M^2 factor of Bessel–Gauss beams

R. Borghi and M. Santarsiero

Dipartimento di Fisica, Università "La Sapienza," Piazzale A. Moro, 2, 00185 Roma, Italy

Received November 11, 1996

We derive the analytical expression of the M^2 factor of a Bessel–Gauss beam of any order and discuss its behavior for high and low values of the width of the Gaussian profile. The case of unapertured Bessel beams can be treated as a limiting case of our analysis.

Many significant parameters can be introduced to characterize a light beam.[1] One of them, the M^2 factor, has gained such wide popularity that it is currently used in data sheets for commercial lasers. This factor, for three-dimensional beams, is defined as[2]

$$M^2 = 2\pi\sigma_0\sigma_\infty\,, \tag{1}$$

where σ_0 and σ_∞ are the second-order moments associated with the intensity distributions at the waist plane and in the far field, respectively. For a fixed width at the waist a better-quality beam is associated with the smallest angular divergence, so M^2 is to be thought of as an inverse quality factor. A minimum value of 1 is reached for the lowest-order Gaussian beam.

The M^2 factor also plays an important role in the framework of the paraxial approximation. It has indeed been proved that the widths of propagated beams, both coherent and partially coherent, obey the same law as a Gaussian beam; i.e., its squared value is a parabolic function of the distance from the waist plane.[2,3] Therefore, it is possible to associate any beam with an equivalent Rayleigh distance along which the width of the transverse section of the beam is approximately invariant. Such a distance turns out to be inversely proportional to M^2.[4,5] It must be stressed that here we refer to a particular definition of width, whereas different definitions are usable.[6,7]

An interesting case is represented by Bessel beams.[8] Because of the nondiffracting behavior of such beams, the range over which the beams' cross section remains unaltered is virtually infinite. Unfortunately, their M^2 factor cannot be computed because such beams would convey an infinite amount of power, and therefore their variance also diverges. By the way, this is why a true beam of this kind cannot be realized. In any practical implementation a windowing profile has to be superimposed upon the Bessel pattern. One wonders whether the presence of a profile can remove divergence problems in the evaluation of M^2 while permitting the ideal unapertured case to be recovered through a suitable limiting procedure. The deceptively simple case of a circular aperture does not help. Not only is the propagation process intractable in simple analytical terms, but also a further cause of divergence is introduced by the discontinuity and at the profile boundary.[9] Among continuous profiles a good candidate for use in M^2 evolution is the Gaussian one, which leads to the Bessel–Gauss beams introduced by Gori *et al.*[10] Bessel–Gauss beams have proved to be useful in several problems,[11,12] often giving rise to closed-form results.

In this Letter we consider the evaluation of M^2 for a Bessel–Gauss beam of any order. As far as we know, the expression of M^2 for such beams is not available in the literature. Taking into account the current interest in Bessel–Gauss beams, this may be surprising but probably is due to the fact that the integrals to be performed involve hypergeometric functions. As we shall see, this difficulty can be circumvented through a procedure that eventually leads to a rather simple closed formula for M^2. Such a formula shows that, for any order of the beam, M^2 is an increasing function of the width of the Gaussian profile. Accordingly, it can be concluded that in the limiting case of unapertured Bessel beams M^2 is to be taken as infinite. On the other hand, we shall see that M^2 tends toward a finite value when the width of the Gaussian profile becomes smaller. The latter result can be easily interpreted in terms of known properties of Laguerre–Gauss beams.

Let us suppose that, at the plane $z = 0$ of a suitable reference frame, the field distribution produced by a Bessel–Gauss beam of order n (BGB_n) is present:

$$U_n(r,\vartheta) = AJ_n(\beta r)\exp(-r^2/{w_0}^2)\exp(in\vartheta)\,, \tag{2}$$

where A is a possibly complex amplitude factor, which will be set to unity without loss of generality, $J_n(\cdot)$ is the nth-order Bessel function of the first kind,[13] and β and w_0 are two positive parameters.

When we use formula 6.633.2 of Ref. 13 and the integral representation of the Bessel function $J_n(x)$,[13] the two-dimensional Fourier transform[14] of $U_n(r,\vartheta)$, namely $\tilde{U}_n(\nu,\varphi)$, turns out to be

$$\tilde{U}_n(\nu,\varphi) = \pi {w_0}^2(-i)^n \exp\left(-\frac{\beta^2 {w_0}^2}{4}\right)\exp(in\varphi) \times \exp(-\pi^2 {w_0}^2\nu^2) I_n(\pi\beta {w_0}^2\nu)\,, \tag{3}$$

with $I_n(\cdot)$ being the nth-order modified Bessel function of the first kind.[13] From Eqs. (2) and (3) it turns out that the intensity distributions $|U_n|^2$ and $|\tilde{U}_n|^2$ are radial functions, so the second-order moments σ_0 and σ_∞ are given by

$${\sigma_0}^2 = \frac{\int_0^\infty |U_n|^2 r^3 \mathrm{d}r}{\int_0^\infty |U_n|^2 r \mathrm{d}r}\,, \tag{4}$$

$${\sigma_\infty}^2 = \frac{\int_0^\infty |\tilde{U}_n|^2 \nu^3 \mathrm{d}\nu}{\int_0^\infty |\tilde{U}_n|^2 \nu \mathrm{d}\nu}\,, \tag{5}$$

0146-9592/97/050262-03$10.00/0

respectively. The evaluation of such quantities involves the use of hypergeometric functions (see formula 6.633.5 of Ref. 13) and appears to be cumbersome. However, it is possible to overcome this difficulty by use of an alternative technique. To this end, let us introduce the functions $F_1(\alpha)$ and $F_2(\alpha)$, defined as

$$F_1(\alpha) = \int_0^\infty J_n^2(\beta r)\exp(-\alpha r^2) r\mathrm{d}r\,, \tag{6}$$

$$F_2(\alpha) = \int_0^\infty I_n^2(\delta \nu)\exp(-\alpha \nu^2) \nu\mathrm{d}\nu\,, \tag{7}$$

with $\delta = \pi\beta w_0^2$, and note that σ_0 and σ_∞ can be written as

$$\sigma_0^2 = -\left[\frac{F_1'(\alpha)}{F_1(\alpha)}\right]_{\alpha=2/w_0^2}, \tag{8}$$

$$\sigma_\infty^2 = -\left[\frac{F_2'(\alpha)}{F_2(\alpha)}\right]_{\alpha=2\pi^2 w_0^2}, \tag{9}$$

where the prime denotes derivation with respect the variable α. The explicit expression of $F_1(\alpha)$ turns out to be

$$F_1(\alpha) = \frac{1}{2\alpha} I_n\left(\frac{\beta^2}{2\alpha}\right)\exp\left(-\frac{\beta^2}{2\alpha}\right), \tag{10}$$

where formula 6.633.2 of Ref. 13 and the equation[13]

$$I_n(x) = i^n J_n(-ix)\,, \tag{11}$$

have been used. Performing the first derivative of function $F_1(\alpha)$ gives us

$$\begin{aligned} F_1'(\alpha) &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{2\alpha}\exp\left(-\frac{\beta^2}{2\alpha}\right) I_n\left(\frac{\beta^2}{2\alpha}\right)\right] \\ &= \frac{1}{\beta^2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\beta^2}{2\alpha}\right)\left\{\frac{\mathrm{d}}{\mathrm{d}x}[x\exp(-x) I_n(x)]\right\}_{x=\beta^2/2\alpha}, \end{aligned} \tag{12}$$

and, when we use[13]

$$x I_n'(x) = x I_{n+1}(x) + n I_n(x)\,, \tag{13}$$

the term in braces in Eq. (12) turns out to be

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[x\exp(-x) I_n(x)] &= I_n(x)\exp(-x) \\ &\quad \times \left\{1 + n + x\left[\frac{I_{n+1}(x)}{I_n(x)} - 1\right]\right\}. \end{aligned} \tag{14}$$

Substituting Eq. (14) into Eq. (12), we obtain, after some algebra,

$$\begin{aligned} F_1'(\alpha) = &-\frac{1}{2\alpha^2}\exp\left(-\frac{\beta^2}{2\alpha}\right) I_n\left(\frac{\beta^2}{2\alpha}\right) \\ &\times \left\{1 + n + \frac{\beta^2}{2\alpha}\left[\frac{I_{n+1}\left(\frac{\beta^2}{2\alpha}\right)}{I_n\left(\frac{\beta^2}{2\alpha}\right)} - 1\right]\right\}, \end{aligned} \tag{15}$$

and then, from Eqs. (8), (10), and (15), we derive

$$\sigma_0^2 = \frac{w_0^2}{2}\left\{1 + n + \gamma\left[\frac{I_{n+1}(\gamma)}{I_n(\gamma)} - 1\right]\right\}, \tag{16}$$

where the parameter $\gamma = \beta^2 w_0^2/4$ has been introduced.

The evaluation of σ_∞ can be performed in an analogous way, starting from Eqs. (7) and (9), yielding

$$\sigma_\infty^2 = \frac{1}{2\pi^2 w_0^2}\left\{1 + n + \gamma\left[\frac{I_{n+1}(\gamma)}{I_n(\gamma)} + 1\right]\right\}. \tag{17}$$

Thus, from Eq. (1), the M^2 factor of BGB_n turns out to be

$$M^2 = \{[1 + n + \gamma R_n(\gamma)]^2 - \gamma^2\}^{1/2}, \tag{18}$$

with

$$R_n(\gamma) = I_{n+1}(\gamma)/I_n(\gamma)\,. \tag{19}$$

Equation (18) is the main result of this study. Curves of M^2 as a function of βw_0 are shown in Fig. 1 for several values of order n. It can be seen that the quality factor tends to the value $(n + 1)$ when βw_0 approaches zero. This is due to the fact that, when $1/\beta$ is much larger than w_0, i.e., when the Gaussian profile is much narrower than the central lobe of the Bessel function, the latter can be approximated by means of a Taylor expansion. In this case we have[13]

$$U_n(r, \vartheta) \approx \frac{A\beta^n}{2^n n!} r^n \exp(-r^2/w_0^2)\exp(in\vartheta)\,, \tag{20}$$

which is proportional to the field at the waist plane of a Laguerre–Gauss beam of indices 0 and n, whose M^2 factor is exactly $(n + 1)$.[2]

On the other hand, we can derive the behavior of M^2 for large values of βw_0 by considering the asymptotic expression of the modified Bessel functions. In this case, indeed, since[13]

$$I_n(x) \to \frac{\exp(x)}{\sqrt{2\pi x}}\left(1 - \frac{4n^2 - 1}{8x}\right), \tag{21}$$

from Eqs. (16) and (17) and expression (21), it can be shown that, if $\beta w_0 \to \infty$, then

$$\sigma_0 \to w_0/2, \qquad \sigma_\infty \to \beta/2\pi\,, \tag{22}$$

regardless of the value of n. As a consequence, from Eq. (1), the value of M^2 tends to $\beta w_0/2$. In such a

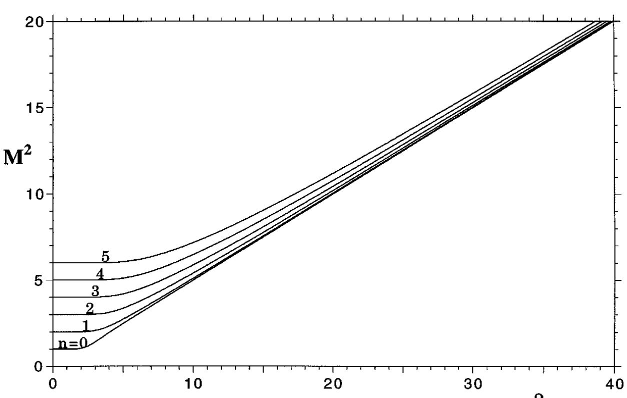

Fig. 1. M^2 factor of a Bessel–Gauss beam of order n.

limit the equivalent Rayleigh distance[5] turns out to be proportional to the ratio w_0/β, in agreement with the estimation based on the geometric model for Bessel and Bessel–Gauss beams.[8,10]

In conclusion, we believe that our analytical result regarding the quality factor of Bessel–Gauss beams could be useful in designing and characterizing optical systems for diffraction-free beams, and, in particular, in studying the shape-invariance properties of such beams[15] or the divergence properties of related partially coherent beams.[16,17]

We thank Franco Gori for many stimulating discussions in preparing this Letter. This research is sponsored by Istituto Nazionale di Fisica della Materia and Ministero dell'Università e della Ricerca Scientifica.

References

1. P. M. Mejías, H. Weber, R. Martínez-Herrero, and A. González-Ureña, eds., *Proceedings of Workshop on Laser Beam Characterization* (Sociedad Espanda de Optica, Madrid, Spain, 1993).
2. A. Siegman, Proc. SPIE **1224,** 2 (1990).
3. R. Simon, N. Mukunda, and E. C. G. Sudarshan, Opt. Commun. **65,** 322 (1988).
4. P. A. Bélanger, Opt. Lett. **16,** 196 (1991).
5. A. Siegman, IEEE J. Quantum Electron. **27,** 1146 (1991).
6. M. W. Sasnett, in *The Physics and Technology of Laser Resonators*, D. R. Hall and P. E. Jackson, eds. (Hilger, London, 1989), p. 132.
7. M. A. Porras and R. Medina, Appl. Opt. **34,** 8247 (1995).
8. J. Durnin, J. J. Miceli, and J. H. Eberly, Phys. Rev. Lett. **58,** 1499 (1987).
9. C. Paré and P. A. Bélanger, Opt. Commun. **123,** 679 (1991).
10. F. Gori, G. Guattari, and C. Padovani, Opt. Commun. **64,** 491 (1987).
11. J. Lu, X. Xu, H. Zou, and J. F. Greenleaf, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **42,** 649 (1995).
12. P. L. Greene and D. G. Hall, J. Opt. Soc. Am. A **13,** 962 (1996).
13. I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
14. R. N. Bracewell, *The Fourier Transform and Its Applications*, Rev. 2nd ed. (McGraw-Hill, New York, 1986).
15. F. Gori, S. Vicalvi, M. Santarsiero, and R. Borghi, Opt. Lett. **21,** 1205 (1996).
16. J. Turunen, A. Vasara, and A. T. Friberg, J. Opt. Soc. Am. A **8,** 282 (1991).
17. F. Gori, G. Guattari, and C. Padovani, Opt. Commun. **64,** 311 (1987).