Shape-invariance range of a light beam

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Received February 12, 1996

A typical axially symmetric light beam on paraxial free propagation maintains the same transverse shape as at the waist plane for a certain range along its axis. We discuss a general procedure for estimating this range.

Light beams of a coherent and a partially coherent nature can exhibit a host of different features in the course of propagation. Some of them, the well-known Hermite–Gauss and Laguerre–Gauss (LG) modes,[1] simply expand at a regular pace while changing their curvature. Others present twisting phenomena. Most of them have a transverse configuration that changes in a more-or-less rapid manner from one plane to another. To account for such varied characteristics, several parameters can be introduced. References 2 and 3 can be consulted for a general review of recent results.

Here we discuss another parameter that could be profitably used to describe a relevant aspect of light-beam propagation, namely, the interval of distances along which, starting from the waist plane,[4] the transverse profile remains unchanged, to a certain approximation, apart from phase and scale factors. For brevity, we call such an interval the shape-invariance range (SIR). We limit ourselves to coherent axially symmetric beams in paraxial propagation. We can observe at once that certain beams have infinite SIR's, as proved by the case of a single Gaussian mode of arbitrary order. For other cases, however, the SIR can be virtually zero, as revealed for example by diffraction of a plane wave by a circular hole.[6] The problem is how to evaluate the SIR of a general beam.

We first discuss the basic idea in an intuitive way and then pass on to a more comprehensive treatment. Suppose that the beam under consideration is expanded into a series of normalized LG modes. The field space distribution $V(r, z)$ is then expressible as[1]

$$V(r,z) = \left[\frac{2}{\pi v^2(z)}\right]^{1/2} \exp[ikz - i\Phi(z)] \times \exp\left\{\left[\frac{ik}{2R(z)} - \frac{1}{v^2(z)}\right] r^2\right\} \times \sum_{n=0}^{\infty} c_n \exp[-2in\Phi(z)] L_n\left[\frac{2r^2}{v^2(z)}\right]. \quad (1)$$

Here k is the wave number, L_n is the nth Laguerre polynomial, c_n are the (generalized) Fourier coefficients of the series expansion, and the quantities $v(z)$, $R(z)$, and $\Phi(z)$ have the usual expressions[1]:

$$v(z) = v_0\left[1 + \left(\frac{\lambda z}{\pi v_0^2}\right)^2\right]^{1/2},$$

$$R(z) = z\left[1 + \left(\frac{\pi v_0^2}{\lambda z}\right)^2\right],$$

$$\tan \Phi(z) = \frac{\lambda z}{\pi v_0^2}, \quad (2)$$

where λ is the wavelength and v_0 is the spot size at the waist.

For simplicity we assume that the field is normalized in such a way that

$$\sum_{n=0}^{\infty} |c_n|^2 = 1. \quad (3)$$

Because of the presence of the terms $\exp[-2in\Phi(z)]$ in Eq. (1) the phase relationships among the modes change with z. This in turn determines a change of the overall transverse pattern. For a certain range of values of z, however, the pattern can be nearly identical to the one seen at $z = 0$. Suppose in fact that only a finite number of coefficients c_n are significantly different from zero, and denote by $(n_{\max} - n_{\min})$ the difference between the highest and the lowest indices of such coefficients. Then, when we move away from the plane $z = 0$, there will be a certain range of values of z across which the condition

$$2\Phi(z)(n_{\max} - n_{\min}) < \varphi \quad (4)$$

is met for an arbitrary choice of $\varphi > 0$. Suppose, in particular, that $\varphi \ll 2\pi$, and note that the left-hand side of inequality (4) is the maximum dephasing introduced among modes by the propagation process. Then, for all values of z for which condition (4) is met, the field profile resulting from the overall mode interference remains essentially the same as at $z = 0$, except for a possible magnification factor. In conclusion, for beams possessing a finite number of nonnegligible coefficients c_n, it should be possible to estimate the SIR through inequality (4).

The above argument is to be somewhat refined. First, the waist spot size v_0 in expansion (1) can in principle be chosen in an arbitrary way. Suppose now that the field under consideration is itself a single LG mode with a spot size w_0. As we have already

0146-9592/96/161205-03$10.00/0

noted, the corresponding SIR must be infinite. On the other hand, if we use a spot size $v_0 \neq w_0$, series (1) contains infinitely many terms. Paradoxically, the reasoning based on inequality (4) would then lead to some finite value of SIR. Evidently, the use of inequality (4) corresponds to a sufficient rather than to a necessary condition. We ask whether a suitable choice of v_0 can circumvent this difficulty. A criterion for such a choice is the following. It is well known[5] that the intensity variance of any beam obeys the law

$$\Delta {r_z}^2 = \Delta {r_0}^2 \left[1 + \left(\frac{M^2 \lambda z}{2\pi \Delta {r_0}^2} \right)^2 \right], \tag{5}$$

where we used the M^2 factor. The quantities Δr_0 and Δr_z are the equivalent widths of the beam (square roots of the intensity variances) at $z = 0$ and $z \neq 0$, respectively. On comparing Eq. (5) with the first relation in Eq. (2) we find that our beam and the LG modes expand at the same rate if the spot size v_0 is chosen as follows:

$$v_0 = \frac{\Delta r_0 \sqrt{2}}{M} \cdot \tag{6}$$

It is easily seen that this choice criterion removes the paradox quoted above. In fact, if the beam under scrutiny is a single LG mode of order m and spot size w_0, the corresponding intensity variance and M^2 factor are[7]

$$\Delta {r_0}^2 = \frac{2m+1}{2} {w_0}^2, \qquad M^2 = 2m + 1\,. \tag{7}$$

On inserting from Eq. (7) into Eq. (6) we find that $v_0 = w_0$. As a consequence only one term appears in series (1), so the difference $(n_{\max} - n_{\min})$ in inequality (4) is to be taken as 0. An infinite SIR is therefore found as required. It should be mentioned that the choice criterion expressed by Eq. (6) turns out to be the same as the one leading to the concept of an embedded Gaussian beam.[5,8]

We next note that, to give inequality (4) a more precise basis, some measure of shape invariance must be specified. We have seen from Eq. (1) that if it were not for the factors $\exp[-2in\Phi(z)]$ within the series the propagated field would simply be an enlarged version of the field at $z = 0$ multiplied by an amplitude and a phase factor. We then define a reference propagated field as follows:

$$V_r(r,z) = \left(\frac{2}{\pi {v_0}^2} \right)^{1/2} \exp\left(-\frac{r^2}{{v_0}^2} \right) \sum_{n=0}^{\infty} c_n \times \exp[-2i(n - \overline{n})\Phi(z)] L_n \left(\frac{2r^2}{{v_0}^2} \right), \tag{8}$$

where, taking Eq. (3) into account, we let

$$\overline{n} = \sum_{n=0}^{\infty} n |c_n|^2. \tag{9}$$

The meaning of Eq. (8) is as follows: V_r represents the effect of propagation when transverse expansion, wavefront curvature, and phase factors are disregarded. The term $\exp[2i\overline{n}\Phi(z)]$ has been introduced to account for the average phase acquired by the beam through propagation. In particular, let us suppose that only one mode, say, the mth, appears in the field expansion. In this case we would have $\overline{n} = m$, and $V_r(r,z)$ would then equal $V(r,0)$ for any z. We know that in such a case the SIR is infinite. Hence the mean-square difference between $V_r(r,z)$ and $V(r,0)$ can be taken as a measure of how much shape invariance is lost on propagation along a distance z. Accordingly, we define the shape-invariance error

$$\epsilon(z) = \left[2\pi \int_0^\infty |V_r(r,z) - V(r,0)|^2 r \mathrm{d}r \right]^{1/2}. \tag{10}$$

We note that, owing to condition (3), $\epsilon(z)$ represents an error relative to the total power carried by the beam. The SIR can now be defined as the distance ζ such that, if $z \leq \zeta$, then $\epsilon(z)$ remains less than some given quantity, say, $\epsilon_{\max}$. As in many similar cases, there is some arbitrariness in the choice of the limiting value to be accepted for $\epsilon_{\max}$. In most instances a sensible value for such a maximum will be dictated by the accuracy with which experiments are performed. On inserting Eqs. (1) and (8) into Eq. (10) and using the orthonormality of LG modes, we obtain the following expression:

$$\epsilon(z) = \left(2 \sum_{n=0}^{\infty} |c_n|^2 \{1 - \cos[2(n - \overline{n})\Phi(z)]\} \right)^{1/2}. \tag{11}$$

Once $\epsilon_{\max}$ is chosen, Eq. (11) can be used for the evaluation of the SIR.

We can obtain a useful relation by replacing each term in Eq. (11) by an upper bound. Indeed, inasmuch as

$$1 - \cos(2\alpha) = 2\sin^2\alpha \leq 2\alpha^2, \tag{12}$$

the following inequality for $\epsilon(z)$ can easily be derived:

$$\epsilon(z) \leq 2\Phi(z) \left[\sum_{n=0}^{\infty} |c_n|^2 (n - \overline{n})^2 \right]^{1/2} = 2\Phi(z)\Delta n\,, \tag{13}$$

where

$$\Delta n = \sqrt{\overline{n^2} - \overline{n}^2}, \qquad \overline{n^2} = \sum_{n=0}^{\infty} n^2 |c_n|^2. \tag{14}$$

By virtue of inequality (13) we can estimate the SIR ζ by requiring that

$$2\Phi(\zeta)\Delta n = \epsilon_{\max}\,. \tag{15}$$

By using Eqs. (2) we obtain

$$\zeta = \frac{\pi {v_0}^2}{\lambda} \tan\left(\frac{\epsilon_{\max}}{2\Delta n} \right) \cdot \tag{16}$$

More precisely, Eq. (16) gives a lower bound for the SIR because the exact expression [Eq. (11)] for $\epsilon(z)$ has been replaced by inequality (13). Of course, the smaller the value of $\epsilon_{\max}$, the better the SIR estimate given by Eq. (16) is. As seen from Eq. (15), the constraint on the error $\epsilon(z)$ has led to a limitation for the maximum dephasing among modes, as in the qualitative argument used for inequality (4).

It might be observed that the use of Eq. (16) can be cumbersome because it requires the explicit evaluation of values of c_n [see Eq. (14)]. We now show that Δn can be computed by an alternative procedure. Let us observe that the LG modes obey the differential equation

$$\hat{H} L_n\left(\frac{2r^2}{v_0{}^2}\right)\exp\left(-\frac{r^2}{v_0{}^2}\right) = nL_n\left(\frac{2r^2}{v_0{}^2}\right)\exp\left(-\frac{r^2}{v_0{}^2}\right), \tag{17}$$

where $\hat{H}$ is the differential operator

$$\hat{H} = \frac{1}{2}\left[-\frac{v_0{}^2}{4}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{r^2}{v_0{}^2} - 1\right]. \tag{18}$$

Equation (17) is easily derived from the Schrödinger equation for a two-dimensional harmonic oscillator,[9] with only circularly symmetric solutions considered.

Using Eqs. (1) and (17), we can easily prove that

$$\overline{n} = 2\pi\int_0^\infty V^*(r,0)\hat{H}V(r,0)r\mathrm{d}r,$$
$$\overline{n^2} = 2\pi\int_0^\infty V^*(r,0)\hat{H}^2V(r,0)r\mathrm{d}r, \tag{19}$$

where the asterisk denotes the complex conjugate. Therefore we can evaluate the value Δn that enters inequality (13) and Eq. (15) through Eq. (19) without actually computing the values of c_n.

Let us summarize our results and add some remarks. For any square-integrable field distribution across the plane $z = 0$ Eqs. (14), (15), and (19) afford a simple method for estimating the SIR of the corresponding beam. In particular, this procedure leads to an infinite SIR for single LG modes, whereas it could be seen to give a vanishing SIR for limiting cases such as a hard-edge flat distribution at $z = 0$ in which the M^2 factor is known to diverge. The procedure outlined could be extended for obtaining estimates of the SIR when the starting plane does not coincide with the waist plane.

This research is sponsored by Instituto Nazionale di Fisica della Materia and Ministero dell'Universitá e della Ricerca Scientifica.

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