Change of energy of photons passing through rotating anisotropic elements

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Abstract. We discuss simple experiments where the concept of angular momentum of a photon can be put to use in order to explain the photon energy changes that are produced by the passage of light beams through rotating anisotropic elements.

Résumé. On montre que le concept de moment angulaire du photon peut etre utilisé pour expliquer certaines phénomènes qui ont lieu lorsque la lumière polarisé passe à travers des milieux anistropes tournantes. Des expériments élémentaires sont discutées.

1. Introduction

The existence of a linear momentum for photons is often invoked to explain simple phenomena of optics in a particle picture. For example, when a light beam is scattered by an acoustic wave, both the direction and the temporal frequency of the scattered field can be found by a simple application of energy and momentum conservation laws (Yariv and Yeh 1984). On the other hand, the existence of an angular momentum for photons does not seem to play a role very often (see the beautiful introduction in Hecht 1987). The best known phenomenon in which the angular momentum of light is involved is the torque exerted by a beam of circularly polarized light on a birefringent half-wave plate. This torque, whose existence was predicted by Sadowsky (1899, 1900), Poynting (1909) and Epstein (1914) was directly measured in celebrated experiments (Beth 1935, 1936, Holbourn 1936, Carrara 1949). Although such a delicate measurement could be repeated today in a much easier way using lasers, as demonstrated by a careful analysis (Padmabandu and Marathay 1992), this seems to be still out of reach for most undergraduate laboratories.

In this paper, we discuss some elementary experiments that involve the use of rotating anisotropic elements such as, for example, wave plates and linear polarizers. It is easily seen through a classical, non-relativistic wave analysis that frequency changes of the light beam passing through the rotating elements may occur. In a particle picture, this entails changes of photon energy. We shall show that such a result can be explained by using the concept of angular momentum of the photon and taking into account the work associated with the torque exerted on the rotating bodies.

In section 2, the basic phenomena produced by rotating anisotropic elements are examined. In section 3, we present some practical schemes for the study of such phenomena and we discuss some experimental details. Finally, in section 4, we outline how the subject matter of this paper occurs in interferometry and in basic research.

2. The effect of a rotating anisotropic element on light

Many types of anisotropic devices are in use in optics. In what follows, however, we limit ourselves

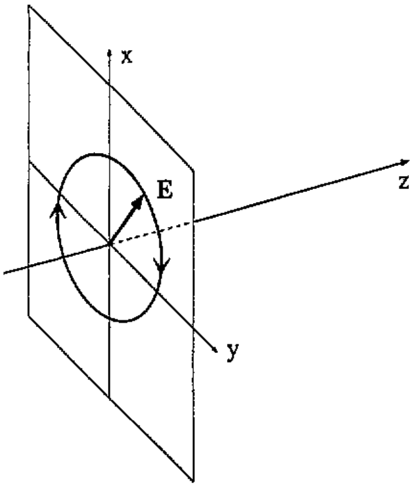


Figure 1. Schematic representation of a right-handed wave propagating in the positive z direction.

to two of the most common devices, namely the half-wave plate (HWP) and the linear polarizer (LP) (Hecht 1987). We shall refer to rather idealized objects. For example, we shall assume that all the reflections are suppressed through antireflection coatings. Similarly, we shall suppose that an LP transmits completely the allowed state of polarization and extinguishes completely the other one. While such hypotheses are not essential in our analysis, they simplify the formulae we are going to derive.

2.1. Rotation of a half-wave plate

We shall deal first with the HWP. Suppose that a circularly polarized light beam impinges on the plane $z = 0$ of a suitable reference frame in the positive z-direction. Let the electric vector $\boldsymbol{E}$ of the wave rotate as shown in figure 1, i.e. clockwise around the z-axis. Looking toward the source, the rotation would be counterclockwise. Therefore, according to the usual convention of optics, this light would be termed left-handed (Hecht 1987, Born and Wolf 1991). As is well known, this terminology is rather unfortunate for the following reason. On the basis of a mechanical analogy, we would expect a left-handed rotation to be associated to a negative angular momentum. On the other hand, both the classical and the quantum values of the angular momentum are positive for this type of light. This contradiction arises of course because of the convention of looking toward the source, i.e. in a direction that is opposite to the one in which the light propagates. At present, the situation in optics textbooks is a bit confusing. While most authors use the traditional terminology, some of them (Yariv and Yeh 1984) adopt the opposite convention. In this paper, we shall use the non-traditional terminology. Accordingly, circularly polarized light of the type of figure 1 will be termed right-handed. For the sake of brevity, we shall use the shorthand notations RH and LH to mean right-handed and left-handed circularly polarized, respectively.

Using real quantities to represent the fields of the wave, we express the x and y components of $\boldsymbol{E}$ in the form

$$\begin{cases} E_x = A\cos(\omega t), \\ E_y = A\sin(\omega t), \end{cases} \tag{1}$$

where A is a constant and the initial phase of the field has been made to vanish through a suitable choice of the time origin. The quantity ω, which properly speaking is the angular frequency of the wave, will be simply termed frequency in the following.

Let the plane $z = 0$ be the entrance plane of an HWP whose principal axes, say x' and y' are rotated through an angle α with respect to x and y (see figure 2). We can make use of the rotation matrix, $\mathbf{R}$, defined as (Goldstein 1980)

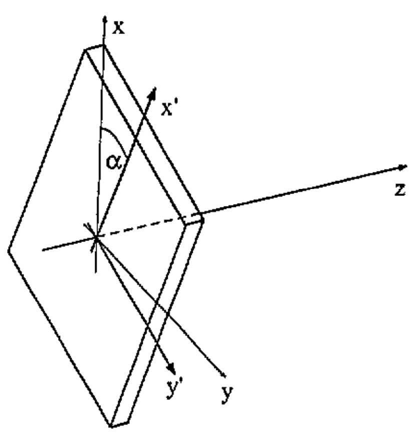
$$\mathbf{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \tag{2}$$

to find the components of $\boldsymbol{E}$ in the rotated frame. After traversing the plate, one of the two components is dephased by π with respect to the other. In order to find the output components $\bar{E}_x$ and $\bar{E}_y$, we simply have to go back to the x, y frame through a rotation by an angle $-\alpha$ (see equation (2)). With simple passages we obtain

$$\begin{cases} \bar{E}_x = A\cos(\omega t - 2\alpha), \\ \bar{E}_y = -A\sin(\omega t - 2\alpha). \end{cases} \tag{3}$$

On comparing equation (3) to equation (1) we see that two differences have occurred. First, we have the well known result that the half-wave plate has transformed RH into LH light. Second, the initial

Figure 2. Rotation by an angle α around the z-axis of the reference frame x', y' attached to the half-wave plate.



phase of the field has changed by twice the angle between the x, y and x', y' frames.

Let us suppose now that the plate is set in a circular uniform motion around the z-axis with angular speed Ω†. As far as the effect of this motion on the phase is concerned, we can assume that a non-relativistic approach can be used, provided that the involved speeds are not too high. Consequently, we simply replace α by $\pm\Omega t$ depending on whether the rotation is positive or negative around the z-axis. Equations (3) then become

$$\begin{cases} \bar{E}_x = A\cos[(\omega \mp 2\Omega)t], \\ \bar{E}_y = -A\sin[(\omega \mp 2\Omega)t]. \end{cases} \quad (4)$$

Therefore, we reach the conclusion that the light emerging from the rotating HWP is downward or upward shifted in frequency with respect to the entering light.

We now want to discuss the origin of this frequency shift in a particle picture. A photon entering the plate has energy $\hbar\omega$ and angular momentum $\hbar\hat{z}$ (for right-handed light), where $\hat{z}$ is the unit vector along the z-axis. When the photon emerges from the plate, it has a different energy, say $\hbar\bar{\omega}$, and angular momentum $-\hbar\hat{z}$. Let us consider a very small time interval Δt and let n be the number of photons passing through the plate during Δt. The energy and angular momentum changes for the n photons are

$$\Delta E_r = n\hbar(\bar{\omega} - \omega), \quad (5)$$

$$\Delta \boldsymbol{L}_r = -2n\hbar\hat{z}, \quad (6)$$

respectively, where the subscript r stands for radiation field. The difference of energy and angular momentum is transferred to the plate. This transfer takes place through the action of the torque exerted by the radiation field on the plate. As we shall see in a moment, we will not need the explicit expression of this torque (Padmabandu and Marathay 1992), so let us simply denote it by $\boldsymbol{\tau}$. In order to transfer angular momentum to the plate in the correct direction, $\boldsymbol{\tau}$ has to be parallel to $\hat{z}$. The change of angular momentum of the plate in the time interval Δt must compensate the analogous quantity for the photons (equation (6)). On applying the laws of dynamics we then find

$$|\boldsymbol{\tau}|\Delta t = 2n\hbar. \quad (7)$$

On the other hand, because of the plate motion, the torque does a (positive or negative) work given by

$$\Delta W = |\boldsymbol{\tau}|\Delta\alpha = \pm|\boldsymbol{\tau}|\Omega\Delta t = \pm 2n\Omega\hbar, \quad (8)$$

where $\Delta\alpha$ is the rotation angle in the time interval Δt, the upper or lower sign is to be chosen depending on whether the plate rotates clockwise or counterclockwise with respect to $\hat{z}$, and relation (7) has been used. Such work corresponds to the opposite of the variation of energy for the radiation field. Using equations (5) and (8) we obtain

$$\bar{\omega} = \omega \mp 2\Omega, \quad (9)$$

where the upper (lower) sign refers to a plate rotating clockwise (counterclockwise). Such a result is in complete agreement with the wave theory argument leading to equation (4) even if, using the classical treatment, we cannot see any energy change.

It is easily seen that if the entering light is LH we reach the same conclusion except that the plus and minus signs are to be interchanged in equation (9). We can, however, encompass both cases in a single rule. On passing through the rotating plate, circularly polarized light will be downward (upward) frequency shifted if the electric field of the wave rotates in the same (opposite) sense of the plate.

What about the case in which a linearly polarized light impinges on the plate? In classical terms, we can think of this case as a superposition of two counter-rotating circular states of polarization. Therefore, also the outcoming beam is a superposition of circularly polarized waves. Their roles interchange with respect to the input in that the RH component becomes the LH one and *vice versa.* On the ground of the above rule about frequency changes we can conclude that one of the two outgoing components is upward shifted in frequency by 2Ω while the other is downward shifted by the same quantity. The overall frequency difference between the two components is then 4Ω.

In quantum terms, the description has to be a bit more subtle. A linear polarization state is not an eigenstate of the angular momentum (Hecht 1987). As a consequence, each photon of the linearly polarized beam is simultaneously in both the states of RH and LH with equal probability amplitudes. The angular momentum has not a definite value and its expectation value is zero. Passing through the rotating plate, the two circular states interchange and frequency shifts are introduced. Therefore, at the output of the plate, neither the angular momentum nor the frequency of the photon have definite values. This means, for example, that on measuring the frequency of the photon either $\omega - 2\Omega$ or $\omega + 2\Omega$ can be found. Before the measurement, however, the photon has, in a sense, both frequencies. This is well known and lies at the very root of quantum mechanics.

2.2. Rotation of a linear polarizer

We turn to the other anisotropic element of our concern: a rotating LP. We first assume that the input beam is RH and that the LP rotates clockwise around the z-axis. We can refer again to figures 1 and 2. Suppose that the axis of the LP is parallel to the

† Here and in the following we assume $\Omega \ll \omega$.

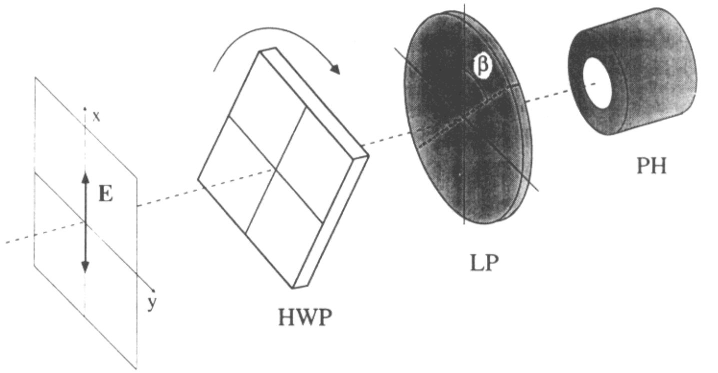


Figure 3. An in-line experimental set-up using a rotating half-wave plate (HWP) and a linear polarizer (LP).

x'-axis (figure 2). The x and y components of the electric field of the incoming wave are given by equation (1). In order to find the transmitted field, say $\bar{E}'$, in the rotating frame, we have to project E_x and E_y on the x'-axis. This gives

$$\bar{E}' = E_x \cos\alpha + E_y \sin\alpha = A\cos(\omega t - \alpha). \qquad (10)$$

The output components in the x, y frame are then

$$\begin{cases} \bar{E}_x = A\cos(\omega t - \alpha)\cos\alpha, \\ \bar{E}_y = A\cos(\omega t - \alpha)\sin\alpha. \end{cases} \qquad (11)$$

Letting $\alpha = \Omega t$ and using trigonometric identities we can write equations (11) as follows:

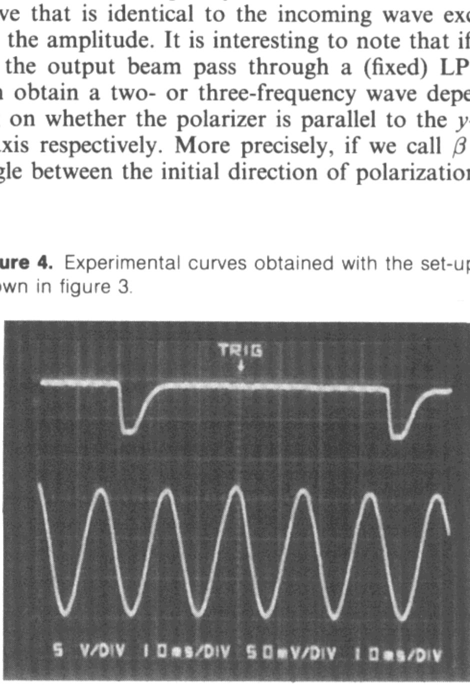
$$\begin{cases} \bar{E}_x = \dfrac{A}{2}\cos(\omega t) + \dfrac{A}{2}\cos[(\omega - 2\Omega)t], \\ \bar{E}_y = \dfrac{A}{2}\sin(\omega t) - \dfrac{A}{2}\sin[(\omega - 2\Omega)t]. \end{cases} \qquad (12)$$

Here, we see that the light emerging from the rotating LP is made up of two components: (a) a RH wave with the same frequency as the incoming wave; (b) a LH wave downward shifted in frequency by 2Ω. As far as (b) is concerned, the frequency shift would be upward for a counterclockwise rotation of the LP ($\alpha = -\Omega t$). We could similarly analyse the case in which LH light impinges on the rotating LP. The following general rules would be found. The circular component that emerges from the LP with the same sense of rotation as the input wave has no frequency shift. The other component is downward (upward) frequency shifted if its electric field rotates in the same (opposite) sense as the LP. We note again that in all cases only half of the incoming intensity emerges from the LP. In fact, for example, the rms value of the field (10) is $1/\sqrt{2}$ times the rms value of the input field (equations (1)).

The quantum interpretation of the previous results is quite similar to the one given for the rotating half-wave plate. Any incoming photon with well defined energy $\hbar\omega$ and angular momentum, say $\hbar\hat{z}$, goes into a superposition of the following three states: (a) energy $\hbar\omega$ and angular momentum $\hbar\hat{z}$; (b) energy $\hbar(\omega \pm 2\Omega)$, depending on the rotation sense of the LP, and angular momentum $-\hbar\hat{z}$; (c) a vacuum state (the photon gets absorbed by the LP). The gain or loss of energy in case (b) can be traced back to the work done by the wave on the rotating LP.

Let us finally briefly consider the case in which linearly polarized light falls on the rotating LP. In this case, following a similar approach, we obtain an output field consisting of two counter-rotating circularly polarized waves with upward and downward shifts of frequency. In addition, we have a wave that is identical to the incoming wave except for the amplitude. It is interesting to note that if we let the output beam pass through a (fixed) LP we can obtain a two- or three-frequency wave depending on whether the polarizer is parallel to the y- or x-axis respectively. More precisely, if we call β the angle between the initial direction of polarization of

Figure 4. Experimental curves obtained with the set-up shown in figure 3.



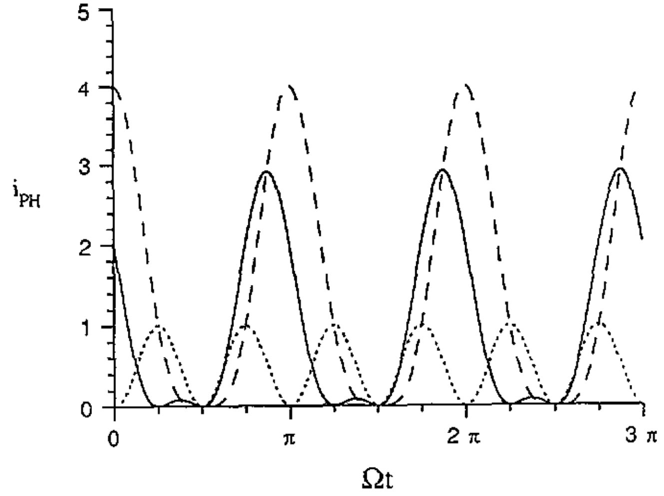


Figure 5. Theoretical curves (in arbitrary units) of the output signal of the photodetector, when the half-wave plate is replaced by a second linear polarizer, for three different values of the angle β (dashed line: $\beta = 0$; full line: $\beta = -\pi/4$; dotted line: $\beta = -\pi/2$).

the laser beam and the transmission axis of the fixed LP, we obtain the following expression for the amplitude E_{out} of the emerging field:

$$E_{out} = \frac{A}{2}\cos(\omega t)[\cos\beta + \cos(2\Omega t - \beta)]. \qquad (13)$$

The quantum interpretation follows much the same pattern as in the cases discussed above.

Before ending this section, we want to discuss another point about energy changes. We have seen that photons emerging from a rotating anisotropic element can possess a different energy relative to the input photons. Refer, for example, to the first case discussed in this section and suppose that the electric field of the input wave and the plate rotate in the same sense. Then, each photon loses an energy $2\hbar\Omega$. If a wavetrain with a certain overall energy enters the plate, the output wavetrain has a lower energy†. This is all right in so far as we think of a bunch of photons. But what about the classical analysis? Although we said there is a torque doing work, the amplitude of the output field seems to be the same as that of the input field. A somewhat similar problem arises when analysing the frequency shift produced by a moving mirror (Becker and Sauter 1964, Pippard 1992). In the present case the rotation of the dielectric, when it is polarized by the electric field of the wave, produces a convection current which introduces an extra term in the dielectric constant; this has an out-of-phase component which is responsible for making the required changes to the field amplitude (Pippard 1993).

† Strictly speaking, a finite extent wavetrain is no longer monochromatic. We shall disregard this effect.

3. Experimental set-ups

The basic phenomena described in the previous section can be observed in the laboratory with rather simple devices. We shall first refer to arrangements that require the minimum effort and we shall give a few samples of results. Subsequently, we shall consider more complete although more demanding set-ups.

A simple and yet significant experiment can be made with the scheme of figure 3. A linearly polarized laser beam is sent through a rotating HWP. Rotation can be achieved employing a circular holder where the HWP can be tightened supported by three ball bearings. A rubber belt inserted in a groove of the holder is connected to a pulley on the axis of a suitable motor. Then, there is a linear polarizer LP. The angle β between the initial direction of polarization (x-axis) of the laser beam and the transmission axis of LP can be regulated at will. Light emerging from LP enters a photodetector PH (a photodiode) whose output is monitored on an oscilloscope. It is useful to have an auxiliary signal giving information about the rotation period of HWP. This can be obtained in various ways. For example, a light emitting diode can send radiation onto a little piece of reflecting tape pasted on the rotating element. Once in a period, a reflected beam flashes towards a photodetector whose output furnishes a marker signal.

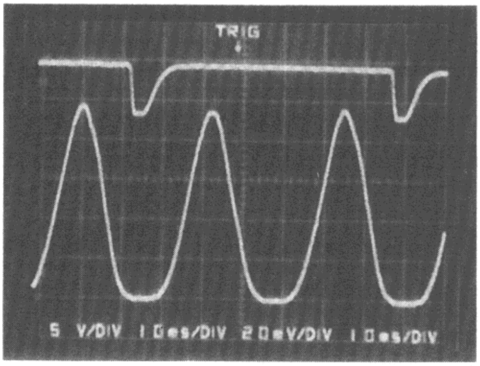
The output of the detector, say i_{PH}, is of the form

$$i_{PH} = p\langle E^2_{out}\rangle, \qquad (14)$$

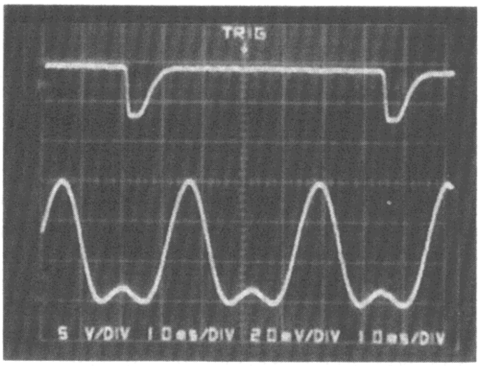
where E_{out} is the field amplitude emerging from LP, the angular brackets denote an average over the response time of the detector and p is a proportionality factor. We assume that the response of PH is fast enough to follow variations at frequencies of the order of Ω.

Thanks to the discussion of section 2, we know that the light emerging from the HWP is the superposition of two counter-rotating circularly polarized waves at frequencies $\omega - 2\Omega$ and $\omega + 2\Omega$. The polarizer LP makes the two waves beat with one another, otherwise, due to their orthogonal polarization states, they could not interfere. It is easily seen that the angular position of LP is immaterial. We then expect a pure beat tone at frequency 4Ω to be exhibited by i_{PH}. This is exactly what can be observed on the oscilloscope as shown in figure 4, where the upper trace gives the marker signal.

Figure 6 Experimental curves corresponding to the curves of figure 5: (a) $\beta = 0$; (b) $\beta = -\pi/4$; (c) $\beta = -\pi/2$.



(*a*)



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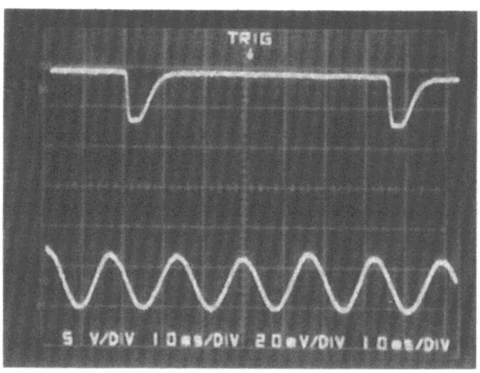

(*c*)

Let us consider now a similar experiment where the rotating HWP is replaced by a rotating LP. In this case, using equations (13) and (14), we obtain

$$i_{PH} = \frac{pA^2}{8}\left[\tfrac{1}{2} + \cos^2\beta + 2\cos\beta\cos(2\Omega t - \beta) + \tfrac{1}{2}\cos(4\Omega t - 2\beta)\right]. \qquad (15)$$

Graphs of equation (15) for a few values of β are given in figure 5. The corresponding oscilloscope traces are shown in figure 6(a)–(c).

The previous experiments do not require a vibration isolated table. Nonetheless, they do not afford any separation of the components of the output beams. This can be attained by suitable interferometric arrangements. We shall discuss one of them.

Let us consider the scheme of figure 7. The linearly polarized beam produced by a laser is transformed into a circularly polarized one by means of a quarter-wave plate QWP. The outgoing beam enters a Mach–Zehnder interferometer. Two HWPs, namely HWP1 and HWP2, are inserted into the arms of the interferometer. One of them, say HWP1, is rotating at angular speed Ω. At the output of the interferometer, we have the superposition of two equally polarized waves (with circular polarization). Because of the rotation of HWP1 a frequency difference 2Ω exists between them. As a consequence, there is a system of fringes moving at constant speed in front of the photodetector PH. We can initially adjust the fringes (when the motor is stopped) in such a way that their width is much greater than the linear extent of the detection area. Powering the motor, a beat tone at 2Ω is then observed on the oscilloscope.

Whether the frequency shift of the light outcoming from HWP1 is upward or downward can be ascertained by detecting the direction in which the fringes move. This can be done by visual inspection by lowering the speed of rotation of the plate to about one revolution per second. The present set-up has to be mounted on a vibration isolated table. In addition, the rotating HWP should not transmit vibrations to the optical bench. This could be achieved by means of some damping cushion. In our experiment we supported the rotating device from outside the table.

4. Concluding remarks

In this paper, we considered the effect of anisotropic

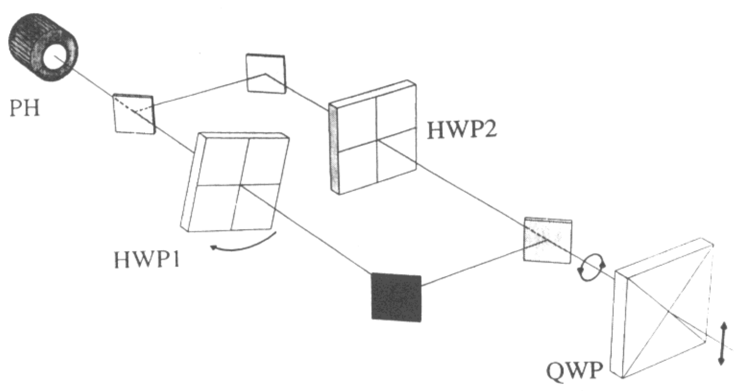

Figure 7. An interferometric set-up using wave plates.

rotating elements on a light beam. We referred to the very simple cases of a half-wave plate and a linear polarizer. It would not be difficult to extend our considerations to other cases like, e.g. a dichroic waveplate with arbitrary thickness. In addition, effects on reflected beams could be considered. We quote a simple example: when a linearly polarized beam falls on a rotating piece of Polaroid, the reflected light exhibits frequency shifts. The experimental test of this phenomenon is as simple as the ones we discussed in the previous section.

It may be useful to outline the link between the subject matter of the present paper and current researches of applicative and fundamental character. A continuous rotation of anisotropic elements is one of the frequency shifting techniques used, e.g. in heterodyne interferometry (Crane 1969, Sommargren 1975, Hu 1983). The phase shift introduced by a proper orientation of the same elements is being used in interferometry to control the relative phase of two light beams (Sommargren 1981, Creath 1988, Jin and Tang 1992, Moreno *et al* 1992). An analogous phase shift has been exploited to inquire about the Pancharatnam phase, that is the phase taken by the radiation field as its polarization changes in a cyclic way (Pancharatnam 1956, Chyba *et al* 1988). Pancharatnam phase is a particular case of the topological phase taken by the state of a quantum system as it changes in a cyclic way, that is the so-called Berry's phase (Berry 1984). The angular momentum of the photons has been used to explain the mechanical rotation induced by a radiation field in the molecules of a liquid crystal (Santamato *et al* 1986), in a dielectric sphere (Chang and Lee 1985) and in a birefringent half-wave plate, as we already remembered (Padmabandu and Marathay 1992). We want to note that in an interesting recent paper (Hariharan and Roy 1992) a ring resonator was used in the study of the Pancharatnam phase. This type of interferometer has very good stability properties. Our previous analyses could be easily extended to such an experimental configuration.

Finally, let us summarize the didactic advantages of using an elementary quantum picture to explain the frequency shift phenomena seen in this paper. First, the concept of angular momentum of a photon is illustrated in a very simple context. Second, the explanation turns out to be even shorter than the classical one. Third, one obtains the correct solution to a problem, namely the energy change of a wave-train passing through the rotating element, that would be more difficult to face in classical terms.

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