

# Orbital angular momentum of light: a simple view

**F Gori<sup>†</sup>, M Santarsiero<sup>†</sup>, R Borghi<sup>‡</sup> and G Guattari<sup>‡</sup>**

<sup>†</sup> Dipartimento di Fisica, Università di Roma Tre and Istituto Nazionale Fisica della Materia, Via della Vasca Navale, 84 I-00146, Rome, Italy

<sup>‡</sup> Dipartimento di Ingegneria Elettronica, Università di Roma Tre and Istituto Nazionale Fisica della Materia, Via della Vasca Navale, 84 I-00146, Rome, Italy

Received 18 May 1998

**Abstract.** We present a simple model for the orbital angular momentum of a light beam. Using heuristic arguments, we evaluate the expectation value of such a quantity for a photon. The resulting expression coincides with that derived from Maxwell's equations. Examples are given to illustrate the main points.

## 1. Introduction

The angular momentum of photons is a subject whose exposition at an elementary level is not trivial. Since Maxwell's equations can be considered as the quantum theory of the electromagnetic field for the single photon (Akhiezer and Berestetskii 1965, Heitler 1984, Marcuse 1980), one can start from the classical definition of the angular momentum of a radiation field. It turns out that this quantity can be divided into two parts, which can be interpreted as *orbital* and *spin* components. Although such a division has to be managed with some caution (Akhiezer and Berestetskii 1965, Heitler 1984, Marcuse 1980) its usefulness is beyond dispute. Unfortunately, the mathematical derivation of this result is rather long and intricate. Consequently, one would like to introduce the main results without embarking upon the mathematical details. As for the photon spin, things go smoothly. Since its eigenstates are states of circular polarization of the light field, one can induce the existence of photon spin by considering the rotary motion induced in electrons of materials illuminated by the wave field (see the beautiful discussion by Hecht (1987)). Furthermore, the analogy with the spin of material particles can be exploited. Finally, convincing experiments can be quoted, starting from Beth's celebrated experiment (Beth 1936) and proceeding to present-day experimental observations (Friese *et al* 1998, Simpson *et al* 1997). On the other hand, the orbital part seems to lack an intuitive meaning and the customary field expansion into spherical harmonics does not add much insight. As a result, one most often omits the concept of orbital angular momentum of light altogether.

In recent times, several researches have been carried out on the orbital angular momentum of light beams (Allen *et al* 1992, Van Enk and Nienhuis 1992, 1994a, Courtial *et al* 1997) in which the paraxial approximation has been used as a good approximation. These investigations have led to many new results and have greatly clarified the subject. Furthermore, significant applications of these results have been demonstrated (Simpson *et al* 1997, Friese *et al* 1998).

Therefore, it seems of interest to propose a simple model for the orbital angular momentum of photons which can be presented with minimal prerequisites. Essentially, the required concepts are that a photon possesses a linear momentum and that the optical intensity can be thought of as proportional to the spatial probability density of the photon. Both ideas are generally familiar to the student from discussions about the photoelectric and Compton effects, as well as from the interpretation of classical intensity distributions in diffraction and interference phenomena in terms of single-photon behaviour (Hecht 1996). An elementary knowledge of the scalar complex representation of optical fields is the only other requirement. As we shall see in section 2, the correct formula for the orbital angular momentum of a photon can be derived within a few steps using a simple, almost pictorial representation of the photon behaviour. Further insight into the meaning of the angular momentum can be gained through the worked examples in section 3.

## 2. A model for the orbital angular momentum of a photon

Let us consider a certain plane, to be taken as  $z = 0$ , illuminated by a monochromatic light beam. As mentioned in the introduction, the photon spin is connected with states of circular polarization. As a consequence, if we assume that the beam is linearly polarized, the expectation value of the spin angular momentum is zero and we can focus our attention on the orbital part only. In the scalar, complex representation of the light beam (Mandel and Wolf 1995), we can describe the field distribution across  $z = 0$  through a function  $V(x, y)$ . There is no need to specify the exact meaning of  $V$ ; it could, for example, represent the complex electric field of the wave. The important point is its probabilistic meaning. Following the idea first put forward by Einstein, we assume that for a single photon, the squared modulus of  $V$  is proportional to the probability density that the photon crosses the plane  $z = 0$  at point  $(x, y)$ . The other idea we need is that a monochromatic photon of frequency  $\nu$  possesses a linear momentum with modulus  $h\nu/c$ , where  $h$  is Planck's constant and  $c$  is the speed of light. We must specify, however, the direction of such a vector. With this aim, let us assume that in the neighbourhood of  $(x, y)$  the wavefront of the beam can be approximated by its tangent plane (see figure 1). In other words, we locally replace the wavefront by a plane wave. Then the linear momentum of a photon passing at  $(x, y)$  can be thought of as directed along the wave vector, say  $\mathbf{k}$ , of such a plane wave. Since  $k = 2\pi\nu/c$ , the linear momentum has a modulus  $\hbar k$ , where  $\hbar = h/(2\pi)$ . Now, let  $k_x$  and  $k_y$  be the transverse components of  $\mathbf{k}$ . Then the  $z$ -component of the angular momentum of the photon is

$$m_z = \hbar(xk_y - yk_x). \quad (1)$$

Of course, we do not know where the photon hits the plane  $z = 0$ . Therefore, we must content ourselves with the expectation value of  $m_z$ , which is to be computed through the probability density for the crossing point. In view of the above remark about the meaning of  $|V(x, y)|^2$ , such a probability density, say  $p(x, y)$ , can be written

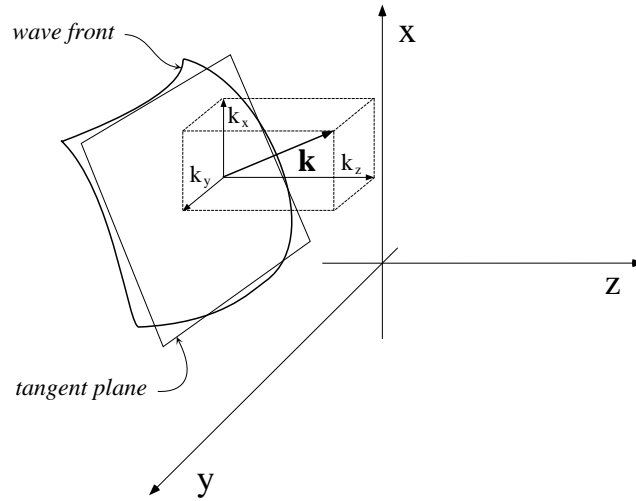
$$p(x, y) = \frac{|V(x, y)|^2}{\iint |V(x, y)|^2 dx dy} \quad (2)$$

and the expectation value for the angular momentum along the  $z$ -axis is

$$\langle m_z \rangle = \hbar \iint (xk_y - yk_x) p(x, y) dx dy. \quad (3)$$

It remains to be seen how  $k_x$  and  $k_y$  can be derived from the knowledge of  $V(x, y)$ . To this end, let us write  $V(x, y)$  in the form

$$V(x, y) = |V(x, y)| \exp[i\phi(x, y)]. \quad (4)$$



**Figure 1.** Local approximation of a wavefront by its tangent plane.

If the wavefront is sufficiently regular, as in the case of paraxial beams, we can expand the phase in a neighbourhood of  $(x, y)$  and consider only the first-order terms, i.e.

$$\phi(x + \xi, y + \eta) = \phi(x, y) + \frac{\partial \phi}{\partial x} \xi + \frac{\partial \phi}{\partial y} \eta \quad (5)$$

where  $\xi$  and  $\eta$  represent small deviations along  $x$  and  $y$ , respectively. When this expression of the phase is compared to the phase distribution, say  $\psi(x, y)$ , produced by a plane wave across the plane  $z = 0$ , namely

$$\psi(x + \xi, y + \eta) = \alpha_0 + k_x(x + \xi) + k_y(y + \eta) \quad (6)$$

where  $\alpha_0$  is an initial phase term, we see that the following equations hold

$$k_x = \frac{\partial \phi}{\partial x} \quad k_y = \frac{\partial \phi}{\partial y}. \quad (7)$$

On inserting equations (2) and (7) into equation (3) we obtain

$$\langle m_z \rangle = \hbar \frac{\iint (x \partial \phi / \partial y - y \partial \phi / \partial x) |V(x, y)|^2 dx dy}{\iint |V(x, y)|^2 dx dy}. \quad (8)$$

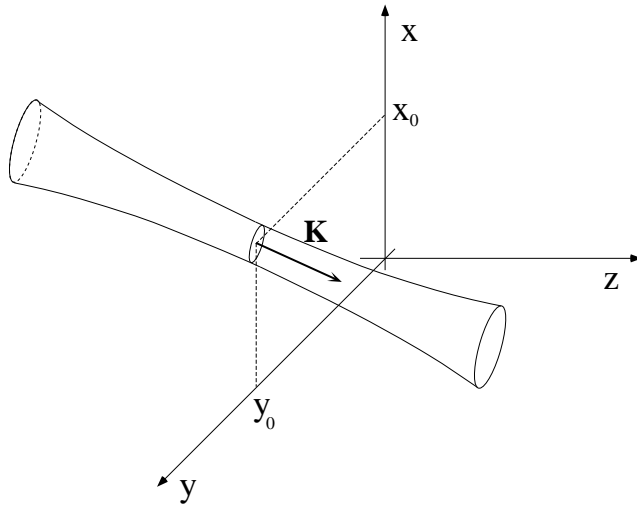
This is the expected value for the  $z$ -component of the orbital angular momentum of a photon. It coincides with the expression derived from Maxwell's equations (Van Enk and Nienhuis 1992, Courial *et al* 1997). It is remarkable that the above heuristic argument can lead in such a simple way to the right result.

A few comments can be of help. We evaluated the expected value of the angular momentum along the  $z$ -axis. A different value can be expected if we refer the angular momentum to an axis parallel to the  $z$ -axis but passing through a typical point  $(x_a, y_a)$  at  $z = 0$ . In this case, equation (1) has to be replaced by

$$m'_z = \hbar [(x - x_a) k_y - (y - y_a) k_x] \quad (9)$$

and equation (3) becomes

$$\langle m'_z \rangle = \langle m_z \rangle - \hbar (x_a \langle k_y \rangle - y_a \langle k_x \rangle) \quad (10)$$



**Figure 2.** A tilted beam impinging on the plane  $z = 0$ .

where

$$\langle k_\alpha \rangle = \iint k_\alpha p(x, y) dx dy \quad (\alpha = x, y) \quad (11)$$

are the expected values of the  $x$ - and  $y$ -components of the wave vector. It may well happen that both  $\langle k_x \rangle$  and  $\langle k_y \rangle$  vanish (we shall see an example later). In this case the orbital angular momentum becomes an intrinsic feature in that it is independent of the coordinates of the point at which the chosen axis crosses the plane  $z = 0$ †.

As a further remark, let us note that for certain types of field distributions polar rather than cartesian coordinates are used. Letting

$$x = r \cos \vartheta \quad y = r \sin \vartheta \quad (12)$$

we can write

$$\frac{\partial}{\partial \vartheta} = \frac{\partial x}{\partial \vartheta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \vartheta} \frac{\partial}{\partial y} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \quad (13)$$

so that (8) becomes

$$\langle m_z \rangle = \hbar \frac{\int_0^{2\pi} \int_0^\infty (\partial \phi / \partial \vartheta) |U(r, \vartheta)|^2 r dr d\vartheta}{\int_0^{2\pi} \int_0^\infty |U(r, \vartheta)|^2 r dr d\vartheta} \quad (14)$$

where  $U$  specifies the field distribution at  $z = 0$  in polar coordinates.

### 3. Examples

Suppose a TEM<sub>00</sub> Gaussian mode (Siegman 1986) impinges on the plane  $z = 0$  centred at a point  $(x_0, y_0)$ . Further, let us assume the axis of the beam to be inclined with respect to the

† The word ‘intrinsic’ is used throughout the paper to say that the orbital angular momentum does not depend on the chosen axis. Of course, it should not be confused with the spin angular momentum, which is sometimes referred to as intrinsic.

$z$ -axis (see figure 2). Neglecting the ellipticity induced by such inclination (assumed to be small) we write the corresponding field distribution as

$$V(x, y) = A \exp\left(-\frac{(x-x_0)^2 + (y-y_0)^2}{v^2}\right) \exp[i(K_x x + K_y y)] \quad (15)$$

where  $A$  is an amplitude term,  $v$  is the spot size, and  $K_x$  and  $K_y$  are the  $x$ - and  $y$ -components of the mean wave vector of the beam. In other words,  $\mathbf{K}$  is directed along the beam axis. On substituting equation (15) into equation (8) we find through simple steps that

$$\langle m_z \rangle = \hbar (x_0 K_y - y_0 K_x). \quad (16)$$

It may be worthwhile to note that equation (16) gives the same value that would pertain to a photon crossing  $z = 0$  at  $(x_0, y_0)$  (see equation (1)). This can be interpreted by saying that such a point plays the role of a centre of mass for the beam. It is seen that the present angular momentum is not an intrinsic feature of the beam. Indeed it can be made arbitrarily high (at least in principle) or vanishing through a suitable choice of  $(x_0, y_0)$  and  $(K_x, K_y)$ .

As a second example, we shall consider a well known class of Laguerre–Gauss beams (Siegman 1986) specified at their waist by the distributions

$$U_n(r, \vartheta) = A r^n \exp(\pm i n \vartheta) \exp\left(\frac{-r^2}{v^2}\right) \quad (17)$$

where  $A$  is a constant and  $n$  is an integer number. On substituting equation (17) into (14) we obtain at once

$$\langle m_z \rangle = \pm n \hbar. \quad (18)$$

This beautiful result was first derived by Allen *et al* (1992). Writing the phase of the field in equation (17) as  $\phi = \pm n \arctan(y/x)$ , we deduce from equation (7) that

$$\begin{aligned} k_x &= \frac{\partial \phi}{\partial x} = \mp \frac{n y}{x^2 + y^2} = \mp n \frac{\sin \vartheta}{r} \\ k_y &= \frac{\partial \phi}{\partial y} = \pm \frac{n x}{x^2 + y^2} = \pm n \frac{\cos \vartheta}{r}. \end{aligned} \quad (19)$$

Using equation (19), it is easily seen through equation (11) that  $\langle k_x \rangle$  and  $\langle k_y \rangle$  vanish. Therefore, the expected value of the angular momentum (18) has an intrinsic meaning, so that Laguerre–Gauss beams, differently from the previous example, possess a well defined orbital angular momentum. This remark can be applied to every field whose phase distribution depends on  $\vartheta$  only through the phase term  $\exp(\pm i n \vartheta)$ , regardless of the dependence on the radial variable. Fields of this type are said to possess a vortex structure (Berry 1981, Indebetouw 1993), because their wavefronts are helicoidally shaped, and have been the subject of much research in recent times.

#### 4. Conclusions

We have shown that an elementary model for the photon behaviour can lead to the correct formula of the angular momentum of paraxial light beams. Obviously, more general radiation fields may show features that would not be revealed by our simplified treatment (Barnett and Allen 1994, Van Enk and Nienhuis 1994b). However, we think that the present model affords a simple introductory view of the phenomena.

## References

- Allen L, Beijersbergen M W, Spreeuw J C and Woerdman J P 1992 *Phys. Rev. A* **45** 8185
- Akhiezer A I and Berestetskii V B 1965 *Quantum Electrodynamics* (New York: Wiley-Interscience)
- Barnett S M and Allen L 1994 *Opt. Commun.* **110** 670
- Berry M 1981 *Physics of Defects* ed R Balian et al (Amsterdam: North-Holland)
- Beth R A 1936 *Phys. Rev.* **50** 115
- Courtial J, Dholakia K, Allen L and Padgett M J 1997 *Opt. Commun.* **144** 210
- Friese M E J, Nienimen T A, Heckenberg N R and Rubinsztein-Dunlop H 1998 *Opt. Lett.* **23** 1
- Hecht E 1987 *Optics* (Reading MA: Addison-Wesley)
- 1996 *Physics: Calculus* (Pacific Grove, CA: Brooks/Cole)
- Heitler W 1984 *The Quantum Theory of Radiation* (New York: Dover)
- Indebetouw G 1993 *J. Mod. Opt.* **40** 73
- Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
- Marcuse D 1980 *Principles of Quantum Electronics* (New York: Academic)
- Siegman A E 1986 *Lasers* (Mill Valley, CA: University Science Books)
- Simpson N B, Dholakia K, Allen L and Padgett M J 1997 *Opt. Lett.* **22** 52
- Van Enk S J and Nienhuis G 1992 *Opt. Commun.* **94** 147
- 1994a *J. Mod. Opt.* **41** 963
- 1994b *Opt. Commun.* **112** 225