The Simon–Mukunda polarization gadget

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Abstract. The universal gadget recently proposed by Simon and Mukunda to synthesize any non-absorbing optical element acting on the polarization of a wave is explained by elementary means. This different approach also leads to alternative synthesis procedures.

Résumé. Le gadget universel, proposé récemment par Simon et Mukunda pour synthétiser chaque élément optique pas-absorbent qui agit sur la polarisation d'une onde, est expliqué par des moyens élémentaires. Cette différente approche conduit aussi à des procédés alternatifs de synthèse.

1. Introduction

Simon and Mukunda discussed some years ago the following problem: can an arbitrary non-absorbing optical element, acting on the polarization of a wave, be synthesized using only quarter wave and half wave plates (QWPs and HWPs, respectively)? They were able to prove (Simon and Mukunda 1989) that such a synthesis is possible if two QWPs and two HWPs are used. Later, they refined their result showing (Simon and Mukunda 1990) that the synthesis process could be made by using two QWPs and only one HWP. Simon and Mukunda reached such conclusions using the powerful and elegant tools of group theory. For readers that are not familiar with this theory the full appreciation of Simon and Mukunda's works can be difficult. In this paper we intend to show how the Simon–Mukunda gadget can be explained in an elementary way. We shall see that the most important feature of the gadget can be physically understood taking into account the effect of the rotation of a HWP on circularly polarized radiation. This not only clarifies the gadget physical basis, but also allows us to connect the phenomena under consideration to other interesting effects, in which rotating anisotropic elements produce frequency shifts (Crane 1969, Sommargren 1975, Hu 1983, Bagini *et al* 1994, Pippard 1994). The mathematical tools of our analysis are Jones matrices (Jones 1941), whose basic properties are briefly recalled in the following section. As an additional bonus, our results suggest the implementation of alternative gadgets using QWPs and rotators.

2. Preliminaries

We recall here a few basic concepts about Jones vectors and matrices. A deeper discussion can be found in Swindell (1975) and Hecht (1987), while a generalization to partially coherent light can be seen in Mandel and Wolf (1995). We shall refer to coherent monochromatic radiation, omitting a temporal factor $\exp(-i\omega t)$, where ω is the angular frequency. The wavefront is supposed to be plane, at least approximately. The general polarization state can be specified by means of a column vector, say $\boldsymbol{v}$, of the form

$$\boldsymbol{v} = \begin{pmatrix} a_x \exp(i\delta_x) \\ a_y \exp(i\delta_y) \end{pmatrix}, \tag{1}$$

where the two column elements are projections on the x- and y-axes of a given reference frame of a vector representing the electromagnetic wave, e.g. the electric field. The positive quantities a_x, a_y are the amplitudes, while the real quantities δ_x, δ_y are initial phases at the considered point. As is known, the polarization state is not affected by the multiplication of a_x, a_y by a common factor. Therefore a normalization can be made by setting equal to one, in suitable units, the quantity $a_x^2+a_y^2$, which is proportional to the radiation intensity. Furthermore, polarization does not change if the same contribution to

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the phase is added to δ_x and δ_y. Accordingly, we could always set δ_x equal to zero and make the corresponding adjustment to the phase constant δ_y.

Linear polarization states are characterized by a phase difference $\delta_x - \delta_y$ equal to 0 or π. The case $\delta_x - \delta_y = \pi$ deserves a comment. This is that, if $\delta_x = 0$ (arbitrarily), then $\delta_y = -\pi$. The x- and y-components of the wave thus become, respectively, $a_x \exp[\mathrm{i}(kz - \omega t)]$ and $a_y \exp[\mathrm{i}(kz - \omega t - \pi)]$. Hecht's comment (Hecht 1987) is that $-\pi$ has been added to the phase of the x-component to get the phase of the y-component leading the x-component by π.

If θ is the angle between the direction of polarization and the x-axis, the corresponding Jones vector, indicated by $\boldsymbol{l}_\theta$, is

$$\boldsymbol{l}_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{2}$$

In particular, we shall be interested in the cases $\theta = \pi/4$ and $\theta = -\pi/4$, in which equation (2) becomes

$$\boldsymbol{l}_{\pi/4} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{l}_{-\pi/4} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3}$$

Circular polarization states, which are characterized by x- and y-components with the same amplitude and phase difference $\delta_x - \delta_y = \pm\pi/2$, will be indicated by $\boldsymbol{c}_\mathrm{r}$ and $\boldsymbol{c}_\mathrm{l}$, depending on whether polarization is right or left circular. The corresponding Jones vectors are

$$\boldsymbol{c}_\mathrm{r} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \qquad \boldsymbol{c}_\mathrm{l} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}. \tag{4}$$

We look now at the variation with time of the successive positions in the xy-plane from the side where z is negative (Yariv and Yeh 1984). The convention to assess the sense of rotation of the electric field vector is not unique (Born and Wolf 1991). We could look at the xy-plane from the side where z is positive. The terms righthanded and lefthanded could be interchanged.

The effect of an optical component acting on the polarization can be described by means of a suitable 2×2 Jones matrix, say $\mathbf{A}$. The vector $\boldsymbol{v}'$, expressing the polarization after passing through the component, is $\boldsymbol{v}' = \mathbf{A}\boldsymbol{v}$, i.e. in an explicit way,

$$\begin{pmatrix} a'_x \exp(\mathrm{i}\delta'_x) \\ a'_y \exp(\mathrm{i}\delta'_y) \end{pmatrix} = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix} \begin{pmatrix} a_x \exp(\mathrm{i}\delta_x) \\ a_y \exp(\mathrm{i}\delta_y) \end{pmatrix}, \tag{5}$$

where ζ_j ($j = 1, \ldots, 4$) are the complex elements of the $\mathbf{A}$ matrix.

We are interested in optical components that do not change the wave intensity. In this case it is easily seen that the following condition,

$$|\det \mathbf{A}| = 1, \tag{6}$$

has to be satisfied, where det stands for determinant. It is then said that the $\mathbf{A}$ matrix characterizes a unitary transformation. In particular, if $\det\mathbf{A}$ is equal to one, the transformations are said to be special unitary. It is not difficult to show that the corresponding matrices take on the form

$$\mathbf{A} = \begin{pmatrix} \zeta_1 & \zeta_2 \\ -\zeta_2^* & \zeta_1^* \end{pmatrix}, \qquad |\zeta_1|^2 + |\zeta_2|^2 = 1, \tag{7}$$

where the star denotes complex conjugation. It may be worthwhile noting that the set of matrices representing unitary transformations forms a group, usually indicated by $U(2)$ (Joshi 1982). Special unitary transformations form a group too, usually denoted by $SU(2)$. Because of the constraint appearing in equation (7), $SU(2)$ elements are specified by three real parameters. This is why such a group is said to be a three-parameter group.

There are several components that are able to change polarization without altering the intensity. The most important examples are, basically, wave plates and rotators. The corresponding Jones matrices are

$$\mathbf{M}_0(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & \exp(\mathrm{i}\varphi) \end{pmatrix}, \qquad \mathbf{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{8}$$

respectively, where the index 0 means that the wave plate axes coincide with the coordinate ones. The effect of $\mathbf{M}_0$ is to increase the phase of the y-component of an amount φ with respect to the x-component. The effect of $\mathbf{R}$ is to rotate the field vector by an angle α. It can be easily controlled; in fact, if the field vector is rotated by such an angle, its x- and y-components are obtained from the old ones through multiplication by $\mathbf{R}(\alpha)$. Here α is assumed to be positive when it represents a counterclockwise rotation in the xy-plane (seen from positive z). Note that $\mathbf{M}_0$ can be written

$$\mathbf{M}_0(\varphi) = \exp(\mathrm{i}\varphi/2)\begin{pmatrix} \exp(-\mathrm{i}\varphi/2) & 0 \\ 0 & \exp(\mathrm{i}\varphi/2) \end{pmatrix}, \tag{9}$$

i.e. it is a special unitary matrix, apart from a phase factor. $\mathbf{R}(\alpha)$ is also special unitary.

An important point for the following developments is the change in the Jones matrix of an optical component when the component is rotated through an angle α in the xy-plane. Its matrix $\mathbf{A}$ before rotation must be replaced by $\mathbf{A}'$, where

$$\mathbf{A}' = \mathbf{R}(\alpha)\mathbf{A}\mathbf{R}(-\alpha). \tag{10}$$

In particular, as it is physically obvious, the matrix associated with a rotator is unchanged. In order to prove equation (10), one can proceed as follows. First, the x- and y-components of the incoming field are found in a rotated reference frame whose axes are aligned to those of the optical component. This is obtained by multiplying the x- and y-components in the original frame by $\mathbf{R}(-\alpha)$. In fact, rotating the frame through the angle α is formally equivalent to rotating the vector through the angle $-\alpha$ while keeping the frame unchanged. The matrix $\mathbf{A}$ then describes the action of the optical component. Finally, we simply have to go back to the original frame by means of the matrix $\mathbf{R}(\alpha)$. We shall indicate by $\mathbf{M}(\varphi, \alpha)$ the matrix of a plate characterized by a delay φ rotated by an angle α. In particular, $\mathbf{M}(\varphi, 0)$ coincides with $\mathbf{M}_0(\varphi)$. Applying equation (10) to equation (8), by simple calculations it is found that

$$\mathbf{M}(\varphi, \alpha) = \exp(\mathrm{i}\varphi/2) \times \begin{pmatrix} \cos\frac{\varphi}{2} - \mathrm{i}\sin\frac{\varphi}{2}\cos 2\alpha & -\mathrm{i}\sin\frac{\varphi}{2}\sin 2\alpha \\ -\mathrm{i}\sin\frac{\varphi}{2}\sin 2\alpha & \cos\frac{\varphi}{2} + \mathrm{i}\sin\frac{\varphi}{2}\cos 2\alpha \end{pmatrix}. \quad (11)$$

For HWPs and QWPs, φ is π and $\pi/2$, respectively. The corresponding matrices, to be denoted by $H(\alpha)$ and $Q(\alpha)$, are

$$\mathbf{H}(\alpha) = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix},$$
$$\mathbf{Q}(\alpha) = \frac{\exp(\mathrm{i}\pi/4)}{\sqrt{2}} \times \begin{pmatrix} 1 - \mathrm{i}\cos 2\alpha & -\mathrm{i}\sin 2\alpha \\ -\mathrm{i}\sin 2\alpha & 1 + \mathrm{i}\cos 2\alpha \end{pmatrix}. \quad (12)$$

In particular we have, for α=0,

$$\mathbf{H}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{Q}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}. \quad (13)$$

We conclude this section by observing that $\mathbf{H}(\alpha)$ can be written in either of the following forms,

$$\mathbf{H}(\alpha) = \mathbf{R}(2\alpha)\mathbf{H}_0, \quad (14)$$
$$\mathbf{H}(\alpha) = \mathbf{H}_0\mathbf{R}(-2\alpha). \quad (15)$$

3. Some effects of wave plates

In this section we show how, using Jones calculus, some well known effects of wave plates are found, together with the effect which is the physical basis of the gadget we study in this paper. By use of equations (3), (4) and (13), we obtain

$$Q_0\boldsymbol{l}_{\pi/4} = \boldsymbol{c}_{\mathrm{r}}, \qquad Q_0\boldsymbol{l}_{-\pi/4} = \boldsymbol{c}_{\mathrm{l}}. \quad (16)$$

This result means that linear polarization at an angle $\pi/4$ ($-\pi/4$) with the x-axis is transformed into circular right (left) polarization by a QWP whose axes are the coordinate ones. Conversely, we easily find the result

$$\mathbf{Q}_0\boldsymbol{c}_{\mathrm{r}} = \boldsymbol{l}_{\pi/4}, \qquad \mathbf{Q}_0\boldsymbol{c}_{\mathrm{l}} = \boldsymbol{l}_{-\pi/4} \quad (17)$$

i.e. the rule of transformation of circularly into linearly polarized light. It is also well known that a HWP converts right circular light into left circular light and vice versa. This is easily checked on applying $\mathbf{H}_0$ (see equation (13)) to the vectors (4). Maybe it is less known how the conversion takes place if the HWP is rotated. Using equations (4) and (12), we find that

$$\mathbf{H}(\alpha)\boldsymbol{c}_{\mathrm{r}} = \exp(2\mathrm{i}\alpha)\boldsymbol{c}_{\mathrm{l}}, \qquad \mathbf{H}(\alpha)\boldsymbol{c}_{\mathrm{l}} = \exp(-2\mathrm{i}\alpha)\boldsymbol{c}_{\mathrm{r}}, \quad (18)$$

i.e. that the effect of rotation is to introduce a phase factor. It is important to note that the phase change has opposite sign for the two types of circular polarization. As we shall see, this phenomenon is essential to explain the compensator proposed by Simon and Mukunda. It is also the basis of the frequency shifts produced by rotating anisotropic elements (Chyba *et al* 1988, Agarwal and Simon 1990, Aravind 1992, Bagini *et al* 1994).

4. Synthesized compensator and rotator

The compensator, or variable wave plate (Born and Wolf 1991), proposed by Simon and Mukunda (Simon and Mukunda 1990), is made of two QWPs and one HWP. Three different sequences of such elements are possible. The simplest one to explain is, with obvious notations, Q–H–Q. It is clear that, if the axes of all the plates have the same direction, the total phase lag is 2π and no polarization change takes place. Indeed, it is easily checked by means of equation (13) that the product matrix is

$$\mathbf{Q}_0\mathbf{H}_0\mathbf{Q}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (19)$$

i.e. the identity matrix. Let us suppose now that the HWP is rotated by an angle α and study the action of the system on arbitrarily polarized radiation. Before examining the matrix representation of the device, let us try to understand its effect using the results we have just seen. Any incoming field can be represented by means of $\boldsymbol{l}_{\pi/4}$ and $\boldsymbol{l}_{-\pi/4}$ in the form

$$\boldsymbol{v} = b_1\boldsymbol{l}_{\pi/4} + b_2\boldsymbol{l}_{-\pi/4}, \quad (20)$$

where b_1 and b_2 are suitable coefficients. It is then sufficient to study the effect of the system on $\boldsymbol{l}_{\pi/4}$ and on $\boldsymbol{l}_{-\pi/4}$ light separately. When the field $b_1\boldsymbol{l}_{\pi/4}$ passes through the first QWP, the field $b_1\boldsymbol{c}_{\mathrm{r}}$ (see equation (16)) is produced. This gives rise to $b_1\boldsymbol{c}_{\mathrm{l}}\exp(2\mathrm{i}\alpha)$ for the effect of the rotated HWP (equation (18)). Finally, the second QWP produces $b_1\boldsymbol{l}_{\pi/4}\exp(2\mathrm{i}\alpha)$ (equation (17)). Therefore, $\boldsymbol{l}_{\pi/4}$ light impinging on the system comes out with unchanged polarization and amplitude, but with a phase change of 2α. This is an example of the Pancharatnam phase, that is the phase taken by the radiation field as its polarization changes in a cyclic way (Pancharatnam 1956, Chyba *et al* 1988, Agarwal and Simon 1990, Aravind 1992). In an analogous way, $\boldsymbol{l}_{-\pi/4}$ light retains its polarization and amplitude, but undergoes a phase change of -2α. In conclusion, a phase difference 4α is introduced between the components $\boldsymbol{l}_{\pi/4}$ and $\boldsymbol{l}_{-\pi/4}$. Therefore the system is equivalent to a 4α wave plate, as is better seen in a reference frame whose x-axis is along $\boldsymbol{l}_{\pi/4}$. In such a frame, the three plates appear to be rotated by $\pi/4$. The validity of the previous analysis can be verified by using Jones calculus. By means of equation (12), we find

$$\mathbf{Q}\left(\frac{\pi}{4}\right)\mathbf{H}\left(\frac{\pi}{4}+\alpha\right)\mathbf{Q}\left(\frac{\pi}{4}\right) = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \times \begin{pmatrix} -\sin 2\alpha & \cos 2\alpha \\ \cos 2\alpha & \sin 2\alpha \end{pmatrix}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} \exp(-2\mathrm{i}\alpha) & 0 \\ 0 & \exp(2\mathrm{i}\alpha) \end{pmatrix}. \quad (21)$$

By comparison with equation (9), we conclude that

$$\mathbf{Q}\left(\frac{\pi}{4}\right)\mathbf{H}\left(\frac{\pi}{4}+\alpha\right)\mathbf{Q}\left(\frac{\pi}{4}\right) = \exp(-2\mathrm{i}\alpha)\mathbf{M}_0(4\alpha). \quad (22)$$

This proves that any wave plate can be synthesized, apart from a phase factor, by the sequence Q–H–Q.

The plate adjustment needs only angular regulation of the HWP. This is very useful for the practical implementation of the device. Furthermore, the phase delay is linear in α and only one eighth of a turn is sufficient to make the delay vary of π, which is the meaningful range of delays. It is physically obvious that the equivalence has to remain valid if the whole device is rotated by an angle γ. In other terms, the equality

$$\mathbf{Q}\left(\gamma+\frac{\pi}{4}\right)\mathbf{H}\left(\gamma+\frac{\pi}{4}+\alpha\right)\mathbf{Q}\left(\gamma+\frac{\pi}{4}\right) = \exp(-2\mathrm{i}\alpha)\mathbf{M}(4\alpha,\gamma) \tag{23}$$

has to hold. This is easy to prove using equation (10).

It has still to be shown that the compensator can be synthesized by the sequences Q–Q–H and H–Q–Q too. We shall discuss this subject later. Now we are interested in the synthesis of a rotator using two QWPs and also one HWP. It appears in equation (14) that such a synthesis is possible, if we recall that two cascaded QWPs produce a HWP, i.e. $\mathbf{Q}_0\mathbf{Q}_0 = \mathbf{H}_0$. Then, on multiplying equation (14) from the right by $\mathbf{Q}_0\mathbf{Q}_0$, we obtain

$$\mathbf{H}(\alpha)\mathbf{Q}_0\mathbf{Q}_0 = \mathbf{R}(2\alpha). \tag{24}$$

Alternatively, the sequence Q–Q–H can be used, according to the relation

$$\mathbf{Q}_0\mathbf{Q}_0\mathbf{H}(\alpha) = \mathbf{R}(-2\alpha), \tag{25}$$

which is easily checked.

5. The general anisotropic optical component

From an optical point of view, as non-absorbing anisotropic elements can only introduce phase differences between the x- and y-components and/or rotate the polarization plane, one can envisage that any element, possibly produced by cascaded devices, is equivalent to a sequence of only two elements, i.e. a suitably oriented wave plate and a rotator. Reasonable as it may sound at the physical level, such an equivalence has to be proved. In mathematical terms, one has to show that the $\mathbf{A}$ matrix describing the system can always be written as the product of a $\mathbf{M}$ matrix and a $\mathbf{R}$ matrix. Neglecting possible phase factors, we shall refer to matrices of the form (7). We have to show that, for any given $\mathbf{A}$, it is possible to find three real parameters φ, ε and λ such that

$$\mathbf{A} = \mathbf{M}(\varphi,\varepsilon)\mathbf{R}(\lambda) = \mathbf{R}(\varepsilon)\mathbf{M}_0(\varphi)\mathbf{R}(\lambda-\varepsilon), \tag{26}$$

or, in an equivalent way, setting $\chi = \lambda - \varepsilon$,

$$\mathbf{R}(-\varepsilon)\mathbf{A} = \mathbf{M}_0(\varphi)\mathbf{R}(\chi). \tag{27}$$

Equation (27), written in an explicit way, gives

$$\begin{pmatrix}\cos\varepsilon & \sin\varepsilon\\ -\sin\varepsilon & \cos\varepsilon\end{pmatrix}\begin{pmatrix}\zeta_1 & \zeta_2\\ -\zeta_2^* & \zeta_1^*\end{pmatrix} = \begin{pmatrix}\exp(-\mathrm{i}\varphi/2) & 0\\ 0 & \exp(\mathrm{i}\varphi/2)\end{pmatrix}\begin{pmatrix}\cos\chi & -\sin\chi\\ \sin\chi & \cos\chi\end{pmatrix}, \tag{28}$$

where a phase factor has been omitted in $\mathbf{M}_0$. Performing the matrix products, we see that the relations

$$\begin{aligned}\zeta_1\cos\varepsilon - \zeta_2^*\sin\varepsilon &= \exp(-\mathrm{i}\varphi/2)\cos\chi,\\ -\zeta_1\sin\varepsilon - \zeta_2^*\cos\varepsilon &= \exp(\mathrm{i}\varphi/2)\sin\chi\end{aligned} \tag{29}$$

have to hold. Equations (29), solved with respect to ζ_1 and ζ_2, give

$$\begin{aligned}\exp(-\mathrm{i}\varphi/2)\cos\varepsilon\cos\chi - \exp(\mathrm{i}\varphi/2)\sin\varepsilon\sin\chi &= \zeta_1,\\ \exp(-\mathrm{i}\varphi/2)\sin\varepsilon\cos\chi + \exp(\mathrm{i}\varphi/2)\cos\varepsilon\sin\chi &= -\zeta_2^*.\end{aligned} \tag{30}$$

We have to prove that, for any choice of ζ_1 and ζ_2, under the condition $|\zeta_1|^2+|\zeta_2|^2 = 1$, a triplet of real numbers φ, ε, χ exists, solving equations (30). This can be seen in a simple way. Setting

$$\zeta_1 = a+\mathrm{i}b, \qquad \zeta_2 = c+\mathrm{i}d, \qquad a^2+b^2+c^2+d^2 = 1, \tag{31}$$

equations (30) become

$$\begin{aligned}\cos(\varepsilon+\chi)\cos\frac{\varphi}{2} &= a, \qquad \cos(\varepsilon-\chi)\sin\frac{\varphi}{2} = -b,\\ \sin(\varepsilon+\chi)\cos\frac{\varphi}{2} &= -c, \qquad \sin(\varepsilon-\chi)\sin\frac{\varphi}{2} = -d.\end{aligned} \tag{32}$$

These equations are easily solved with respect to φ, ε, χ. In fact we can set

$$\cos\frac{\varphi}{2} = \sqrt{a^2+c^2}, \qquad \sin\frac{\varphi}{2} = \sqrt{b^2+d^2}. \tag{33}$$

These relations agree with the condition $a^2+b^2+c^2+d^2 = 1$ and account for the fact that the physically meaningful range for φ is $[0,\pi]$. From equations (32) ε and χ are easily obtained.

6. The Simon–Mukunda universal gadget

We have seen in equation (26) that, apart from phase factors, we can always set

$$\mathbf{A} = \mathbf{M}(\varphi,\varepsilon)\mathbf{R}(\lambda). \tag{34}$$

According to equation (23), equation (34) can be written

$$\mathbf{A} = \exp(\mathrm{i}\varphi/2)\mathbf{Q}\left(\varepsilon+\frac{\pi}{4}\right)\mathbf{H}\left(\varepsilon+\frac{\pi+\varphi}{4}\right)\times\mathbf{Q}\left(\varepsilon+\frac{\pi}{4}\right)\mathbf{R}(\lambda). \tag{35}$$

$\mathbf{H}$ and $\mathbf{Q}$ can be exchanged (see equation (A5) in the appendix) obtaining

$$\mathbf{A} = \exp(\mathrm{i}\varphi/2)\mathbf{Q}\left(\varepsilon+\frac{\pi}{4}\right)\mathbf{Q}\left(\varepsilon+\frac{\pi}{4}+\frac{\varphi}{2}\right)\times\mathbf{H}\left(\varepsilon+\frac{\pi+\varphi}{4}\right)\mathbf{R}(\lambda). \tag{36}$$

Furthermore, **H** and **R** can be compacted (see equation (A10)). Accordingly, equation (36) becomes

$$\mathbf{A} = \exp(\mathrm{i}\varphi/2)\mathbf{Q}\left(\varepsilon + \frac{\pi}{4}\right)\mathbf{Q}\left(\varepsilon + \frac{\pi}{4} + \frac{\varphi}{2}\right) \times\mathbf{H}\left(\varepsilon + \frac{\pi+\varphi}{4} - \frac{\lambda}{2}\right). \quad (37)$$

This proves the Simon and Mukunda general result: apart from phase factors, any element **A**, specified by the parameters φ, ε, λ, can be synthesized by two QWPs and a HWP. The order of the elements can be easily modified. In fact, using equation (A6), we can give equation (37) the form

$$\mathbf{A} = \exp(\mathrm{i}\varphi/2)\mathbf{Q}\left(\varepsilon + \frac{\pi}{4}\right)\mathbf{H}\left(\varepsilon + \frac{\pi+\varphi}{4} - \frac{\lambda}{2}\right) \times\mathbf{Q}\left(\varepsilon + \frac{\pi}{4} - \lambda\right). \quad (38)$$

Finally, using equation (A6) of the appendix once again, equation (38) can be put in the form

$$\mathbf{A} = \exp(\mathrm{i}\varphi/2)\mathbf{H}\left(\varepsilon + \frac{\pi+\varphi}{4} - \frac{\lambda}{2}\right) \times\mathbf{Q}\left(\varepsilon + \frac{\pi}{4} + \frac{\varphi}{2} - \lambda\right)\mathbf{Q}\left(\varepsilon + \frac{\pi}{4} - \lambda\right). \quad (39)$$

In section 4, we have shown that a compensator, or variable wave plate, can be synthesized by the sequence Q–H–Q. Indeed, letting in equation (38)

$$\lambda = 0, \qquad \varphi = 4\alpha, \qquad \varepsilon = \gamma, \quad (40)$$

equation (23) is found again. We are now able to answer the question posed in section 4, i.e. how can the compensator be synthesized by the sequences Q–Q–H and H–Q–Q. Making the substitutions of equation (40) into equation (37), we obtain

$$\mathbf{M}(4\alpha, \gamma) = \exp(2\mathrm{i}\alpha)\mathbf{Q}\left(\gamma + \frac{\pi}{4}\right)\mathbf{Q}\left(\gamma + \frac{\pi}{4} + 2\alpha\right) \times\mathbf{H}\left(\gamma + \frac{\pi}{4} + \alpha\right). \quad (41)$$

The same substitutions, performed in equation (39), give

$$\mathbf{M}(4\alpha, \gamma) = \exp(2\mathrm{i}\alpha)\mathbf{H}\left(\gamma + \frac{\pi}{4} + \alpha\right) \times\mathbf{Q}\left(\gamma + \frac{\pi}{4} + 2\alpha\right)\mathbf{Q}\left(\gamma + \frac{\pi}{4}\right). \quad (42)$$

Taking into account equations (23), (41) and (42), we conclude that the compensator can be synthesized by any combination involving two QWPs and a HWP. However, it is worthwhile noting that the sequence given in equation (23) seems the most useful for a practical implementation of the device, as it requires only angular regulation of the HWP. The alternative gadgets in equations (41) and (42) need control over two relative angles and their realization is somehow trickier.

7. An alternative gadget using rotators

Let us consider now a different identity,

$$\exp(\mathrm{i}\alpha)\mathbf{Q}\left(\frac{\pi}{4}\right)\mathbf{R}(\alpha) = \mathbf{M}_0(2\alpha)\mathbf{Q}\left(\frac{\pi}{4}\right), \quad (43)$$

whose validity is easily checked by using equations (22) and (24). As all the identities given so far, it has been derived (with a method that differs from ours) by Simon and Mukunda, but it has some interesting consequences that were not employed in the original papers. Multiplying both sides of equation (43) by $\mathbf{Q}(-\pi/4)$ from the right or from the left, we obtain

$$\mathbf{Q}\left(\frac{\pi}{4}\right)\mathbf{R}(\alpha)\mathbf{Q}\left(-\frac{\pi}{4}\right) = \exp(-\mathrm{i}\alpha)\mathbf{M}_0(2\alpha), \quad (44)$$

$$\mathbf{Q}\left(-\frac{\pi}{4}\right)\mathbf{M}_0(2\alpha)\mathbf{Q}\left(\frac{\pi}{4}\right) = \exp(\mathrm{i}\alpha)\mathbf{R}(\alpha). \quad (45)$$

These relations show that, apart from a phase factor, a rotator is changed into a wave plate and *vice versa* when such elements are sandwiched between two crossed QWPs. Equation (45) has also been derived in a recent paper (Ye 1995) where an experimental realization was also presented. Let us now refer to equation (44). Notice that, in order to synthesize a 2α wave plate, the rotator must give exactly the rotation α. In other terms, a fixed rotator can be used to synthesize only one wave plate. The question arises of whether it is possible to obtain a greater flexibility by rotating one (or both) of the QWPs. It is easy to calculate that

$$\begin{aligned}\mathbf{Q}\left(\frac{\pi}{4}\right)\mathbf{R}(\alpha)\mathbf{Q}\left(\gamma - \frac{\pi}{4}\right) &= \mathbf{Q}\left(\frac{\pi}{4}\right)\mathbf{R}(\alpha+\gamma)\mathbf{Q}\left(-\frac{\pi}{4}\right)\mathbf{R}(-\gamma) \\ &= \exp[-\mathrm{i}(\alpha+\gamma)]\mathbf{M}_0(2\alpha+2\gamma)\mathbf{R}(-\gamma),\end{aligned} \quad (46)$$

having used equation (44). As one can see, any phase lag can be obtained acting on γ. The price to be paid is that the synthesized plate is preceded by a rotator (it must be kept in mind that, in matrix multiplication, the rightmost term represents the first element encountered by the light). An analogous procedure is used to prove the identity

$$\mathbf{Q}\left(\varepsilon + \frac{\pi}{4}\right)\mathbf{R}(\alpha)\mathbf{Q}\left(-\frac{\pi}{4}\right) = \exp[-\mathrm{i}(\alpha - \varepsilon)] \times\mathbf{R}(\varepsilon)\mathbf{M}_0(2\alpha - 2\varepsilon), \quad (47)$$

providing us with a synthesis process in which the rotator $\mathbf{R}(\varepsilon)$ follows the plate. Equations (46) and (47) have some practical usefulness because obviously the effect of the rotation can be compensated by a suitable rotation of part of the apparatus. Notice that equations (46) and (47) also hold in the case $\alpha = 0$. Therefore any wave plate, apart from a rotation, can be synthesized using only two QWPs.

8. Conclusions

The Simon–Mukunda gadget is a simple device that can be usefully exploited in the laboratory for synthesizing any non-absorbing anisotropic element. As we have shown its operation can be explained through elementary considerations. An additional result of the technique used in the present paper is that, apart from a rotation, any wave plate can be synthesized using only two quarter-wave plates. We finally note that, following an inverse procedure with respect to Simon and Mukunda, the acquired familiarity with combination of optical elements could be used as a didactical tool to explore some properties of the group $SU(2)$, whose structure can be represented in terms of Jones matrices.

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Appendix. Some useful identities

We discuss here some simple identities that allow us to change the positions of HWP and QWP. We begin with the following observation. If the Jones matrices representing three optical elements **A**, **B** and **C** verify the relation

$$\mathbf{A} = \mathbf{B}\mathbf{C}, \tag{A1}$$

then also the matrices obtained by rotating every element by one and the same angle γ verify an analogous relation, i.e.

$$\mathbf{A}' = \mathbf{B}'\mathbf{C}'. \tag{A2}$$

This property is physically obvious. It means that, if two cascaded optical elements are equivalent to a third one, the equivalence cannot depend on the reference frame. We have already used this idea to obtain equation (23) from equation (22). In a formal way, equation (A2) is proved by multiplying equation (A1) by $\mathbf{R}(\gamma)$ from the left and by $\mathbf{R}(-\gamma)$ from the right and inserting between **B** and **C** the identity operator $\mathbf{R}(-\gamma)\mathbf{R}(\gamma)$. Let us consider now the equation

$$\mathbf{H}_0\mathbf{Q}(\alpha) = \mathbf{Q}(-\alpha)\mathbf{H}_0. \tag{A3}$$

As it is easily checked using equations (12) and (13), this is an identity that holds for any α. Taking into account equation (A2), equation (A3) is generalized in the following way,

$$\mathbf{H}(\gamma)\mathbf{Q}(\gamma + \alpha) = \mathbf{Q}(\gamma - \alpha)\mathbf{H}(\gamma), \tag{A4}$$

or, changing the variables,

$$\mathbf{H}(\gamma)\mathbf{Q}(\beta) = \mathbf{Q}(2\gamma - \beta)\mathbf{H}(\gamma), \tag{A5}$$

$$\mathbf{Q}(\beta)\mathbf{H}(\gamma) = \mathbf{H}(\gamma)\mathbf{Q}(2\gamma - \beta). \tag{A6}$$

Another pair of identities that derives from equation (14) is

$$\mathbf{R}(\alpha)\mathbf{H}_0 = \mathbf{H}\left(\frac{\alpha}{2}\right), \tag{A7}$$

$$\mathbf{H}_0\mathbf{R}(\alpha) = \mathbf{H}\left(-\frac{\alpha}{2}\right). \tag{A8}$$

Rotating by an angle γ, as $\mathbf{R}(\alpha)$ remains unchanged under any rotation, we finally obtain

$$\mathbf{R}(\alpha)\mathbf{H}(\gamma) = \mathbf{H}\left(\gamma + \frac{\alpha}{2}\right), \tag{A9}$$

$$\mathbf{H}(\gamma)\mathbf{R}(\alpha) = \mathbf{H}\left(\gamma - \frac{\alpha}{2}\right). \tag{A10}$$

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