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Twisted Gaussian Schell-model beams: a superposition model

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Abstract. We present a simple model in which a twisted Gaussian Schell-model beam is produced by an incoherent superposition of ordinary Gaussian beams. The meaning and some direct consequences of the model are given.

1. Introduction

Astigmatic Gaussian beams of the general type [1] have elliptically shaped lines of equal intensity at any cross-section. Upon propagation, the beam twists around its axis in that the ellipses rotate progressively. This is perhaps the oldest example of a coherent twisting beam. In recent times, the twisting phenomenon has acquired new relevance in the problem of characterizing actual laser beams [2] as well as in the study of optical vortices [3]. Recently, Simon and Mukunda [4] have introduced a type of partially coherent field that they called the twisted Gaussian Schell-model (TGSM) beam. This is an axially symmetric beam in which the cross spectral density [5] across the beam possesses a phase term called the twist phase whose presence has relevant effects on the beam behaviour upon propagation. A thorough analysis of the TGSM beam has been given in [4, 6]. In addition, Wolf's [7] modal expansion for such a field has been determined.

Simon and Mukunda were led to include the twist phase term in the cross spectral density by a symmetry argument. Although they also suggested a practical scheme for synthesizing a TGSM beam starting from an anisotropic Li–Wolf [8] source, the meaning of the twist phase remains rather difficult to understand. In this paper, we present a simple model in which a TGSM beam is produced by an incoherent superposition of ordinary Gaussian beams. It will be seen that the main properties of the TGSM beam acquire an immediate meaning once such a model is adopted.

The material is organized as follows. In section 2 we review briefly the TGSM beam and we discuss the meaning as well as the consequences of our model. A mathematical description of the superposition model is given in section 3. A generalized version of the superposition scheme is described in section 4.

2. The twisted Gaussian Schell-model beams: a superposition model

TGSM beams have been introduced by Simon and Mukunda [4] as the partially coherent fields having the most general axially symmetric Gaussian cross spectral density. Denoting by (x_i, y_i), $i = 1, 2$, the Cartesian coordinates of two typical points in the waist plane, such a cross spectral density, say W_0, has the form

$$W_0(x_1, y_1; x_2, y_2) = I_0 \exp\left(-\frac{x_1^2 + y_1^2 + x_2^2 + y_2^2}{4\sigma_1^2} - \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2\sigma_\mu^2} - iku(x_1 y_2 - x_2 y_1)\right), \tag{1}$$

where σ_1 denotes the effective beam width, σ_μ is the (transverse) coherence length, k is the wavenumber and u is a real parameter. Equation (1) shows that such fields are genuinely two dimensional in the sense that they are not separable into a product of two one-dimensional fields [6]. As far as propagation is concerned, a TGSM beam differs from an untwisted beam for a substantial increase in the angular divergence. The parameter u is referred to as the twist parameter, because it is related to the twisting property that could be revealed by breaking the axial symmetry. It disappears in the coherent limit, $\sigma_\mu^2 \to \infty$, as is shown by the inequality

$$k|u| \leq \frac{1}{\sigma_\mu^2} \tag{2}$$

that Simon and Mukunda derived from the positive semidefiniteness of the cross spectral density, and that will turn out to be automatically satisfied in our model.

Now, we shall show that a TGSM field can be synthesized by a suitable superposition of independent laser beams. This basic idea has already been used in [9] to produce a Gaussian Schell-model (GSM) source, also called a Collett–Wolf [10] source, by means of parallel beams. In [9], the field radiated by such a source was thought of as made by an infinite number of uncorrelated Gaussian beams, propagating in directions that were all parallel to the mean propagation axis of the field, taken as the z axis (figure 1). The cross spectral density of the total field depends on the relative weights given to the superimposed beams. In particular, using a Gaussian weight function, an ordinary (radially symmetric) GSM source is obtained. In this model, the divergence of the total field is determined by the characteristics of the single beam and does not depend on the weight function.

The model that we described can be extended to TGSM beams if the propagation axes of the superimposed beams are inclined with respect to one another (figure 2), in a way that depends on the shift of the beam centre. In particular, the TGSM beams studied by Simon and Mukunda are obtained when the inclination is proportional to the distance of the beam centre from the z axis, that is when the projection on the x–y plane of the mean wave-vector of the single beam, say $\mathbf{k}_\perp$, is proportional to the beam shift and its direction is as shown in figure 3, where the field lines of $\mathbf{k}_\perp$ are shown as broken circles. The coefficient relating the two quantities turns out to be proportional to the twist parameter. As in the previous case, the weight function has to be a Gaussian.

This model explains intuitively some characteristics of the TGSM beams. In particular, it is understood that the field divergence depends also on the inclination of the superimposed beams, that is on the twist parameter: the larger this parameter, the more rapidly the total beam widens out. If the inclination is set equal to zero, a GSM source is obtained. The twisting properties can be made evident using an asymmetric source field distribution. A way to realize this is to superimpose astigmatic beams, or to use a Gaussian weight function having different variances on the two axes.

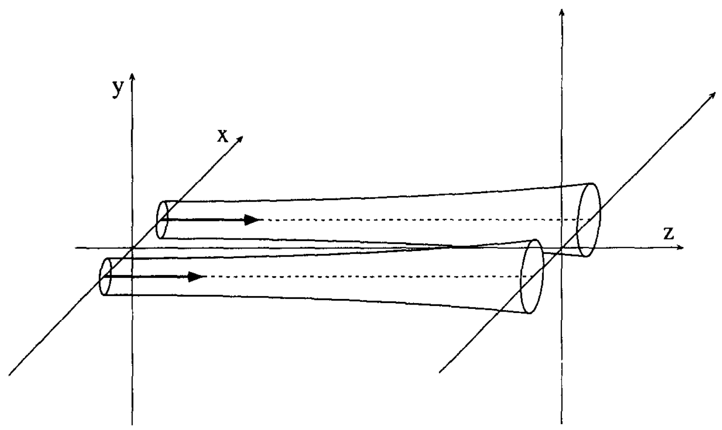

Figure 1. Two typical beams of the superposition giving a GSM source.

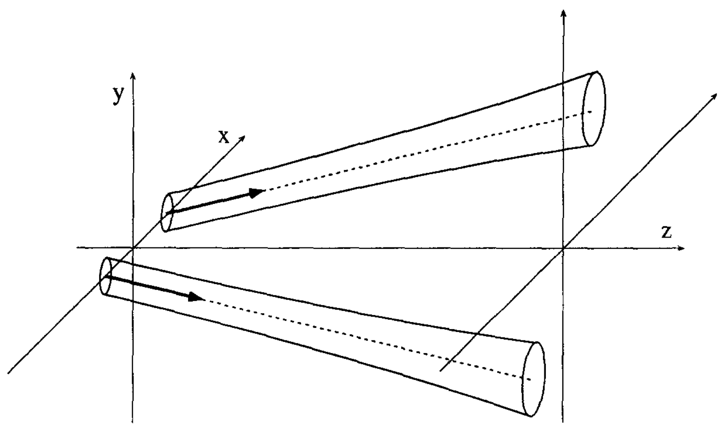

Figure 2. Two typical beams of the superposition giving a TGSM beam.

A model rather similar to the model discussed above has also been used by Friberg *et al.* [11] to explain an experimental procedure by which they synthesized a TGSM beam.

3. Mathematical description of the model

Let us consider, on the plane $z = 0$, a field distribution of the following type:

$$V_0(x, y; x_0, y_0) = \exp\left(-\frac{(x - x_0)^2 + (y - y_0)^2}{X^2} - 2\pi i\alpha(x_0 y - y_0 x) \right), \tag{3}$$

where X is a positive constant and α is a real parameter. Equation (3) represents an ordinary Gaussian beam with its centre at (x_0, y_0). The linear phase term accounts

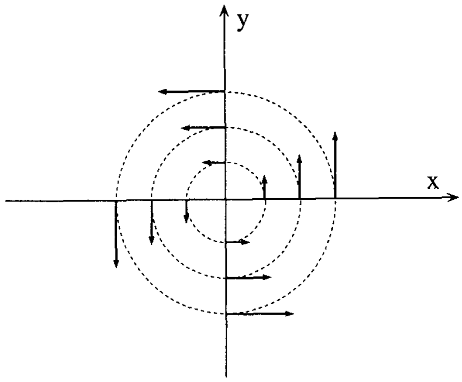

Figure 3. The transverse components of the mean wave-vectors of the Gaussian fields giving rise to a TGSM beam.

for an inclination of the beam. More precisely, if $\mathbf{k}$ is the wave-vector of a plane wave propagating along the beam axis, then its transverse components k_x and k_y are

$$k_x = 2\pi\alpha y_0, \tag{4}$$

$$k_y = -2\pi\alpha x_0. \tag{5}$$

The cross spectral density of such a field is [7]

$$V_0(x_1, y_1; x_0, y_0) V_0^*(x_2, y_2; x_0, y_0) = \exp\left(-\frac{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (x_2 - x_0)^2 + (y_2 - y_0)^2}{X^2} \right) \times \exp\{ -2\pi i\alpha[x_0(y_2 - y_1) - y_0(x_1 - x_2)]\}, \tag{6}$$

with the asterisk denoting the complex conjugate.

Let us consider now a superposition of fields of the form (3), each of them being characterized by a different pair (x_0, y_0). We assume that the various fields are mutually incoherent. As a consequence, the overall cross spectral density of the resulting field is obtained by summing expressions of the form (6) with a weight function, say $P(x_0, y_0)$, proportional to the relative power contributed by the beam centred at (x_0, y_0). We shall use a continuous superposition with the Gaussian weight function

$$P(x_0, y_0) = A \exp\left(-\frac{x_0^2 + y_0^2}{2\sigma^2} \right), \tag{7}$$

where A and σ are positive constants.

The cross spectral density to be evaluated is then

$$W_0(x_1, y_1; x_2, y_2) = \iint V_0(x_1, y_1; x_0, y_0) V_0^*(x_2, y_2; x_0, y_0) P(x_0, y_0)\, dx_0\, dy_0. \tag{8}$$

The integral is extended to the whole plane x_0, y_0. Because of equations (4) and (5) this entails the presence of evanescent waves. It has been shown in [12] that this does not impair the validity of the paraxial approximation, which will be used later, provided that the linear dimensions of the coherence area are large enough with respect to the wavelength.

After some algebra we obtain

$$W_0(x_1, y_1; x_2, y_2) = A\,\frac{2\pi\sigma^2 X^2}{4\sigma^2 + X^2}\exp\left(4\pi i\alpha\,\frac{2\sigma^2}{4\sigma^2 + X^2}(x_1 y_2 - x_2 y_1)\right)$$

$$\times\exp\left(-\frac{x_1^2 + y_1^2 + x_2^2 + y_2^2}{4\sigma^2 + X^2} - \frac{2\sigma^2(1 + \pi^2 X^4 \alpha^2)}{X^2(4\sigma^2 + X^2)}[(x_1 - x_2)^2 + (y_1 - y_2)^2]\right). \quad (9)$$

It may be seen that this equation is equivalent to equation (1) if we set

$$I_0 = \frac{2A\pi\sigma^2 X^2}{4\sigma^2 + X^2}, \qquad \frac{1}{4\sigma_I^2} = \frac{1}{4\sigma^2 + X^2},$$

$$\frac{1}{2\sigma_\mu^2} = \frac{2\sigma^2(1 + \pi^2 X^4 \alpha^2)}{X^2(4\sigma^2 + X^2)}, \qquad ku = -\frac{8\pi\alpha\sigma^2}{4\sigma^2 + X^2}. \quad (10)$$

The upper bound on the twist phase (2) is implicit in our model. Imposing the inequality (2) on the parameters given by the relations contained in equation (10) leads to the condition

$$(1 - \pi|\alpha|X^2)^2 \geqslant 0$$

that is always satisfied.

The optical intensity corresponding to the cross spectral density (9) is

$$I_0(x, y) = W_0(x, y; x, y) = A\,\frac{2\pi\sigma^2 X^2}{4\sigma^2 + X^2}\exp\left(-\frac{2(x^2 + y^2)}{4\sigma^2 + X^2}\right). \quad (11)$$

The effect of the slope connected to the parameter α is the increase in the angular divergence exhibited by a TGSM beam. This may be seen by calculating the intensity for the propagated field in our model. A simple way of performing the calculus is to sum, using the weight (7), the propagated intensities produced by the various uncorrelated Gaussian fields whose superposition gives rise to the TGSM beam.

In the paraxial approximation, a field of the form (3) leads, on the plane $z = \text{constant} > 0$, to the optical intensity [13]

$$I_z(x, y; x_c, y_c) = \left(\frac{X}{X_z}\right)^2 \exp\left(-\frac{2}{X_z^2}[(x - x_c)^2 + (y - y_c)^2]\right), \quad (12)$$

where $X_z^2 = X^2[1 + (z\lambda/\pi X^2)^2]$ is the usual Gaussian beam spot size, λ is the radiation wavelength, and x_c, y_c are the coordinates of the beam centre, given by

$$x_c = x_0 + \alpha\lambda z y_0, \quad (13)$$

$$y_c = y_0 - \alpha\lambda z x_0. \quad (14)$$

If the intensities (12) are superimposed using the weight functions (7), we obtain

$$I_z(x, y) = 2A\pi\sigma^2\left(\frac{X}{w(z)}\right)^2 \exp\left(-\frac{2(x^2 + y^2)}{w^2(z)}\right), \quad (15)$$

where $w^2(z)$ is given by

$$w^2(z) = X_z^2 + 4\sigma^2[1 + (\alpha\lambda z)^2] = w^2(0) + \lambda^2 z^2\left(\frac{1}{\pi^2 X^2} + \alpha^2[w^2(0) - X^2]\right). \tag{16}$$

The existence of sources has been recently pointed out [14, 15] that are characterized by different coherence properties but nevertheless produce the same intensity throughout the space. They are rather peculiar in that the fields which they generate cannot be distinguished at the intensity level. Yet the physical differences would appear as soon as those fields are used in diffraction or interference experiments. We find here another example of sources endowed with such a property. Because of equations (15) and (16), once $w^2(0)$ has been fixed, the same spatial intensity distribution can be obtained with an infinite number of pairs α, X. It could be seen, however, that such values lead to different results for the coherence length σ_μ and the twist parameter computed through equation (10).

The result given in equations (15) and (16) means that a TGSM beam has an intensity distribution analogous to that produced by a GSM source but characterized by a larger effective width, even in the far field. In this limit, we have

$$\left(\frac{w(z)}{X_z}\right)^2 = 1 + 4(\pi\alpha\sigma X)^2, \tag{17}$$

and the intensity remains different from that produced by a single Gaussian beam.

Equations (13) and (14) show that the centre of a single Gaussian beam, during propagation, moves away from the origin and rotates. Passing from Cartesian coordinates (x_c, y_c) to polar coordinates (r_c, ϑ), it is easy to prove that

$$r_c^2 = (x_0^2 + y_0^2)[1 + (\alpha\lambda z)^2], \tag{18}$$

$$\tan\vartheta = \tan(\vartheta_0 + \delta), \tag{19}$$

where, in equation (19), $\tan\vartheta_0 = y_0/x_0$ and $\tan\delta = -\alpha\lambda z$. Equation (18) explains the reason for the larger widening of the beam compared with a GSM source that we obtain if $\alpha = 0$. In this case, the single beams widen but their centres remain fixed, while in the general case the centres spread, causing a larger spread in the intensity. Equation (19) shows the twisting phenomenon, that is the progressive rotation of the beam centre around the mean propagation axis. Coherent beams that exhibit such a twisting property were treated in [16].

Starting from equation (15), it is possible to calculate the M^2 factor [17], defined by the formula

$$w^2(z) = w^2(0) + \frac{(\lambda z M^2)^2}{\pi^2 w^2(0)}, \tag{20}$$

if, as in our case, $z = 0$ is the waist plane.

In our notation, it turns out to be

$$M^2 = \left[(4\sigma^2 + X^2)\left(\frac{1}{X^2} + 4\pi^2\alpha^2\sigma^2\right)\right]^{1/2}. \tag{21}$$

Using the relations given in equation (10), this result may be expressed as follows:

$$M^2 = \left(1 + \frac{4\sigma_1^2}{\sigma_\mu^2} + 4\sigma_1^4(ku)^2\right)^{1/2}. \tag{22}$$

In view of our previous remarks, it is not surprising that the M^2 factor is an increasing function of the twist parameter u.

4. Generalizations of the superposition scheme

The superposition model that we introduced for TGSM beams may be generalized by considering fields of the form

$$V_0(x,y;x_0,y_0)=\exp\left(-\frac{(x-x_0)^2}{X^2}+\frac{(y-y_0)^2}{Y^2}-2\pi\mathrm{i}(\alpha x_0 y-\beta y_0 x)\right), \tag{23}$$

that is Gaussian beams that are shifted and inclined with respect to the z axis, such as the fields given by equation (3), but have different spot sizes and different twist parameters along the two transverse coordinates. An incoherent superposition of the beams (23) with a Gaussian weight function that is no longer circularly symmetric, but has different variances along the two axes, namely

$$P(x_0,y_0)=A\exp\left(-\frac{x_0^2}{2\sigma_x^2}-\frac{y_0^2}{2\sigma_y^2}\right), \tag{24}$$

gives, after some algebra, a generalized version of the cross spectral density (1):

$$W_0(x_1,y_1;x_2,y_2)=I_0\exp\left(-\frac{x_1^2+x_2^2}{4\sigma_{\mathrm{I}x}^2}-\frac{y_1^2+y_2^2}{4\sigma_{\mathrm{I}y}^2}-\frac{(x_1-x_2)^2}{2\sigma_{\mu x}^2}\right.$$
$$\left.-\frac{(y_1-y_2)^2}{2\sigma_{\mu y}^2}-\mathrm{i}ku(x_1y_2-x_2y_1)+\mathrm{i}kv(x_1y_1-x_2y_2)\right). \tag{25}$$

The quantities introduced in equation (25) are

$$I_0=\frac{2A\pi XY\sigma_x\sigma_y}{[(4\sigma_x^2+X^2)(4\sigma_y^2+Y^2)]^{1/2}},$$

$$\frac{1}{4\sigma_{\mathrm{I}x}^2}=\frac{1}{4\sigma_x^2+X^2},\qquad \frac{1}{4\sigma_{\mathrm{I}y}^2}=\frac{1}{4\sigma_y^2+Y^2},$$

$$\frac{1}{2\sigma_{\mu x}^2}=\frac{2\sigma_x^2}{X^2(4\sigma_x^2+X^2)}+\frac{2\pi^2\beta^2\sigma_y^2Y^2}{4\sigma_y^2+Y^2},\qquad \frac{1}{2\sigma_{\mu y}^2}=\frac{2\sigma_y^2}{Y^2(4\sigma_y^2+Y^2)}+\frac{2\pi^2\alpha^2\sigma_x^2X^2}{4\sigma_x^2+X^2},$$

$$ku=-4\pi\left(\frac{\alpha\sigma_x^2}{4\sigma_x^2+X^2}+\frac{\beta\sigma_y^2}{4\sigma_y^2+Y^2}\right),\qquad kv=-4\pi\left(\frac{\alpha\sigma_x^2}{4\sigma_x^2+X^2}-\frac{\beta\sigma_y^2}{4\sigma_y^2+Y^2}\right). \tag{26}$$

The generalized cross spectral density (25) gives rise to a TGSM beam if $X=Y$, $\alpha=\beta$ and $\sigma_x=\sigma_y$, but it may show very different behaviours. In particular, it is worthwhile to note that, by superimposing asymmetric intensities with a suitable asymmetric weight, a circular intensity may still be obtained. Another interesting case is the superposition of symmetric intensities with a symmetric weight, but letting $\alpha=-\beta$. As a consequence, the parameter u vanishes and a value different from zero is obtained for v, which can be thought of as an astigmatism parameter of the beam. The physical difference with respect to the TGSM beam is that, while for $\alpha=\beta$ the mean wave-vectors of the elementary beams have transverse components whose field lines are circumferences, in this case such components are given by

$$k_x=2\pi\alpha y_0, \tag{27}$$

$$k_y=2\pi\alpha x_0, \tag{28}$$

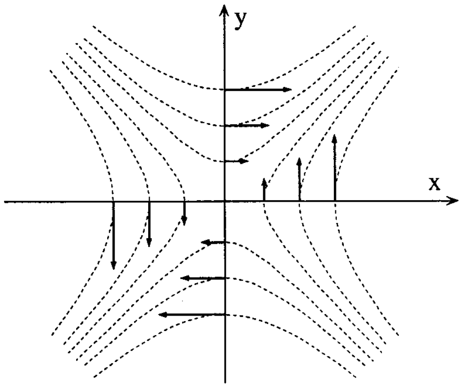

Figure 4. The transverse components of the mean wave-vectors of the Gaussian fields giving rise to an astigmatic beam.

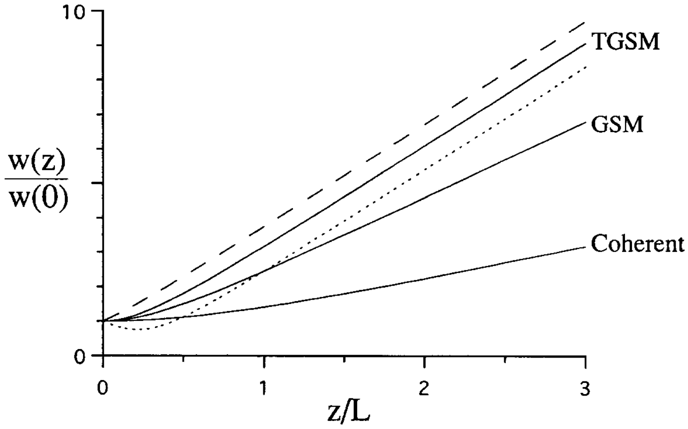

Figure 5. The different behaviours of the normalized spot sizes for various types of field as a function of z/L, where L is the Rayleigh distance pertaining to a coherent Gaussian beam whose spot size at the waist is $w(0)$: (——), spot sizes of a coherent beam, a GSM beam with $\sigma_\mu = \sigma_I$, and a TGSM beam with $\sigma_\mu = \sigma_I$ and $k|u| = \sigma_\mu^{-2}$: (---), (·····), spot sizes of an astigmatic beam whose intensity is given by equation (29) with $\sigma_\mu = \sigma_I$ and $k|v| = \sigma_\mu^{-2}$.

so that their field lines are hyperbolae, as shown in figure 4. The effect is no longer a rotation of the field pattern, but a deformation, as can be seen by calculating the propagated intensity of the total field. The calculation follows the same scheme described in section 3, giving the result

$$I_z(x', y') = 2A\pi\sigma^2 \frac{X^2}{w_+(z)w_-(z)} \exp\left[-2\left(\frac{x'^2}{w_+^2(z)} + \frac{y'^2}{w_-^2(z)}\right)\right], \tag{29}$$

where x' and $y'$are the coordinates in a system rotated of $+\frac{1}{4}\pi$ with respect to xy and the effective spot sizes

$$w_\pm^2(z) = X_z^2 + 4\sigma^2[(\alpha\lambda z) \pm 1]^2, \tag{30}$$

have been introduced. The other quantities appearing in the calculation have already been defined in section 3. It is easy to check that on the waist plane the intensity (29) has the form (11) and that, in the far field

$$\left(\frac{w_{\pm}(z)}{X_z}\right)^2 = 1 + 4(\pi\alpha X\sigma)^2, \tag{31}$$

so that the field distribution that we are considering gives rise to the same intensity produced by a TGSM beam in both the $z = 0$ and the $z = \infty$ planes but has very different properties. In particular, its optical intensity at any finite z is elliptical and the two effective spot sizes (30) are connected to the spot size of a TGSM beam, given by equation (16), by the relation

$$w^2(z) = \frac{w_+^2(z) + w_-^2(z)}{2}. \tag{32}$$

In figure 5 we show the behaviour of the normalized spot sizes for various types of field as a function of z/L, where L is the Rayleigh distance pertaining to a coherent Gaussian beam whose spot size at the waist is $w(0)$. The full solid curves represent the spot sizes of a coherent beam, a GSM beam with $\sigma_\mu = \sigma_\mathrm{I}$, and a TGSM beam with $\sigma_\mu = \sigma_\mathrm{I}$ and $k|u| = \sigma_\mu^{-2}$. The broken and the dotted curves represent the spot sizes of an astigmatic beam whose intensity is given by equation (29), with $\sigma_\mu = \sigma_\mathrm{I}$ and $k|v| = \sigma_\mu^{-2}$.

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