. CONCLUSIONS

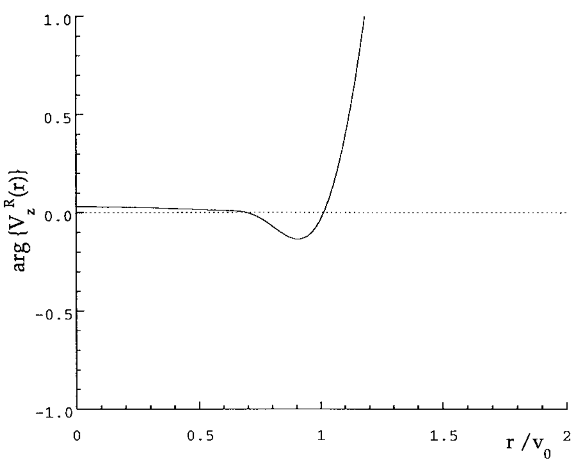
For a certain interval along the propagation axis, the field cross section of a typical light beam approximately maintains the same shape, except for a transverse scale factor and a change of curvature of the wavefront. To quantify such behavior, we used beam parameters called *shape-invariance error* (SIE) and *shape-invariance range* (SIR). Such parameters can be evaluated for a general axially symmetric beam, starting from the modal expansion of the field distribution at the waist in terms of LG functions with a suitable spot size.

In order to verify the correctness of the theory and to highlight the possibilities of this approach, we presented examples pertinent to some classes of beams of practical interest.

In view of the increasing interest of partially coherent light beams, an extension of the above analysis in this sense could be considered.



(a)



(b)

Fig. 5. The same as in Fig. 4 but with $N = 25$. In this case the value of the SIE of 5% is reached at $z = 97$.

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