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Optimization of Laguerre-Gauss truncated series

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Abstract

When a circularly symmetric function with finite support is expanded into a series of Laguerre-Gauss functions the pertaining spot-size can be chosen at will. However, if the series is truncated the choice of the spot-size affects the truncation error. We suggest a simple rule for minimizing the error. The application to the circ function is discussed in some detail.

1. Introduction

Laguerre-Gauss beams [1,2] afford a simple means for studying propagation of virtually any field in the paraxial regime. Let us refer to the field distributions across the waist plane of these beams, say the plane $z = 0$, as the Laguerre-Gauss functions (LG from now on). It is well known that the set of these functions is complete in L^2, namely the space of square integrable functions in the plane. As a consequence, any prescribed field at $z = 0$ can be expanded into a series of LG functions provided only that it belongs to L^2, a fact which can be taken for granted in most cases. In order to evaluate the field at a typical plane $z = \text{const} \neq 0$, we simply sum up the propagated LG beams with the same expansion coefficients as at $z = 0$.

A typical set of LG beams is characterized by a particular value of the spot-size, say v_0, at the waist. As far as the completeness of the LG functions is concerned the choice of v_0 is immaterial. Nonetheless, when the series expansion of a given function is to be truncated, which is usually the case for computational needs, we expect that the value of v_0 should affect the truncation error. Let us give a self- evident example. A purely Gaussian distribution with a given spot-size w_0 can indeed be expanded into a series of LG functions with arbitrarily chosen spot-size v_0. Consider now the limiting case in which the series is truncated to the first term only. Then, while the choice $v_0 = w_0$ ensures zero error any other value of v_0 leads to a truncation error different from zero.

In this paper we shall restrict our attention to the case of circularly symmetric field distributions, for which the subset of LG functions that do not depend on the angular coordinate is enough. We discuss some features of LG expansions that can be of help for finding the optimum value of v_0 to be used for a given function. After some general remarks we restrict our attention to functions with finite support, for which a simple optimization rule is suggested. Such a rule is substantiated and numerically well confirmed for the important case of the circ function.

2. Laguerre-Gauss expansions

The basis functions that we are going to use are

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$$\psi_n(r;v_0)=\sqrt{\frac{2}{\pi}}\frac{1}{v_0}L_n\left(\frac{2r^2}{v_0^2}\right)\exp\left(-r^2/v_0^2\right),\quad n=0,1,2,\ldots, \tag{1}$$

where L_n denotes the nth Laguerre polynomial [4], and $r=|\boldsymbol{r}|$ is the length of the position vector in $\mathcal{R}^2$. They obey the orthonormality condition

$$\int_{\mathcal{R}^2}\psi_n(r;v_0)\psi_m(r;v_0)\,\mathrm{d}\boldsymbol{r}=\delta_{nm},\quad n,m=0,1,2,\ldots \tag{2}$$

where δ_{nm} is the Kronecker symbol. Once the first two functions

$$\psi_0(r;v_0)=\sqrt{\frac{2}{\pi}}\frac{1}{v_0}\exp\left(-\frac{r^2}{v_0^2}\right),$$
$$\psi_1(r;v_0)=\sqrt{\frac{2}{\pi}}\frac{1}{v_0}\left(1-\frac{2r^2}{v_0^2}\right)\exp\left(-\frac{r^2}{v_0^2}\right), \tag{3}$$

have been computed, any other function can be generated by the recurrence relation

$$\psi_{n+1}(r;v_0)=\frac{1}{n+1}\left[\left(2n+1-\frac{2r^2}{v_0^2}\right)\psi_n(r;v_0)-n\psi_{n-1}(r;v_0)\right],\quad n\geq 1. \tag{4}$$

Let us consider any circularly symmetric function $f(r)\in L^2\left(\mathcal{R}^2\right)$. Its series expansion reads

$$f(r)=\sum_{n=0}^{\infty}f_n(v_0)\,\psi_n(r;v_0), \tag{5}$$

where the coefficients are evaluated as follows

$$f_n(v_0)=\int_{\mathcal{R}^2}f(r')\psi_n(r';v_0)\,\mathrm{d}\boldsymbol{r}'. \tag{6}$$

We used the notation $f_n(v_0)$ to point out that the coefficients depend on v_0. Suppose now that the series is truncated after the Nth term. This gives rise to an Nth order approximation of $f(r)$, denoted by $f^{(N)}(r;v_0)$,

$$f^{(N)}(r;v_0)=\sum_{n=0}^{N}f_n(v_0)\,\psi_n(r;v_0). \tag{7}$$

It is possible to think of $f^N(r;v_0)$ as the result of passing $f(r)$ through a linear system characterized by the kernel [5]

$$K^{(N)}(r,r';v_0)=\sum_{n=0}^{N}\psi_n(r;v_0)\,\psi_n(r';v_0). \tag{8}$$

In fact we have

$$f^{(N)}(r;v_0)=\int_{\mathcal{R}^2}K^{(N)}(r,r';v_0)f(r')\,\mathrm{d}\boldsymbol{r}', \tag{9}$$

as can be easily seen by taking into account Eqs. (5)–(8). The advantage of this approach is that we can gain some insight into the quality of the Nth approximation in a rather general way, i. e. without making reference to a particular form of $f(r)$. This can be obtained by shifting our attention to the behaviour of $K^{(N)}$ as a function of r, r' and v_0.

Although the functions of interest are defined on $\mathcal{R}^2$ the problem is actually one-dimensional because of the restriction to circularly symmetric functions. Accordingly, we perform the trivial integration on the angular coordinate and write

$$f^{(N)}(r;v_0)=\int_0^{\infty}H^{(N)}(r,r';v_0)f(r')\,\mathrm{d}r', \tag{10}$$

where

$$H^{(N)}(r,r';v_0)=2\pi r'K^{(N)}(r,r';v_0). \tag{11}$$

From now on we shall refer to $H^{(N)}$ as the impulse response of the truncated series expansion.

On inserting Eqs. (1) and (8) into Eq. (11) we have

$$H^{(N)}(r,r';v_0)=\frac{4}{v_0}\left(\frac{r'}{v_0}\right)\exp\left(-\frac{r^2+r'^2}{v_0^2}\right)\times\sum_{n=0}^{N}L_n\left(\frac{2r^2}{v_0^2}\right)L_n\left(\frac{2r'^2}{v_0^2}\right). \tag{12}$$

We see that, apart from a proportionality factor, the role of v_0 is simply to specify the scale of the variables r and r'. Accordingly, for any value of N, we can get an idea of the behaviour of $H^{(N)}$ by drawing some graphs of it as a function of r' for fixed values of r and

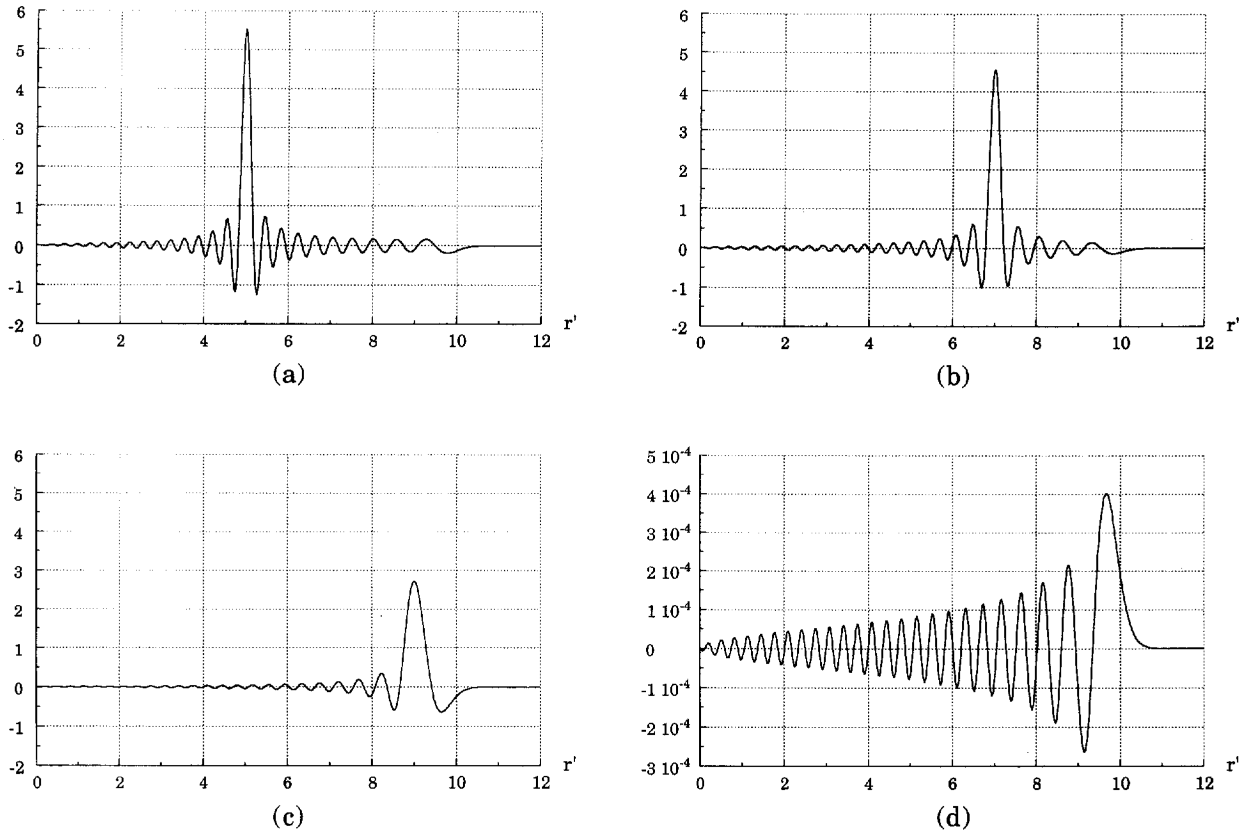

Fig. 1. Behaviour of the kernel $H^{(N)}(r,r';v_0)$ versus r' for $N=49$. Values of r are (a) 5, (b) 7, (c) 9 and (d) 11.

for $v_0=1$. This is done for $N=49$ and $r=5,7,9,11$ in Figs. 1a–d.

Fig. 1 illustrates some features of the impulse response that could be seen on any set of similar curves drawn for other values of N. Roughly speaking, the interval $[0,\infty)$ can be divided into three regions. The first one is $[0,\sqrt{N}]$. Here (see Figs. 1a,b) the impulse response is reasonably good (according to the usual assessment criteria) although its width, no matter how we define it precisely, is a slowly increasing function of r. The second region is $[\sqrt{N},\sqrt{2N}]$. The impulse response now becomes markedly nonsymmetrical with respect to r (Fig. 1c) and its maximum value gets smaller than in the first region. Finally, we have the region $[\sqrt{2N},\infty)$. Here (Fig. 1d) the impulse response is unacceptably bad. Not only is its maximum value drastically reduced but the main peak itself is no longer centered at r.

Although this behaviour may sound surprising when compared with that of the more or less shift-invariant impulse responses of most optical systems it can be easily explained. To this aim, we recall that a LG function has an oscillatory behaviour in which the distance between adjacent zeroes increases progressively [6]. This accounts for the width increase exhibited by the impulse response as a function of r. What is more, a LG function of order n becomes negligibly small when r' exceeds some maximum value. Roughly speaking such a value is $r'=\sqrt{2n}$ [4]. For this reason, when the point r is beyond $\sqrt{2N}$ the impulse response itself is exceedingly small around such a point.

Suppose now that the series expansion of a typical function $f(r)$ is replaced by its Nth order approximation $f^{(N)}(r;v_0)$. Taking into account the previous discussion we expect $f(r)$ to be reproduced by $f^{(N)}(r;v_0)$ with a varying degree of approximation. More specifically the quality of $f^{(N)}(r;v_0)$ should be acceptable up to a certain value of r and then become very poor. The interval where the truncated expansion should work in a satisfactory way will be loosely referred to as the useful interval. In view of the behaviour of the impulse response described above we estimate the useful interval of r' to be $[0, \sqrt{N}]$. Needless to say, such an estimate contains a certain degree of arbitrariness. Quantitative results will be seen in the next section.

Let us discuss how the previous estimate can be of help in dealing with LG truncated series expansions. We shall limit ourselves to the case of functions with finite support of the form $[0, a]$. If we want any such function to be represented by its Nth order approximation with an acceptable uniformity the entire support of the function must fall within the useful interval. On the other hand, there is no point in making this support smaller than the useful interval. This would only reduce the resolution with which the function is reproduced. In conclusion, the best choice seems to be the following: make the useful interval coincide with the function support. As an example we give in Fig. 2 the curves obtained with the truncated series expansion of

$$\mathrm{circ}\left(\frac{r}{a}\right) = \begin{cases} 1, & 0 \le r \le a \\ 0, & r > a \end{cases} \tag{13}$$

for $N = 49$ and a few values of a letting $v_0 = 1$. The series expansion of this function will be discussed in the next section. For the moment we content ourselves with a glance at the graphical results. In Fig. 2 the 49th order approximation of $\mathrm{circ}(r/a)$ is given for (a) $a = 1$, (b) $a = 2$, (c) $a = 5$, (d) $a = 7$, (e) $a = 10$ and (f) $a = 14$. It is seen that these curves agree with our previous predictions. In particular, it seems that the better approximation is obtained when $a = \sqrt{N} = 7$ (Fig. 2d). Furthermore, it should be noted that the curve for $a = 14$ (Fig. 2f) is not only highly oscillating but also fails to reproduce correctly the width of the circ, the transition to very small values occurring at $r \simeq 10$ instead of $r = 14$.

Any finite support $[0, a]$ can be scaled to fit into the interval $[0, 1]$ by a suitable choice of the unit length. Let us assume that such a scaling operation is performed. Then, for any fixed N the matching between $[0, 1]$ and the useful interval is obtained by properly changing v_0. Hence the above discussion leads to the following rule of thumb: *"When a function $f(r)$ with finite support $[0, 1]$ is to be approximated by a LG series truncated to the Nth term the spot-size v_0 should equal to $1/\sqrt{N}$".*

3. Truncated optimization of circ

The circ function is particularly significant in Optics. In fact it describes, up to some constant factor, the field emerging from a circular hole under illumination with an orthogonal plane wave. The evaluation of the diffracted field in the paraxial approximation has defied the ingenuity of researchers since Fresnel's days [3,7–11]. Further, the circ plays a similar role in antenna theory [12]. Therefore, its expansion into LG functions deserves some attention. Examples can be found in Refs. [13,14] while numerical estimates of the best v_0 are given in Ref. [15] for small values of N. In this section we shall test the rule established above by an explicit evaluation of the error induced by a truncation of the series expansion of circ. According to the choice made at the end of the previous section we let $a = 1$ and write

$$\mathrm{circ}(r) = \sum_{n=0}^{\infty} f_n(v_0)\psi_n(r; v_0), \tag{14}$$

where the coefficients f_n are defined by (see Eq. (6))

$$f_n(v_0) = \int_{\mathcal{R}^2} \mathrm{circ}(r)\, \psi_n(r; v_0)\, \mathrm{d}\boldsymbol{r}. \tag{15}$$

The f_n are easily evaluated, and the following recurrence relation can be established [16]

$$\begin{aligned} f_0(v_0) &= \sqrt{2\pi}\, v_0 [1 - \exp(-1/v_0^2)], \\ f_1(v_0) &= \sqrt{2\pi}\, v_0 \left[\left(1 - \frac{2}{v_0^2}\right) \exp\left(-\frac{1}{v_0^2}\right) - 1\right], \\ f_{n+1}(v_0) &= \frac{1}{n+1}\left[\sqrt{2\pi}\, v_0 L_n\left(\frac{2}{v_0^2}\right) \exp\left(-\frac{1}{v_0^2}\right) \right. \\ &\qquad \left. - f_n(v_0) + n f_{n-1}(v_0)\right], \\ &\qquad n = 1, 2, \ldots . \end{aligned} \tag{16}$$

For any fixed value of v_0, the Nth order approximation obtained on inserting Eq. (16) into Eq. (7) minimizes the mean squared error [17]

$$\varepsilon_N(v_0) = \|\mathrm{circ} - f^{(N)}\|^2, \tag{17}$$

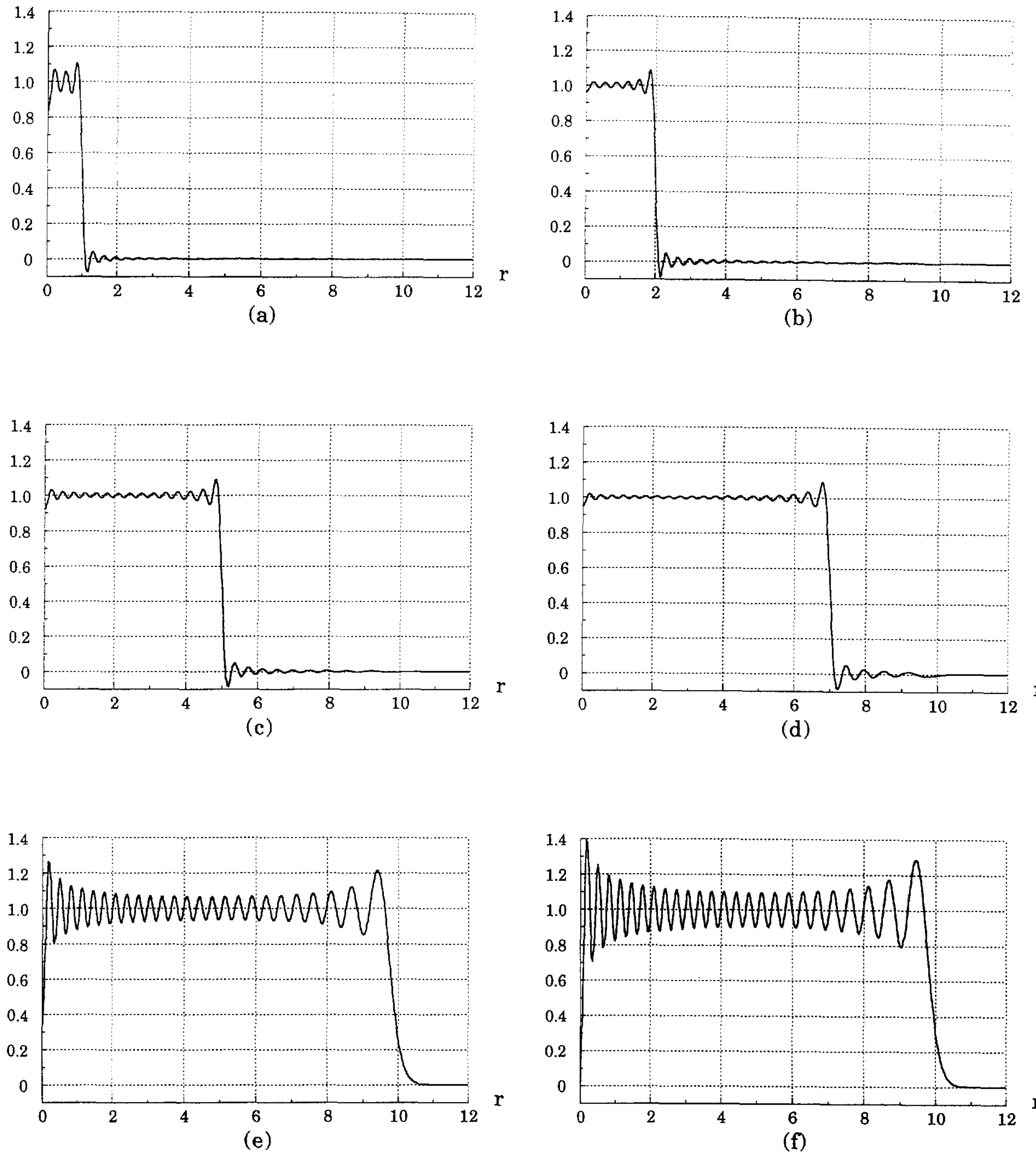

Fig. 2. Approximation of circ(r/a) by Nth order Laguerre-Gauss expansion for $N = 49$ and: (a) $a = 1$, (b) $a = 2$, (c) $a = 5$, (d) $a = 7$, (e) $a = 10$ and (f) $a = 14$.

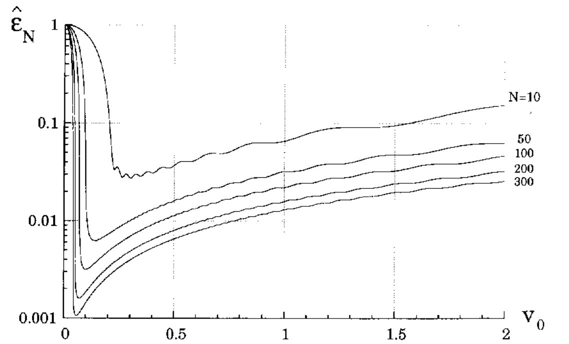

Fig. 3. Behaviour of $\hat{\varepsilon}_N$ as a function of v_0 for different values of N.

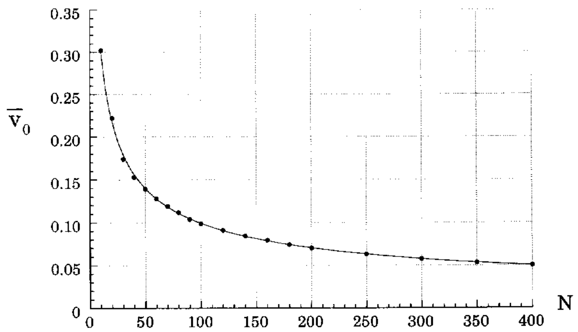

Fig. 4. Plot of $\bar{v}_0$ as a function of N (dots) and fitting curve $1/\sqrt{N}$ (full line).

where for a typical function $g(r)$ the norm $\|g\|$ is defined through the relation

$$\|g\|^2 = \int_{\mathcal{R}^2} |g(r)|^2 \mathrm{d}\boldsymbol{r}. \tag{18}$$

It is convenient to introduce a normalized mean squared error, $\hat{\varepsilon}_N$, which turns out to be

$$\hat{\varepsilon}_N(v_0) = \frac{\varepsilon_N(v_0)}{\|\mathrm{circ}\|^2} = 1 - \frac{1}{\pi}\sum_{n=0}^{N} f_n^2(v_0), \tag{19}$$

where Eqs. (13) and (18) have been used, and the Bessel equality [17] has been taken into account. Incidentally, with reference to the propagation problem mentioned in the Introduction, we note that $\hat{\varepsilon}_N$ is independent of z, because the free-space propagation operator is unitary. In other words, this means that propagated Laguerre-Gauss beams keep their orthonormality on each $z > 0$ plane [1,2,16].

Once N has been given the truncation error (19) can be easily computed as a function of v_0. Examples are given in Fig. 3 for a few values of N. It should be noted that especially for large N small changes of v_0 can produce drastic variations of $\hat{\varepsilon}_N$. As was expected, due to the presence of a discontinuity at $r = 1$, a rather high number of modes ψ_n is required to obtain small values of $\hat{\varepsilon}_N$.

It is also seen that for each N an absolute minimum of $\hat{\varepsilon}_N$ can be found. Let us denote by $\bar{v}_0$ the corresponding value of v_0. In order to find $\bar{v}_0$ as a function of N some minimization algorithm must be adopted. Because of the presence of local minima, classical methods based on gradient search (steepest descent, conjugate directions, etc.) [18] cannot be used in a profitable way, whereas global optimization techniques can be successfully employed. In our work, we used the simulated annealing method [18]. The results are shown in Fig. 4, where $\bar{v}_0$ is plotted as a function of N (dots). For the sake of comparison the curve $\bar{v}_0 = 1/\sqrt{N}$ predicted by our rule is drawn as a full line. It is seen that the agreement is remarkably good.

4. Conclusions

In this paper we suggested a simple rule for choosing the spot-size of LG functions that minimizes the truncation error in the expansion of a function with finite support. In the case of circ function such a rule is in very good agreement with the result obtained by direct numerical search. When the number of terms to be kept in the truncated series is given the rule allows us to make the most efficient use of such terms. On the other hand, once the requested error level has been fixed, the rule leads to utilize the minimum number of terms in the orthonormal expansion.

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