Converting states of a particle under uniform or elastic forces into free particle states

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Received 24 November 1994

Abstract. We show that there is an explicit one-to-one correspondence between the wavefunctions of a free particle and those of a particle subjected to a uniform force field. A similar correspondence can be established between free particle and harmonic oscillator or, more generally, displaced harmonic oscillator. Simple applications to the evaluation of propagators are presented and elementary examples are discussed.

Résumé. Nous montrons qu'il y a une correspondance explicite entre les fonctions d'onde de la particule libre et celles d'une particule dans un champ uniforme. Une correspondance similaire peut être déduite aussi entre les fonctions d'onde d'une particule libre et celles d'un oscillateur linéaire éventuellement déplacé. Nous presentons des applications simples au calcul des propagateurs et nous discutons des exemples élémentaires.

1. Introduction

The free particle (FP) is the simplest system to be studied in quantum mechanics. In particular, the time evolution of its wavefunction can be determined by evaluating a Fresnel transform (Feynman and Hibbs 1965). Unfortunately, its relevance for physical problems is rather limited. The particle under a uniform force (PUF) is a slightly more important system. Nonetheless, it does not receive much attention in textbooks, the usual treatment, if any, being restricted to stationary states described by the rather cumbersome Airy function (Landau and Lifshitz 1965). Therefore it seems of some interest to show that any wavefunction of the FP can be transformed into one of the PUF (and vice versa) by means of a simple correspondence rule. What is more, a similar rule also connects the states of the FP to those of the harmonic oscillator (HO) which is of the utmost importance. Furthermore, combining these results it is easily shown that the FP states can be uniquely mapped onto those of the displaced oscillator (DHO), i.e. of a particle under the action of both a uniform and an elastic force.

The above correspondence rules will be proved by exploiting suitable changes of variables suggested by classical analogies.

As a simple application we will show that the propagators of the PUF, HO and DHO can be derived at once from that of the FP. Specific examples of corresponding states will be worked out starting from the well known Gaussian wavefunction for the FP. Some connection between our quantum results and problems of classical optics will also be outlined.

Throughout the paper we shall refer to the one-dimensional case. The extension to two and three dimensions is straightforward. The acronyms introduced above will also be used as subscripts. For example, ψ_{HO} stands for a wavefunction of the harmonic oscillator.

2. Classical remarks

This section is a reminder that simple changes of variables can formally convert the classical laws of motion of both the PUF and the HO into the law pertaining to the FP. These changes will be the starting point for establishing the correspondence rules in the quantum case. We add that classical solution variables have been used recently to derive new states of the FP (Fradkin 1994).

Let us begin with the law of motion of a particle with mass m subjected to a uniform force f (the

PUF system)

$$x = x_i + v_i t + \frac{ft^2}{2m}, \tag{2.1}$$

where x_i and v_i denote the initial values of the position and the velocity respectively. By introducing the new variables

$$\xi(x,t) = x - \frac{ft^2}{2m}; \qquad \tau(t) = t, \tag{2.2}$$

equation (2.1) is immediately converted into the FP law

$$\xi = x_i + v_i \tau. \tag{2.3}$$

A similar result can be obtained for a HO with angular frequency ω. Indeed if the variables

$$\xi(x,t) = \frac{x}{\cos(\omega t)}; \qquad \tau(t) = \frac{\tan(\omega t)}{\omega}, \tag{2.4}$$

are inserted into the law of motion

$$x = x_i \cos(\omega t) + \frac{v_i}{\omega}\sin(\omega t), \tag{2.5}$$

we arrive again at equation (2.3). In this case, however, singularities occur for $t = \pm T/4$, where $T = 2\pi/\omega$. For our purposes, it is sufficient to consider only the interval

$$-\tfrac{1}{4}T < t < \tfrac{1}{4}T. \tag{2.6}$$

3. The FP–PUF correspondence

We start from the Schrödinger equation for the PUF

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + i\hbar\frac{\partial\psi}{\partial t} + fx\psi = 0. \tag{3.1}$$

where of course $\hbar$ stands for the Planck constant divided by 2π and ψ denotes the wavefunction. Using the classical analogy as a guide we introduce the variables (2.2) and write ψ in the form

$$\psi(x,t) = \phi(\xi,\tau)F(x,t). \tag{3.2}$$

We then ask whether it is possible to find an explicit expression for F such that ϕ satisfies the FP equation

$$\frac{\hbar}{2m}\frac{\partial^2\phi}{\partial\xi^2} + i\frac{\partial\phi}{\partial\tau} = 0. \tag{3.3}$$

If this occurs then any FP solution furnishes a PUF solution by means of equation (3.2). We shall show in a moment that this is indeed the case. To this aim, we make the tentative hypothesis that F itself is a solution of equation (3.1). Using such a hypothesis as well as equation (3.3) we easily find that when (3.2) is inserted into (3.1) the latter reduces to

$$\frac{\partial F}{\partial x} = \frac{i}{\hbar} ftF, \tag{3.4}$$

whose solution is

$$F(x,t) = \exp\left[\frac{i}{\hbar} ftx + g(t)\right], \tag{3.5}$$

where a possible proportionality factor has been included into $\exp[g(t)]$, $g(t)$ being an arbitrary function of time. Now the problem is: can a function $g(t)$ be found such that F is actually a solution of equation (3.1) as we assumed? On inserting equation (3.5) into (3.1) we see that the answer is positive provided that g satisfies the equation

$$\frac{dg}{dt} = -\frac{i}{\hbar}\frac{f^2t^2}{2m}. \tag{3.6}$$

Therefore, we have proved that the function

$$F(x,t) = \exp\left[\frac{i}{\hbar}\left(ftx - \frac{f^2t^3}{6m}\right)\right], \tag{3.7}$$

has the required properties so that equation (3.2) holds true with ϕ satisfying (3.3). We let $g(0) = 0$ to simplify further analysis. We can express our result in a more transparent form by writing

$$\psi_{PUF}(x,t) = \psi_{FP}\left(x - \frac{ft^2}{2m}, t\right)\exp\left[\frac{i}{\hbar}\left(ftx - \frac{f^2t^3}{6m}\right)\right]. \tag{3.8}$$

This means that there is a one-to-one correspondence between the sets of solutions of the Schrödinger equation for the FP and the PUF. Equation (3.8) expresses the correspondence rule. It is worthwhile to note that, as a consequence of the choice $g(0) = 0$,

$$\psi_{PUF}(x,0) = \psi_{FP}(x,0). \tag{3.9}$$

Accordingly, the two systems, namely the PUF and the FP, start from the same initial state. Their different time evolution is accounted for by equation (3.8) through a shift of the space variable and a phase factor. It is also seen that there is a very simple relation between the position probability densities for the two systems

$$|\psi_{PUF}(x,t)|^2 = \left|\psi_{FP}\left(x - \frac{ft^2}{2m}, t\right)\right|^2. \tag{3.10}$$

Therefore, in the course of time the PUF density maintains the same shape as the FP density except that it moves at the constant classical acceleration with respect to the FP. Of course, once the shape equality has been ascertained the additional PUF motion can be thought of as a consequence of Ehrenfest's theorem (Merzbacher 1970).

4. The FP–HO correspondence

The correspondence rule for the HO can be established by repeating the same steps as in section 3. Before doing so we recall (Schiff 1968) that the wavefunction $\psi(x,t)$ for the HO is a periodic function of time except for a phase factor $\exp(-\mathrm{i}\omega t/2)$. In addition, it is easily shown that

$$\psi(x, t+\tfrac{1}{2}T) = -\mathrm{i}\psi(-x,t). \tag{4.1}$$

Accordingly, it is only necessary to know the wavefunction for half a period. As in section 2, we shall limit the time variable to the interval $-T/4 < t < T/4$. The Schrödinger equation for the HO reads

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\mathrm{i}}{\omega x_0^2}\frac{\partial\psi}{\partial t} - \frac{x^2}{4x_0^4}\psi = 0, \tag{4.2}$$

where

$$x_0^2 = \frac{\hbar}{2m\omega}. \tag{4.3}$$

We now introduce the variables (2.4) and let

$$\psi(x,t) = \phi(\xi,\tau)G(x,t). \tag{4.4}$$

Next, we inquire whether a form of G can be found such that ϕ satisfies the FP equation (3.3). On assuming tentatively that G is a solution of equation (4.2) and inserting (4.4) into (4.2) we arrive at the equation

$$\frac{\partial G}{\partial x} = -\mathrm{i}\frac{x}{2x_0^2}\tan(\omega t)G, \tag{4.5}$$

whose solution is of the form

$$G(x,t) = \exp\left[-\mathrm{i}\frac{x^2}{4x_0^2}\tan(\omega t) + g(t)\right]. \tag{4.6}$$

The function $g(t)$ is to be determined by requiring that G satisfies equation (4.2). This happens when g is a solution of the following equation

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\omega}{2}\tan(\omega t). \tag{4.7}$$

Integrating equation (4.7) and inserting the result into equation (4.6) we have

$$G(x,t) = \frac{\exp[-\mathrm{i}(x^2/4x_0^2)\tan(\omega t)]}{\sqrt{|\cos(\omega t)|}}. \tag{4.8}$$

In conclusion, the following correspondence rule has been proved within the interval (2.6)

$$\psi_{\mathrm{HO}}(x,t) = \psi_{\mathrm{FP}}\left(\frac{x}{\cos(\omega t)}, \frac{\tan(\omega t)}{\omega}\right) \times \frac{\exp[-\mathrm{i}(x^2/4x_0^2)\tan(\omega t)]}{\sqrt{\cos(\omega t)}}. \tag{4.9}$$

Equations (3.8) and (3.9) are now replaced by

$$\psi_{\mathrm{HO}}(x,0) = \psi_{\mathrm{FP}}(x,0), \tag{4.10}$$

$$|\psi_{\mathrm{HO}}(x,t)|^2 = \frac{1}{\cos(\omega t)}\left|\psi_{\mathrm{FP}}\left(\frac{x}{\cos(\omega t)}, \frac{\tan(\omega t)}{\omega}\right)\right|^2. \tag{4.11}$$

In this case too we have two different systems that start from the same initial state. Their different time evolution is described by equation (4.9). We note that the factor $1/\cos(\omega t)$ in equation (4.11) ensures probability conservation. In other words, if ψ_{FP} is normalized the same is true for ψ_{HO}. As far as the space dependence is concerned equation (4.11) shows that the density pertaining to the HO is obtained from that of the FP by a suitable scale compression along the x-axis. This can be justified by observing that the wavefunction of a FP generally tends to spread out as time progresses whereas the HO wavefunction remains somewhat more concentrated as a result of the action of the elastic force. Time scales are also different. We see in fact that it takes an infinite time span (from $-\infty$ to ∞) for the FP to simulate the time evolution of the HO in $(-T/4, T/4)$. This again is to be ascribed to the elastic force. Because of its presence the HO wavefunction is forced to trace back its own behaviour periodically up to a phase factor. In addition, owing to equation (4.1) the whole time evolution is specified by half a period only. No such effects are present for the FP and this is why its time evolution can only account for half a period of the HO.

5. The FP–DHO correspondence

Let us consider a particle that is subjected to both a uniform and an elastic force (the DHO system). We want to extend our previous correspondence rules to the present case. The task is rather easy. The proper Schrödinger equation is

$$\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \mathrm{i}\hbar\frac{\partial\psi}{\partial t} + \left(fx - \frac{m\omega^2}{2}x^2\right)\psi = 0. \tag{5.1}$$

We let

$$X = \frac{f}{m\omega^2}, \tag{5.2}$$

and use equation (4.3). Then equation (5.1) can be written

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\mathrm{i}}{\omega x_0^2}\frac{\partial\psi}{\partial t} + \frac{(x-X)^2}{4x_0^4}\psi + \frac{X^2}{4x_0^2}\psi = 0. \tag{5.3}$$

Now, it is easily shown that the wavefunctions ψ_{DHO} satisfying equation (5.3) can be derived from those of the simple HO by means of the relation

$$\psi_{\mathrm{DHO}}(x,t) = \psi_{\mathrm{HO}}(x-X,t)\exp\left[\mathrm{i}\frac{X^2}{4x_0^2}\omega t\right]. \tag{5.4}$$

Therefore, using equation (4.9) we obtain

$$\psi_{\mathrm{DHO}}(x,t) = \psi_{\mathrm{FP}}\left(\frac{x-X}{\cos(\omega t)}, \frac{\tan(\omega t)}{\omega}\right) \times \frac{\exp\{-(\mathrm{i}/4x_0^2)[(x-X)^2\tan(\omega t) - X^2\omega t]\}}{\sqrt{\cos(\omega t)}}, \tag{5.5}$$

where again the time variable is contained within the interval (2.6). On comparing this result to equation (4.9) it is seen that except for a time-dependent phase factor the main difference is the displacement of the centre of the oscillatory motion.

6. Evaluation of the propagators

Let us recall that the time evolution of the wavefunction $\psi_{FP}(x,t)$ for the FP is fully determined by the initial distribution $\psi_{FP}(x,0)$ by means of the integral (Feynman and Hibbs 1965)

$$\psi_{FP}(x,t)=\int_{-\infty}^{\infty}\psi_{FP}(y,0)K_{FP}(x,y,t)\,dy, \tag{6.1}$$

where the propagator K_{FP} is given by

$$K_{FP}(x,y,t)=\sqrt{\frac{m}{iht}}\exp\left[i\frac{m\pi}{ht}(x-y)^2\right]. \tag{6.2}$$

Such explicit expression for the propagator can be established in a rather simple way (Merzbacher 1970). Of course, an evolution integral similar to equation (6.1) also holds for the PUF, HO and DHO but the evaluation of the corresponding propagators is not obvious. An immediate derivation is afforded by the correspondence rules. This is because the propagator is a particular solution of the pertaining Schrödinger equation (it is in fact its Green function). As a consequence, in order to find the required propagators we have only to apply the correspondence rules to equation (6.2). For example, equation (3.8) gives at once for the PUF

$$K_{PUF}(x,y,t)=\sqrt{\frac{m}{iht}}\exp\left[i\frac{m\pi}{ht}\left(x-\frac{ft^2}{2m}-y\right)^2\right]\times\exp\left[\frac{i}{\hbar}\left(ftx-\frac{f^2t^3}{6m}\right)\right]. \tag{6.3}$$

For the HO we use equation (4.9) taking also equation (4.3) into account. The result is

$$\begin{aligned}K_{HO}(x,y,t)&=\sqrt{\frac{m\omega}{ih\sin(\omega t)}}\times\exp\left[i\frac{m\omega\pi}{h\tan(\omega t)}\left(\frac{x}{\cos(\omega t)}-y\right)^2-i\frac{x^2}{4x_0^2}\tan(\omega t)\right]\\&=\sqrt{\frac{m\omega}{ih\sin(\omega t)}}\exp\left[i\frac{m\omega\pi}{h\tan(\omega t)}\times\left(x^2+y^2-2\frac{xy}{\cos(\omega t)}\right)\right].\end{aligned} \tag{6.4}$$

Finally, for the DHO *we obtain from equation (5.5)*

$$\begin{aligned}K_{DHO}(x,y,t)&=\sqrt{\frac{m\omega}{ih\sin(\omega t)}}\times\exp\left\{i\frac{m\omega\pi}{h\tan(\omega t)}\left(\frac{x-X}{\cos(\omega t)}-y\right)^2-\frac{i}{4x_0^2}[(x-X)^2\tan(\omega t)-X^2\omega t]\right\}\\&=\sqrt{\frac{m\omega}{ih\sin(\omega t)}}\exp\left[i\frac{m\omega\pi}{h\tan(\omega t)}\times\left((x-X)^2+y^2-\frac{2(x-X)y}{\cos(\omega t)}\right)^2+i\frac{X^2\omega t}{4x_0^2}\right].\end{aligned} \tag{6.5}$$

7. Examples of corresponding states

We now discuss the application of the correspondence properties to a specific state of the FP. Let us consider the following well known solution of the Schrödinger equation for the FP (Schiff 1968)

$$\psi_{FP}(x,t)=\frac{(2\pi)^{-1/4}}{\sqrt{\Delta x(1+i\Omega t)}}\times\exp\left[-\frac{(x-v_it)^2}{4\Delta x^2(1+i\Omega t)}+i\frac{mv_i}{\hbar}\left(x-\frac{v_i}{2}t\right)\right], \tag{7.1}$$

where

$$\Omega=\frac{\hbar}{2m\Delta x^2}. \tag{7.2}$$

The symbol v_i now stands for the expected value of the velocity whereas Δx is the initial width of the wavepacket. Both of them can be chosen arbitrarily. *The corresponding probability density, namely*

$$|\psi_{FP}(x,t)|^2=\frac{1}{\sqrt{2\pi\Delta x^2(1+\Omega^2t^2)}}\times\exp\left[-\frac{(x-v_it)^2}{2\Delta x^2(1+\Omega^2t^2)}\right], \tag{7.3}$$

describes a Gaussian wavepacket whose centre moves with the velocity v_i. For increasing values of t the width of the packet gets larger and larger.

The probability density of the corresponding state of the PUF is obtained at once from equation (3.10)

$$|\psi_{PUF}(x,t)|^2=\frac{1}{\sqrt{2\pi\Delta x^2(1+\Omega^2t^2)}}\times\exp\left\{-\frac{[x-v_it-(ft^2/2m)]^2}{2\Delta x^2(1+\Omega^2t^2)}\right\}. \tag{7.4}$$

This is a very simple example of how the correspondence rule allows us to give a non-trivial solution of equation (3.1). In view of its simplicity, equation (7.4) should help the student to appreciate the quantum features of the particle behaviour as distinguished from the classical ones. In this case, the quantum characteristic is the spreading of the wavepacket, which can be traced back to the initial uncertainty about the momentum. In addition, equation (7.4) exemplifies Ehrenfest's theorem in a clear manner.

Even more significant are the states that are obtained by applying rules (4.9) and (4.11) to equation (7.1). Limiting ourselves to the position probability density for the HO we have

$$|\psi_{\mathrm{HO}}(x,t)|^2 = \frac{1}{\sqrt{2\pi\Delta x^2_{\mathrm{HO}}(t)}}\exp\left[-\frac{(x-v_i\sin(\omega t))^2}{2\Delta x^2_{\mathrm{HO}}(t)}\right], \quad (7.5)$$

where

$$\Delta x^2_{\mathrm{HO}}(t) = \Delta x^2\left[\cos^2(\omega t) + \frac{x_0^2}{\Delta x^4}\sin^2(\omega t)\right]. \quad (7.6)$$

Here, use has been made of equations (4.3) and (7.2) to express the ratio Ω/ω. It is seen that equation (7.5) accounts for an oscillating Gaussian packet. As far as the width of the packet is concerned let us write equation (7.6) in the form

$$\Delta x^2_{\mathrm{HO}}(t) = \Delta x^2 \frac{1+R^4\tan^2(\omega t)}{1+\tan^2(\omega t)}, \quad (7.7)$$

where

$$R = \frac{x_0}{\Delta x}. \quad (7.8)$$

Of course Δx_{HO} remains constant if $\Delta x = x_0$. Therefore, in this case the correspondence rule (4.9) applied to equation (7.1) gives rise to the famous coherent states of the HO (Glauber 1963). Generally speaking Δx_{HO} passes monotonically from x_0/R to Rx_0 when t tends from 0 to $T/4$ (or $-T/4$). Accordingly, an initial squeezing of the packet gives rise to a later increase of the width and vice versa. In conclusion, we reach both coherent and squeezed states (Yariv 1989) with a single procedure. Similar results could be obtained for the DHO.

Some intuitive insights into the above correspondences can be gained through an optical analogy. The FP equation (3.3) is formally identical to the classical equation that describes paraxial propagation of light in a homogeneous medium (Yariv 1989) when the light field depends only on one transverse coordinate (x). The time variable is replaced by an abscissa along the mean direction of propagation and other simple substitutions are to be made. Similarly, equation (3.1) corresponds to light propagation within a medium with a transverse gradient of the refraction index (prism-like medium) while equation (4.2) accounts for a medium with a parabolic transverse index profile (lens-like medium) (Yariv 1989). The rather well known propagation features of light in such media can afford a model for the time evolution of the quantum wavefunction. We will not pursue this matter further but we mention that the connection between paraxial free propagation and HOs has been noted recently (Nienhuis and Allen 1993).

Finally, we want to notice that our present results could be connected to a recent investigation (Agarwal and Simon 1994) about the relationship between the HO and the fractional Fresnel transform (Namias 1980, Mendlovic and Ozaktas 1993, Gori *et al* 1994).

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