JOURNAL OF MODERN OPTICS, 1996, VOL. 43, NO. 6, 1155–1166

Generalized Bessel–Gauss beams

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(Received 23 May 1995; revision received 16 October 1995)

Abstract. In this paper we describe a superposition model for Bessel–Gauss beams, in which higher orders are included. An analogous model leads to a different class of beams, namely the modified Bessel–Gauss beams. Then a generalized set of beams, containing the previous beams as particular cases, is introduced. The behaviour of these beams upon propagation is investigated, both analytically and numerically.

1. Introduction

Bessel–Gauss beams were introduced [1] to overcome the difficulties of physical realization of diffraction-free beams or Bessel beams [2–9], in which the disturbance in a transverse plane varied like a Bessel function of zero order. A Gaussian modulation had been added so that these fields do not carry an infinite amount of energy. Optical methods have been suggested [10, 11] to generate practically a Bessel–Gauss beam of zero order.

We review in section 2 how the Bessel–Gauss beam of zero order may be viewed as a coherent superposition of ordinary Gaussian beams having a common waist plane, whose axes are uniformly distributed on a cone. This was pointed out in [1, 10]. Bessel–Gauss beams of higher order are obtained when the superimposed beams have amplitudes varying with the azimuthal angle.

In section 3 we consider a different set of fields, which may be generated by superimposing ordinary Gaussian beams whose axes are on a cylinder. Since in this case the amplitude of the resulting disturbance involves the modified Bessel functions of the various orders, we call these fields modified Bessel–Gauss beams. We remark that the modified Bessel–Gauss beam of zero order has been recently introduced in [12].

In section 4 we show that all the previous cases are contained in a generalized set of fields, where the elementary beams constitute a frustum of cone at the waist plane.

The properties of the considered beams are studied both analytically and numerically. Results of computations are presented for the radial and longitudinal behaviour of the fields and are theoretically justified.

In particular, the modified Bessel–Gauss beam of zero order shows, for a suitable choice of the superposition parameters, a central region of uniform intensity in the waist plane. This property would be of use in exploiting the volume of an active medium in a more efficient way than with respect to an ordinary Gaussian beam. Moreover, this field may be used in some applications (medical and industrial) as an alternative to other configurations, such as flattened Gaussian beams [13], super-Gaussian beams [14, 15], or fields obtained by beam integrators [16] and holographic beam formers [17].

2. Bessel–Gauss beams of order *n*

Let us consider, in the plane $z = 0$ of a cylindrical reference frame r, ϑ, z, a field of the form

$$v_0(r, \vartheta; \alpha) = A(\alpha) \exp\left(-\frac{r^2}{w_0^2}\right) \exp\left[i\beta r \cos(\alpha - \vartheta)\right]. \tag{1}$$

Equation (1) represents a Gaussian beam whose mean wave-vector has a projection β on the $z = 0$ plane, forming the angle α with respect to the x axis (figure 1). The amplitude A is a function of α. When α varies, the wave-vector describes a cone of semiaperture φ such that $\sin\varphi = \beta/k$, where k is the wavenumber. We note that the ellipticity of the beam section on the $z = 0$ plane has been neglected in equation (1), since we assume that β is sufficiently small with respect to k. In this approximation, w_0 may be considered as the beam spot size at the waist and the effects of ellipticity upon propagation may be neglected as well.

First, let us suppose that the amplitude A does not depend on α. In this case, superimposing beams of the form (1) with respect to α, we obtain a total field

$$V_0(r) = \int_0^{2\pi} v_0(r, \vartheta; \alpha)\, d\alpha = 2\pi A \exp\left(-\frac{r^2}{w_0^2}\right) J_0(\beta r), \tag{2}$$

where the integral expression of the Bessel functions of the first kind of integer order n [18],

$$J_n(u) = \frac{1}{2\pi i^n} \int_0^{2\pi} \exp\left[i(u \cos\psi - n\psi)\right] d\psi, \tag{3}$$

has been employed, letting $n = 0$.

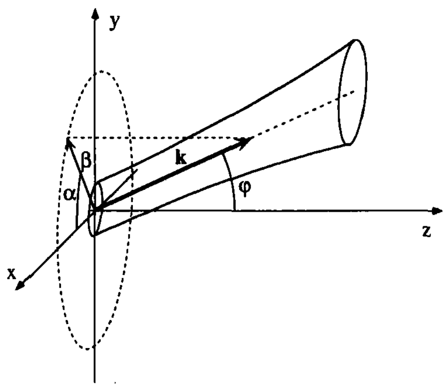

Figure 1. Geometry for Bessel–Gauss beams.

If we now let A be a generic (periodic) function of the angle α, using its Fourier series expansion with coefficients A_n,

$$A(\alpha) = \sum_{n=-\infty}^{+\infty} A_n \exp(\mathrm{i}n\alpha), \tag{4}$$

the superposition gives, according to equation (3),

$$V_0(r, \vartheta) = \int_0^{2\pi} v_0(r, \vartheta; \alpha)\, \mathrm{d}\alpha = 2\pi \exp\left(-\frac{r^2}{w_0^2}\right) \sum_{n=-\infty}^{+\infty} A_n \mathrm{i}^n \exp(\mathrm{i}n\vartheta)\, J_n(\beta r). \tag{5}$$

The field generated at any plane $z = \text{constant}$ by the distribution (5) may be obtained by letting fields of the form (1) propagate separately. In the paraxial approximation, which holds since β is small with respect to k, the elementary contribution v_z is given by [19]

$$\begin{aligned} v_z(r, \vartheta; \alpha) = A(\alpha) \frac{w_0}{w(z)} &\exp\left(\mathrm{i}[kz - \Phi(z)] - \mathrm{i}\frac{\beta^2}{2k} z\right) \\ &\times \exp\left\{-F(z)\left[r^2 + \left(\frac{\beta z}{k}\right)^2 - 2\frac{\beta z r}{k}\cos(\alpha - \vartheta)\right]\right\} \\ &\times \exp[\mathrm{i}\beta r \cos(\alpha - \vartheta)]. \end{aligned} \tag{6}$$

In equation (6) we have introduced the functions characterizing a Gaussian beam [19], taking into account the fact that the fields propagate at an angle φ with respect to the z axis:

$$w^2(z) = w_0^2\left[1 + \left(\frac{z}{L\cos\varphi}\right)^2\right], \quad R(z) = \frac{z}{\cos\varphi} + \frac{L^2\cos\varphi}{z}, \quad \Phi(z) = \arctan\left(\frac{z}{L\cos\varphi}\right), \tag{7}$$

$$F(z) = \frac{1}{w^2(z)} - \frac{\mathrm{i}k}{2R(z)}, \qquad L = \frac{\pi}{\lambda} w_0^2. \tag{8}$$

We note that the phase factor accounting for the propagation in equation (6) contains a term that is quadratic in β, coming from a series expansion due to paraxial approximation. The same result may be obtained using properties of the Fresnel transform [20].

The final result for the total field V_z is

$$\begin{aligned} V_z(r, \vartheta) = 2\pi \frac{w_0}{w(z)} \exp\left(\mathrm{i}[kz - \Phi(z)] - \mathrm{i}\frac{\beta^2}{2k} z\right) \exp\left\{-F(z)\left[r^2 + \left(\frac{\beta z}{k}\right)^2\right]\right\} \\ \times \sum_{n=-\infty}^{+\infty} A_n \mathrm{i}^n \exp(\mathrm{i}n\vartheta)\, J_n\left[\beta r\left(1 - \mathrm{i}F(z)\frac{2z}{k}\right)\right]. \end{aligned} \tag{9}$$

We may call the nth term of equation (9) a Bessel–Gauss beam of order n. Note that, in the limit $w_0 \to \infty$, such a beam becomes the diffraction-free beam of order n, which has been studied [21]. We note that, as far as intensity is concerned, each of these beams is circularly symmetric, but of course this is not true for the superposition field (9). In fact the nth term of the sum contains the phase factor $\exp(\mathrm{i}n\vartheta)$, giving rise to a spiral wave front, rotating upon propagation. Phase singularities of this kind are known as optical vortices [22, 23]. An effect of the phase pattern is that, if we have only the $\pm n$th-order beams with $n \neq 0$, and we set $A_n = \pm A_{-n}$, a sinusoidal modulation, that is conserved in propagation, is given to the field, since $J_{-n}(u) = (-1)^n J_n(u)$. This result holds also for the modified and generalized Bessel–Gauss beams, which are treated in the following sections.

It is worth noting from equation (9) that, while the argument of the Bessel functions at the waist is real, during the propagation it becomes complex. For the case $A(\alpha) =$ constant, the previous expressions coincide with those given in [1].

3. Modified Bessel–Gauss beams

Let us now consider a superposition of Gaussian beams, having mean wave-vectors parallel to the longitudinal z direction, but whose centres are placed on a circumference of radius a around the z axis. We call γ the angle that the x axis forms with the segment joining the centre of the generic beam to the origin (figure 2). When γ varies, the wave-vector describes a cylinder. At the waist plane, the elementary contribution to the total field has the form

$$v_0(r, \vartheta; \gamma) = A(\gamma) \exp\left(- \frac{r^2 + a^2 - 2ar\cos(\gamma - \vartheta)}{w_0^2} \right). \tag{10}$$

If we use for $A(\gamma)$ the general equation (4), we obtain a superposition of beams that we call modified Bessel–Gauss beams of order n at their waist, that is

$$V_0(r, \vartheta) = 2\pi \exp\left(- \frac{r^2 + a^2}{w_0^2} \right) \sum_{n=-\infty}^{+\infty} A_n \exp(in\vartheta)\, I_n\left(\frac{2ar}{w_0^2} \right), \tag{11}$$

where the integral expression for the modified Bessel function of the first kind of integer order n,

$$I_n(u) = \frac{1}{2\pi} \int_0^{2\pi} \exp(u\cos\psi - in\psi)\, d\psi, \tag{12}$$

has been employed.

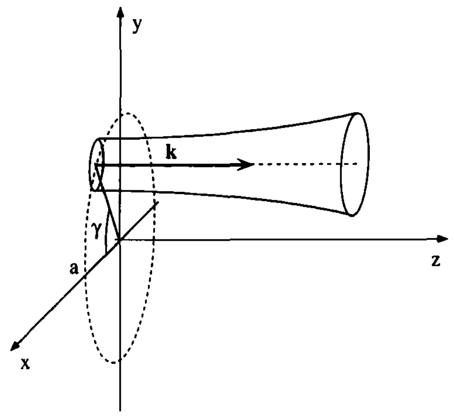

Figure 2. Geometry for modified Bessel–Gauss beams.

The field generated at any plane z = constant by the distribution (11) is given, employing the same procedure as before, by (paraxial regime)

$$V_z(r, \vartheta) = 2\pi \frac{w_0}{w(z)} \exp\{i[kz - \Phi(z)]\} \exp\{-F(z)(r^2 + a^2)\} \times \sum_{n=-\infty}^{+\infty} A_n \exp(in\vartheta)\, I_n[2arF(z)], \tag{13}$$

where the definitions (7) have been employed, with $\cos\varphi = 1$. The result pertaining to the zero-order beam has already been obtained [12] using a different approach.

In this case too, we note from equation (13) that, while the argument of the modified Bessel functions at the waist is real, it becomes complex upon propagation. In particular, for very large z, it can be shown that it tends to the purely imaginary value $-ikar/z$, that is the far field pertaining to the n-th term in the beam (13) contains a factor

$$I_n\left(-i\frac{ka}{z}r\right) = i^n J_n\left(-\frac{ka}{z}r\right), \tag{14}$$

because, as can be seen from equations (3) and (12),

$$I_n(u) = i^n J_n(-iu). \tag{15}$$

Thus a modified Bessel–Gauss beam of order n generates in the far zone an ordinary Bessel–Gauss beam of the same order, as those appearing in equation (5). The inverse relation holds too, as it can be shown that, for large z, the argument of the Bessel functions appearing in equation (9) tends to the limiting form $-ikw_0^2\beta r/2z$, that is the far field of the nth term in such a beam contains a factor

$$J_n\left(-i\frac{kw_0^2}{2z}\beta r\right) = (-i)^n I_n\left(\frac{kw_0^2}{2z}\beta r\right), \tag{16}$$

where equation (15) has been used again.

The connection between ordinary and modified Bessel–Gauss beams of the same order, which transform into each other propagating in the far zone, may be easily understood, since it is well known that the far zone is simulated by a lens, which transforms parallel rays lying on a cylinder into converging rays lying on a cone, and vice versa.

4. Generalized Bessel–Gauss beams

Let us finally superimpose at their waist plane Gaussian beams having the centres on a circumference of radius a and the mean axes on a cone of semiaperture φ (figure 3). Now the angle γ that the x axis forms with the segment joining the centre of the generic beam to the origin coincides with the angle α that the projection β of the wave-vector on the $z = 0$ plane forms with the x axis. This model encompasses both the situations considered in the previous sections as particular cases, and the same approximations will be used.

The expression for the elementary contribution to the field is

$$v_0(r, \vartheta; \gamma) = A(\gamma) \exp\left(-\frac{r^2 + a^2 - 2ar\cos(\gamma - \vartheta)}{w_0^2}\right) \exp[i\beta r \cos(\gamma - \vartheta)]. \tag{17}$$

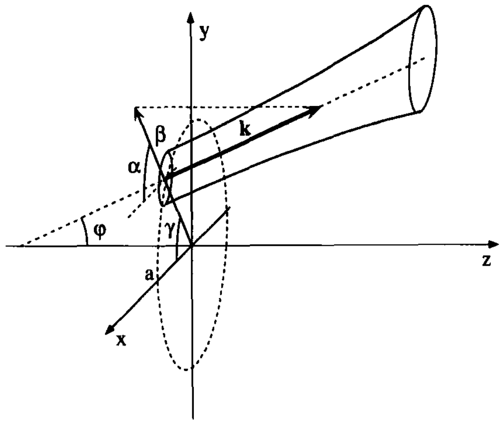

Figure 3. Geometry for generalized Bessel–Gauss beams.

At a distance z, the centre of this beam will be radially shifted by an amount $\beta z/k$. Apart from this, the shape of the field is of the form (6), that is

$$v_z(r,\vartheta)=A(\gamma)\frac{w_0}{w(z)}\exp\left(\mathrm{i}[kz-\Phi(z)]-\mathrm{i}\frac{\beta^2}{2k}z\right)\exp\left[\mathrm{i}\beta r\cos(\gamma-\vartheta)\right]$$
$$\times\exp\left\{-F(z)\left[r^2+\left(a+\frac{\beta z}{k}\right)^2-2r\left(a+\frac{\beta z}{k}\right)\cos(\gamma-\vartheta)\right]\right\},\quad(18)$$

where the definitions (7) have been employed.

If $A(\gamma)=$ constant, the superposition of the various contributions gives

$$V_z(r,\vartheta)=A\frac{w_0}{w(z)}\exp\left(\mathrm{i}[kz-\Phi(z)]-\mathrm{i}\frac{\beta^2}{2k}z\right)\exp\left\{-F(z)\left[r^2+\left(a+\frac{\beta z}{k}\right)^2\right]\right\}$$
$$\times 2\pi I_0\left[F(z)2r\left(a+\frac{\beta z}{k}\right)+\mathrm{i}\beta r\right].\quad(19)$$

If we now use for $A(\gamma)$ an expansion of the form (4), we have

$$V_z(r,\vartheta)=2\pi\frac{w_0}{w(z)}\exp\left(\mathrm{i}[kz-\Phi(z)]-\mathrm{i}\frac{\beta^2}{2k}z\right)\exp\left\{-F(z)\left[r^2+\left(a+\frac{\beta z}{k}\right)^2\right]\right\}$$
$$\times\sum_{n=-\infty}^{+\infty}A_n\exp(\mathrm{i}n\vartheta)\,I_n\left[F(z)2r\left(a+\frac{\beta z}{k}\right)+\mathrm{i}\beta r\right].\quad(20)$$

If we let $a=0$ in equation (20), we obtain the expression (9), exploiting equation (15). Instead, if we let $\beta=0$, equation (13) is given.

Note that in this general case the argument of the modified Bessel functions is always complex, even at the waist plane.

Moreover, comparing a generalized Bessel–Gauss beam with a propagated ordinary Bessel–Gauss beam, it turns out that the two fields are always physically different from each other, owing to the effect of the curvature in the latter. The situation is depicted in figure 4, where it is suggested that the same disturbance

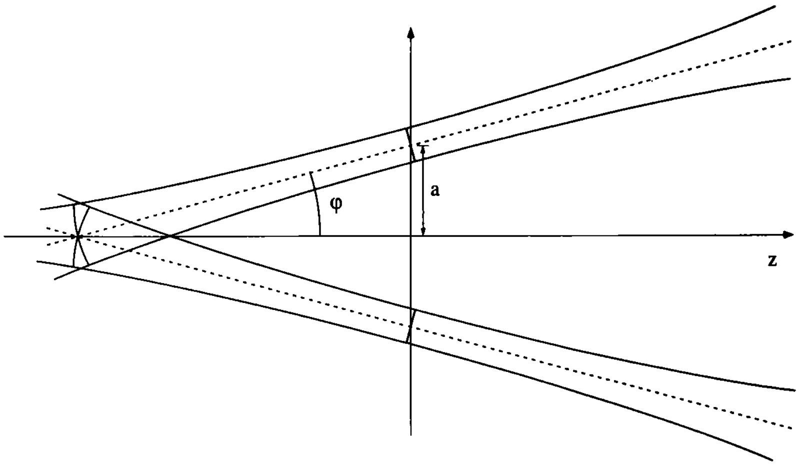

Figure 4. Equivalence between a generalized Bessel–Gauss beam and a superposition of Gaussian beams with same centre, axes placed on a cone, and suitable curvature.

of equation (20) can be obtained by superimposing Gaussian beams as in section 2 with suitable curvature radius and spot size given by equation (7) with $z = -a/\tan\varphi$.

5. Numerical results and discussion

As pointed out in the introduction, the modified Bessel–Gauss beam of zero order may show a central region of uniform intensity in the waist plane. More precisely, this happens when the spot size w_0 of the elementary beam equals the radius a of the cylinder. This can be easily explained considering the intensity $\mathscr{I}$ pertaining to the term of zero order in equation (11). Apart from a constant factor, we have

$$\mathscr{I}(r) = \exp\left(-2\frac{r^2+a^2}{w_0^2}\right) I_0^2\left(\frac{2ar}{w_0^2}\right). \tag{21}$$

In the neighbourhood of the origin we can assume that $r \ll w_0$, a. So we can make the following approximation, employing Taylor expansions for both factors in equation (21) [18]:

$$\mathscr{I}(r) \approx \exp\left(-2\frac{a^2}{w_0^2}\right)\left[1 + 2\frac{r^2}{w_0^2}\left(\frac{a^2}{w_0^2} - 1\right)\right]. \tag{22}$$

From this expression it is clear that, if the radius a equals w_0, the intensity near the axis is constant. Instead, taking the derivative of equation (22) with respect to r, we see that, if w_0 is greater (less) than a, the intensity in the origin has a maximum (minimum). This can be easily understood by considering that on increasing a the maxima of the constituent Gaussian beams recede from one another, so that a central dip appears.

These observations are confirmed by the numerical results presented in figure 5, where the three previous cases are shown. Here and in the following, the intensity is measured in arbitrary units.

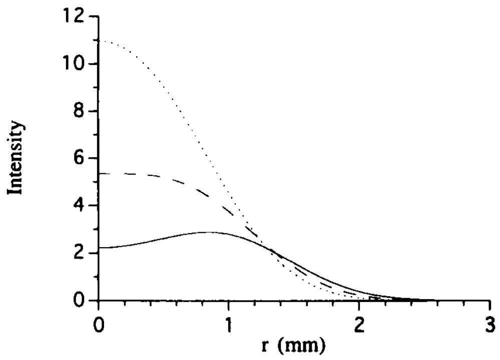

Figure 5. Intensity distribution of the modified beam of zero order on the waist plane as a function of r, for $w_0 = 1$ mm, $\lambda = 0{\cdot}6328$ µm and $a = 0{\cdot}8$ mm (------), $a = 1{\cdot}0$ mm (– – –) and $a = 1{\cdot}2$ mm (——).

Since, as is well known, Gaussian beams widen upon propagation, it may be shown that, even if the intensity has a central dip at the waist, it has a plateau near $r = 0$ at a suitable plane z = constant, and eventually a maximum, when z is further increased.

To this aim, let us consider the zero-order term in equation (13). The corresponding intensity may be written as

$$\mathscr{I}(r, z) = \left(\frac{w_0}{w(z)}\right)^2 \exp\left(-2\frac{r^2 + a^2}{w^2(z)}\right)|I_0[2arF(z)]|^2. \tag{23}$$

Employing Taylor expansions for both factors in equation (23), we can use for the intensity in the neighbourhood of the origin the expression

$$\mathscr{I}(r, z) = \left(\frac{w_0}{w(z)}\right)^2 \exp\left(-2\frac{a^2}{w^2(z)}\right)[1 + 2r^2\Delta(z)], \tag{24}$$

where

$$\Delta(z) = \frac{1}{w^2(z)}\left(\frac{a^2}{w^2(z)} - 1\right) - \left(\frac{ka}{2R(z)}\right)^2. \tag{25}$$

Defining

$$\zeta = \frac{z}{L}, \quad \eta = \frac{a}{w_0}, \tag{26}$$

we obtain from equation (25) the relation

$$\Delta(L\zeta) = \frac{1}{w_0^2}\frac{\eta^2 - 1 - \zeta^2(\eta^2 + 1)}{(1 + \zeta^2)^2}. \tag{27}$$

From equation (24) it is seen that the intensity shows a maximum, a plateau or a minimum near $r = 0$ depending on whether Δ is negative, null or positive respectively, that is whether ζ^2 is greater than, equal to or less than $(\eta^2 - 1)/(\eta^2 + 1)$. Therefore for a suitable value of z a plateau is always obtained. This behaviour is shown by the curves in figure 6.

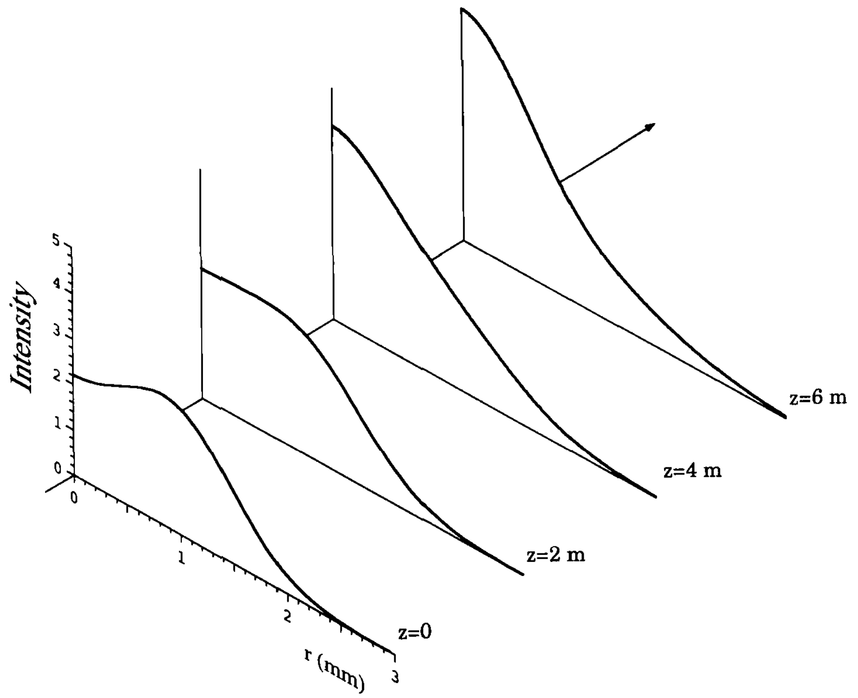

Figure 6. Intensity distribution upon propagation of the modified beam of zero order as a function of r, for $w_0 = 1$ mm, $\lambda = 0{\cdot}6328$ μm and $a = 1{\cdot}2$ mm on the waist plane and on the planes $z = 2$ m, $z = 4$ m and $z = 6$ m.

Still with reference to the modified beam of zero order, let us now consider the intensity at $r = 0$ as a function of the longitudinal coordinate z. If we take the zero-order term in equation (13) and we set $r = 0$, the intensity turns out to be

$$\mathscr{I}(\zeta) = G(\zeta) \exp\left[-2\eta^2 G(\zeta)\right], \tag{28}$$

where we have used the definitions (26), together with

$$G(\zeta) = \left(\frac{w_0}{w(L\zeta)}\right)^2 = \frac{1}{1+\zeta^2}. \tag{29}$$

Taking the derivative with respect to ζ and indicating the derivation with a prime, we obtain

$$\mathscr{I}' = (1 - 2\eta^2 G) \exp(-2\eta^2 G)\, G'. \tag{30}$$

Since $G' = -2\zeta/(1+\zeta^2)^2$, $\mathscr{I}'$ tends to zero in the far zone and vanishes for $\zeta = 0$ (waist plane). It can also vanish when $(1 - 2\eta^2 G) = 0$, that is $\zeta = (2\eta^2 - 1)^{1/2}$, which is real for $2\eta^2 > 1$, which means that $w_0 < 2^{1/2}a$. This value of ζ must correspond to a maximum of the intensity. Accordingly, there must be a minimum of the intensity in the origin. On the other hand, if $w_0 > 2^{1/2}a$, we have only a maximum point at the origin. These behaviours are shown by the curves in figure 7.

Let us now refer to the case of generalized Bessel–Gauss beams. In figure 8 we show the intensity distribution on the $z = 0$ plane for the generalized beam

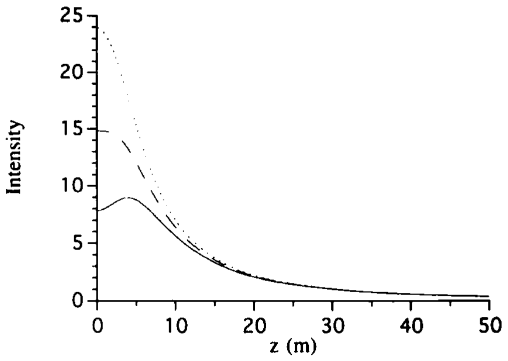

Figure 7. Intensity distribution of the modified beam of zero order on the axis as a function of z, for $w_0 = 1$ mm, $\lambda = 0{\cdot}6328$ μm and $a = 0{\cdot}5$ mm (······), $a = 0{\cdot}7$ mm (---) and $a = 0{\cdot}9$ mm (——).

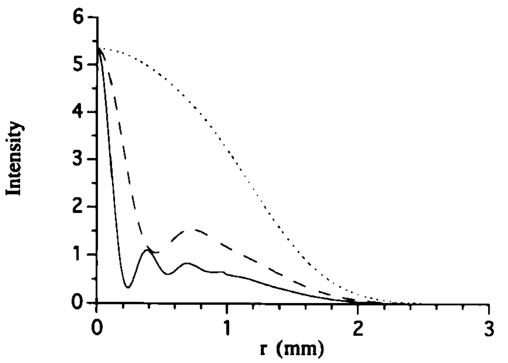

Figure 8. Intensity distribution of the generalized beam of zero order on the waist plane as a function of r, for the values $a = w_0 = 1$ mm, $\lambda = 0{\cdot}6328$ μm, and $\varphi = 10^{-4}$ rad (······), $\varphi = 5 \times 10^{-4}$ rad (---) and $\varphi = 10^{-3}$ rad (——).

of zero order. The parameters are such that, if $\beta = 0$, we have the same result as in the dashed curve in figure 5. We see that, increasing the semiaperture of the cone, an oscillating behaviour appears and becomes more and more evident. From a mathematical point of view, this corresponds to the transition in the modified Bessel function from a purely real argument to a complex argument, whose imaginary part is increasing. From a physical point of view, comparing equations (10) and (17), we see that in the latter an additional phase factor appears, which is different for the various elementary beams. Therefore the elementary contributions to the total field have different phase relationships for different values of r and this causes oscillating behaviour.

Finally, in figure 9 we show the intensity distribution of a generalized beam of zero order as a function of r, at various planes. We see that, for the given choice of the parameters, the distribution resembles that of the ordinary Bessel–Gauss beam [1]. As the distance increases, the transverse distribution assumes a ring shape, like that of the modified beam, with a minimum at the centre.

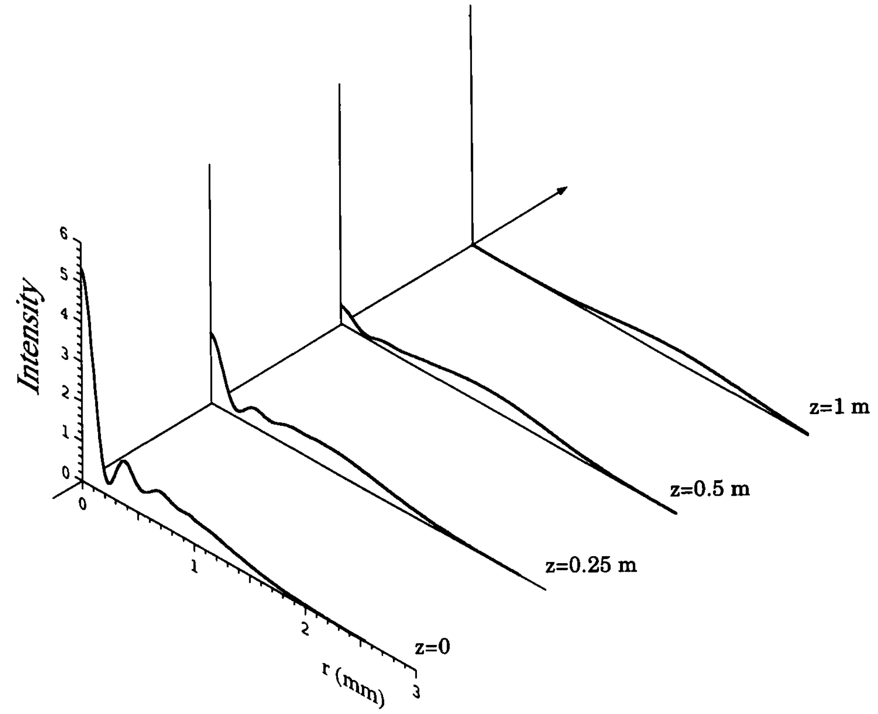

Figure 9. Intensity distribution upon propagation of the generalized beam of zero order as a function of r, for $a = w_0 = 1$ mm, $\lambda = 0{\cdot}6328$ µm and $\varphi = 10^{-3}$ rad at the planes $z = 0$, $z = 0{\cdot}25$ m, $z = 0{\cdot}50$ m and $z = 1$ m.

6. Conclusions

Any coherent light beam may be analytically characterized by an expansion in orthogonal Gaussian modes. Anyway, a superposition of non-orthogonal fields may be physically meaningful too; a classical example is given by the expansion in Gaussian wave packets [24, 25]. The behaviour of the elementary components may give an intuitive explanation about features of the overall field. Incidentally, we recall that geometrical schemes quite similar to those employed here, but involving incoherent superpositions, furnished intuitive models for different classes of partially coherent beams [26–28].

In this paper we have presented some coherent superpositions of non-orthogonal Gaussian beams, giving rise to rather different amplitude profiles and propagation behaviours. A particular superposition generates the well known Bessel–Gauss beam of zero order, but the model accounts in a simple form also for higher-order beams. A slightly different superposition results in modified Bessel–Gauss beams, having an amplitude modulation given by modified Bessel functions. Both the previous cases are contained in a more general superposition scheme.

An efficient system to produce ordinary Bessel–Gauss beams of zero order, making use of a holographic optical element, has been recently proposed [10]. In this paper, we shall limit ourselves to give a few hints about generating in the laboratory some of the fields that we studied. In order to do this, we stress that the field distribution produced by a modified Bessel–Gauss beam in the $z = 0$ plane, as it takes on only real and positive values, can be easily generated by means of pure amplitude transparencies, as photographic emulsions. The same transparencies, may be used in connection with a lens to generate an ordinary Bessel–Gauss beam of the same order. This experimental disposition exploits the Fourier transform relation between the two fields, that we pointed out at the end of section 3. Such a relation, for the beams of zero order, has already been demonstrated by Sheppard [5] and Sheppard and Wilson [6].

Acknowledgment

The authors are very grateful to Professor F. Gori for useful discussions about the topic of this work.

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