Fractional Fourier transform and Fresnel transform (*)

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SUMMARY. - *We show that there is a simple relation between the Fourier transform of fractional order and the Fresnel transform.*

The Fourier transform of fractional order (or, fractional Fourier transform) (1, 2) was introduced by Namias as a mathematical tool for solving problems in quantum mechanics. In some interesting papers (3, 4) Lohmann, Mendlovic and Ozaktas have shown that such a transform can be usefully applied to optical problems. In this note we show that the fractional Fourier transform can be expressed in a rather simple way by means of the Fresnel transform (5, 6). The latter describes many phenomena in paraxial optics (7) as well as in quantum mechanics (8). Therefore, the link that we establish can help to appreciate the role that the fractional Fourier transform can play in the quoted subjects. For the sake of simplicity, we shall refer to the one-dimensional case. Extension to two dimensions is straightforward. Furthermore, we shall limit ourselves to functions belonging to $L^2(R)$.

Let us denote by $\mathbf{F}\{f\}(\nu)$ the Fourier transform of a typical function $f(x)$, namely:

$$\mathbf{F}\{f\}(\nu) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi i \nu x)\, dx \,. \tag{1}$$

Let us further introduce the Hermite-Gauss functions, defined as:

$$G_n(x) = \frac{(2)^{1/4}}{\sqrt{2^n\, n!}} H_n(\sqrt{2\pi}\, x) \exp(-\pi x^2)\,; \quad (n = 0,1,2,\ldots)\,, \tag{2}$$

where H_n is the n-th Hermite polynomial. The G_n belong to the class of

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self-Fourier functions recently introduced by Caola (9) and discussed by several authors (10-13). This means that the G_n are self-reproducing under Fourier transformation. More explicitly, we have:

$$\mathbf{F}\{G_n\}(\nu) = \exp\left(-i n \frac{\pi}{2}\right) G_n(\nu) \; ; \qquad (n = 0,1,2,\ldots). \tag{3}$$

Because of the completeness of the Hermite-Gauss function in the L^2 space (7) the function $f(x)$ can be expanded into a series of the form:

$$f(x) = \sum_{n=0}^{\infty} c_n G_n(x) \; , \tag{4}$$

so that taking equation [3] into account we obtain at once:

$$\mathbf{F}\{f\}(\nu) = \sum_{n=0}^{\infty} c_n G_n(\nu) \exp\left(-i n \frac{\pi}{2}\right) . \tag{5}$$

One can now introduce (1, 3) the fractional Fourier transform of order a, where $0 \leq a \leq 1$, to be denoted by $\widetilde{f}_a(x)$ as follows:

$$\widetilde{f}_a(x) = \sum_{n=0}^{\infty} c_n G_n(x) \exp\left(-i n a \frac{\pi}{2}\right) . \tag{6}$$

On comparing with equations [4] and [5] it is seen that $\widetilde{f}_a$ is the identity transformation for $a = 0$ whereas it gives the Fourier transform for $a = 1$. It is to be said that the present definition is slightly different from those adopted in refs. (1) and (3), which in turn differ from one another. We shall disregard these differences as they only involve scale factors on the x-axis and multiplicative known terms.

Let us now recall that the Fresnel transform of order γ, to be denoted by $\widehat{f}_\gamma(x)$ or $E_\gamma\{f\}(x)$ is defined as (6):

$$\widehat{f}_\gamma(x) \equiv E_\gamma\{f\}(x) = \sqrt{-i\gamma} \int_{-\infty}^{\infty} f(\xi) \exp[\pi i \gamma (x-\xi)^2] \, d\xi \tag{7}$$

where γ a is real number different from zero. On applying this definition to the Hermite-Gauss functions G_n one finds:

$$E_\gamma\{G_n\}(x) = \frac{\sqrt{|\gamma|}}{\sqrt[4]{1+\gamma^2}} \exp\left[i \frac{\pi \gamma x^2}{1+\gamma^2} - i\left(n + \frac{1}{2}\right) \Phi(\gamma)\right] \times \times G_n\left(\frac{x|\gamma|}{\sqrt{1+\gamma^2}}\right) \tag{8}$$

where:

$$\Phi(\gamma) = \tan^{-1}\left(\frac{1}{\gamma}\right) . \tag{9}$$

Equations [8] and [9] can be deduced by well known formulas relating to Gaussian beams (7). Then, on evaluating the Fresnel transform of both sides of equation [4] we obtain:

$$\widehat{f}_\gamma(x) = \frac{\sqrt{|\gamma|}}{\sqrt[4]{1+\gamma^2}} \exp\left[i\frac{\pi\gamma x^2}{1+\gamma^2} - \frac{i}{2}\Phi(\gamma)\right] \times \tag{10}$$
$$\times \sum_{n=0}^{\infty} c_n G_n\left(\frac{x|\gamma|}{\sqrt{1+\gamma^2}}\right) \exp[-i n \Phi(\gamma)] \; .$$

The series on the right-hand side is quite similar to that appearing in equation [6]. More precisely, if we let:

$$a(\gamma) = \frac{2}{\pi}\Phi(\gamma) \; , \tag{11}$$

and change the variable x to $x\sqrt{1+\gamma^2/|\gamma|}$ equation [10] becomes:

$$\widehat{f}_\gamma\left(\frac{x\sqrt{1+\gamma^2}}{|\gamma|}\right) = \frac{\sqrt{|\gamma|}}{\sqrt[4]{1+\gamma^2}} \exp\left[i\frac{\pi x^2}{\gamma} - \frac{i\pi}{4}a(\gamma)\right] \widetilde{f}_{a(\gamma)}(x) \; . \tag{12}$$

This is the result that we wanted to prove. It shows that there is a simple link between the Fresnel and the fractional Fourier transforms.

The applications of the above mathematical tools are outside the scope of the present paper. Nonetheless, let us make a qualitative remark about an implication of equation [12] for optics. As shown by Mendlovic and Ozaktas (3, 4) the fractional Fourier transform is particularly suited to describe light propagation in quadratic index fibres. On the other hand, the Fresnel transform is the standard tool for describing light propagation in a homogeneous medium. Equation [12] shows that each of the two propagation phenomena can be traced back to the other.

Let us finally note that several theorems hold for the fractional Fourier transform (1-4). The same is true for the Fresnel transform (5, 6). While certain theorems of the two sets can be put into a mutual correspondence, it is likely that new results for one transform can be deduced from known properties of the other.

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