Optics Communications 102 (1993) 495–504
North-Holland

OPTICS
COMMUNICATIONS

Full length article

Space intensity distribution and projections of the cross spectral density

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Received 18 January 1993

It is shown that, in the paraxial approximation, the intensity distribution produced throughout the space by a partially coherent field depends only on certain projection integrals of the cross spectral density. These projections, in turn, can be recovered from knowledge of the spatial distribution of the optical intensity. This result can enlighten the recently evidentiated fact that two fields in different states of coherence can give rise to the same intensity everywhere. Some examples are discussed.

1. Introduction

In partially coherent fields, at each temporal frequency, information is carried by the space correlation function of the field, namely, the cross spectral density [1]. Thanks to the existence of propagation formulas, the cross spectral density can be evaluated everywhere in space once it is known for any pair of points across a certain plane, say the plane $z=0$. This requires the knowledge of a four-dimensional complex function. Unfortunately, the experimental determination of such a huge quantity of information is far from being trivial. On the contrary, measuring the optical intensity even across a multitude of planes is a relatively simple task. With this in mind, we ask: what information about the cross spectral density can be gained from the sole knowledge of the optical intensity throughout the space? Could the cross spectral density itself be retrieved on the ground of that knowledge?

Let us start by the second question. At first sight, the following argument could be put forward. We can hardly recover a four-dimensional complex function (the cross spectral density at $z=0$) starting from a three-dimensional real non-negative one (the space distribution of the optical intensity). Actually, a pure dimensionality argument like this is not necessarily conclusive. Because of its very nature, the cross spectral density cannot be chosen at will in the form of an arbitrary complex four-dimensional function. For example, when considered as a kernel, it has to be a hermitian positive one [2]. Therefore we could even hypothize that, by virtue of certain internal constraints, the space correlation function across $z=0$ is completely determined by the space distribution of the optical intensity. In addition, the dimensionality argument does not hold in the so called one-dimensional case, i.e., when the space correlation function at two points depends only on one transverse coordinate for each point. To avoid misunderstanding, we stress that the cross spectral density depends also on the z-coordinate of our reference. So what we call one-dimensional case actually corresponds to a two-dimensional field and the general case to a three-dimensional one.

As a matter of fact, it was proved elsewhere [3] that the space distribution of the optical intensity fully specifies the cross spectral density in the one-dimensional case. On the other hand, it was shown through explicit examples that in the general case fields in different states of coherence can produce the same intensity everywhere. A striking example of this is offered by the so-called doughnut modes that can exhibit rather different coherence properties and yet are indistinguishable on a coherence basis [3,4]. Therefore a recovery of the cross spectral density, if any, cannot be unique.

Then, what about the first question? Is there any coherence feature that can be unambigously recovered from the optical intensity?

In this paper, we give an answer to that question. In order to synthesize our results, we need to define certain projections of the cross spectral density. The precise definition will be given in the next section. For the moment, the following loose definition will suffice. Let us consider the cross spectral density at two typical points, say Q_1 and Q_2, on a plane $z=\text{const}$. If in this plane we move from Q_1 and Q_2 by equal lengths along two straight lines orthogonal to the segment that joins Q_1 and Q_2, the cross spectral density will take on different values. The integral of these values is called the *projection of the cross spectral density* along the two lines.

In the following, we shall prove that, in the paraxial approximation, the projections of the cross spectral density along all the possible pairs of (parallel) lines across a plane $z=\text{const}$ uniquely determine the optical intensity everywhere. Conversely, we shall prove that such projections can be recovered from knowledge of the spatial distribution of the optical intensity. These results show that projections should enjoy a relevant status for a partially coherent field. We shall further prove that, in general, the projections do not specify completely the spatial coherence properties of a field, because their knowledge is insufficient for a unique recovery of the cross spectral density except in the one-dimensional case. To illustrate this point, we shall work out two simple examples where fields in different states of coherence exhibit identical projections along any pair of lines and therefore give rise to the same optical intensity everywhere. Beyond qualifying projections as significant coherence features of a field, the above results afford an alternative procedure to assess whether two fields possess the same spatial distribution of intensity. In certain cases, in fact, it is easier to use the projections than the expression of the propagated intensity to solve such a problem.

2. The projections of the CSD

In this section, we define more precisely the projections of the cross spectral density (CSD from now on) along any pair of parallel lines. Furthermore, we show that projections at different planes are connected by a suitable propagation law. As we shall see, the form of this law suggests that the projections are the basic coherence features in determining the optical intensity throughout the space.

Using the radius vectors shown in fig. 1, we denote by $W_z(\boldsymbol{r}_1, \boldsymbol{r}_2)$ the CSD across a typical plane $z=\text{const}$. The explicit dependence on the temporal frequency is omitted. At the chosen plane we consider two parallel lines, say m_1 and m_2. Take an arbitrary straight line orthogonal to m_1 and m_2 and denote by Q_1 and Q_2 the corresponding intersection points (fig. 1). As we said in the introduction, the projection is obtained by integrating the values taken on by W_z when Q_1 and Q_2 sweep the lines m_1 and m_2, respectively, under parallel displacement.

In order to give an explicit expression for these projections, let (x_1, y_1) and (x_2, y_2) be the coordinates of Q_1 and Q_2, respectively, and let ϕ be the angle between the line Q_1Q_2 and the x-axis (see fig. 2). We introduce a new reference frame, say $x'y'$, rotated by an angle ϕ with respect to xy. The new coordinates of Q_1 and Q_2 are of the form (d_1, y') and (d_2, y'), respectively, where d_1 and d_2 are the distances of the two lines from the origin. The relationships with the old coordinates are

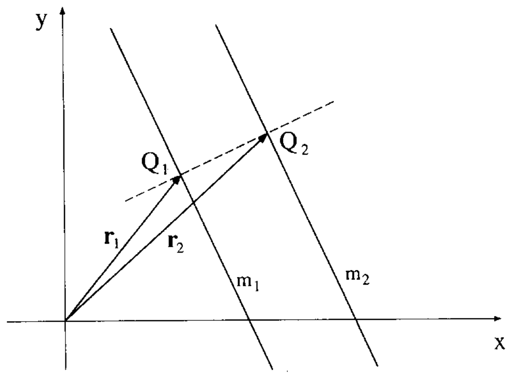

Fig. 1. A generic pair of parallel lines on a plane z=const. We show a typical couple of points that, sweeping the two lines, give the values of the CSD we integrate to obtain the projection.

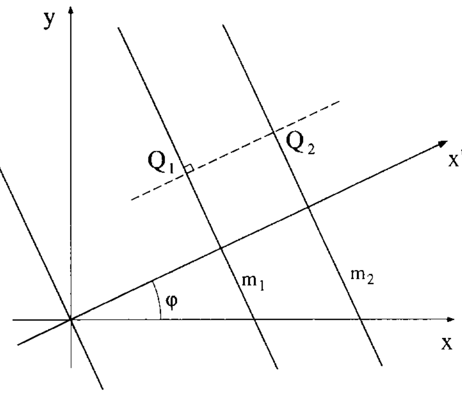

Fig. 2. The rotated frame we use to give an explicit form to the projection.

$$x_1=d_1C-y'S\,,\quad y_1=d_1S+y'C\,;\quad x_2=d_2C-y'S\,,\quad y_2=d_2S+y'C\,, \tag{2.1}$$

where $C=\cos\phi$ and $S=\sin\phi$. If we change y', the points Q_1 and Q_2 move along the lines m_1 and m_2. According to the previously given definition, the projection of the CSD along m_1 and m_2 is given by

$$P_z(d_1,d_2,\phi)=\int W_z(d_1C-y'S,\,d_1S+y'C;\,d_2C-y'S,\,d_2S+y'C)\,\mathrm{d}y'\,, \tag{2.2}$$

where, to avoid the introduction of new symbols, we use again the letter W_z to denote the CSD at z=const even if the arguments are now four scalar variables instead of two vectorial ones. It is seen that such a projection is a function of three variables: the distances of the selected lines from the origin and the angle between the line direction and the y-axis. The points Q_1 and Q_2, which were used to introduce the concept, actually play no role.

It is easy to show that

$$P_z(d_2,d_1,\phi)=P_z^*(d_1,d_2,\phi)\,, \tag{2.3}$$

because the CSD has to be hermitian [2], and that

$$P_z(-d_1,-d_2,\phi+\pi)=P_z(d_1,d_2,\phi)\,, \tag{2.4}$$

because such values of the variables would define the same pair of lines.

For our purposes, the most relevant property of the projections is the way they propagate in space. We shall presently show that the projection along a pair of lines at a certain angle on the plane z=const is connected by a propagation integral to the whole set of projections at the same angle on the plane $z=0$. We recall that, in the paraxial approximation [1], the following propagation formula holds

$$W_z(\boldsymbol{r}_1,\boldsymbol{r}_2)=\frac{1}{(\lambda z)^2}\iint W_0(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)\exp\{\mathrm{i}(\pi/\lambda z)[(\boldsymbol{r}_1-\boldsymbol{\rho}_1)^2-(\boldsymbol{r}_2-\boldsymbol{\rho}_2)^2]\}\,\mathrm{d}^2\rho_1\,\mathrm{d}^2\rho_2\,, \tag{2.5}$$

where λ is the wavelength corresponding to the selected temporal frequency and W_0 is the CSD on the plane $z=0$. It is to be noted that eq. (2.5) can be used for both positive z (direct propagation) and negative z (inverse propagation). On a typical plane $z=\text{const}\neq 0$, let us consider two parallel lines with slope $-\text{cotan}\,\phi$, whose distances from the origin are d_1 and d_2. The value of the CSD at the two points whose coordinates are given by eqs. (2.1) can be evaluated by mean of eq. (2.5) as follows

$$W_z(d_1C-y'S, d_1S+y'C; d_2C-y'S, d_2S+y'C) = \frac{1}{(\lambda z)^2} \iiiint W_0(\xi'_1C-\eta'_1S, \xi'_1S+\eta'_1C; \xi'_2C-\eta'_2S, \xi'_2S+\eta'_2C)$$

$$\times \exp\{i(\pi/\lambda z)[(d_1-\xi'_1)^2-(d_2-\xi'_2)^2+(y'-\eta'_1)^2-(y'-\eta'_2)^2]\}\, d\xi'_1\, d\xi'_2\, d\eta'_1\, d\eta'_2 . \quad (2.6)$$

A change of variables has been used in the plane $z=0$, passing from two axes, say ξ and η, parallel to x and y, respectively, to a new frame, ξ', η', whose axes are rotated by an angle ϕ. On integrating both sides of eq. (2.6) with respect to y' and taking into account eq. (2.2) we obtain

$$P_z(d_1, d_2, \phi) = \frac{1}{\lambda|z|} \iint \exp\{i(\pi/\lambda z)[(d_1-\xi'_1)^2-(d_2-\xi'_2)^2]\}\, P_0(\xi'_1, \xi'_2, \phi)\, d\xi'_1\, d\xi'_2 . \quad (2.7)$$

This is the propagation law for the projections. It is worth noting that in the one-dimensional case the CSD's on the planes $z=\text{const}\neq 0$ and $z=0$ are connected by a propagation formula with exactly the same structure as eq. (2.7).

As a consequence of eq. (2.7), two fields possessing the same projections of the CSD along all the pairs of lines defined by a certain angle on the plane $z=0$, also have the same projections along equally oriented pairs on any plane $z=\text{const}$.

The projections of the CSD, just like the optical intensity, are three-dimensional functions satisfying certain constraints (see eqs. (2.3) and (2.4)). In addition, they propagate in the same way as a one-dimensional CSD, which is completely recovered by knowledge of the optical intensity [3]. Accordingly, we may guess that the projections and the optical intensity carry the same amount of information, and this is what we show in the next section.

3. The role of the projections

In this section, we shall establish the relationship between the spatial distribution of the optical intensity and the projections of the CSD.

Letting $\boldsymbol{r}_1=\boldsymbol{r}_2=\boldsymbol{r}$ in eq. (2.5) we obtain the optical intensity $I(\boldsymbol{r}, z)$ at the point $\boldsymbol{r}$, z in the paraxial approximation:

$$I(\boldsymbol{r},z) = \frac{1}{(\lambda z)^2} \iint W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \exp\{(2\pi i/\lambda z)[(\rho_1^2-\rho_2^2)/2-\boldsymbol{r}\cdot(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)]\}\, d^2\rho_1\, d^2\rho_2 . \quad (3.1)$$

This is the basic link between the space intensity distribution and the CSD at $z=0$. It is mainly on the consequences of this formula that we shall work in the following. We shall first inquire about the coherence features that can be retrieved starting from knowledge of the spatial distribution of the optical intensity. For any $z\neq 0$ we introduce the new variables

$$\boldsymbol{\sigma} = (\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)/2 , \quad \boldsymbol{\tau} = (\boldsymbol{\rho}_2-\boldsymbol{\rho}_1)/\lambda z . \quad (3.2)$$

Then, eq. (3.1) can be written

$$I(\boldsymbol{r}, z) = \int \exp(2\pi i \boldsymbol{r}\cdot\boldsymbol{\tau})\, d^2\tau \int L_0\left(\boldsymbol{\sigma}, \frac{\lambda z}{2}\boldsymbol{\tau}\right) \exp(-2\pi i \boldsymbol{\sigma}\cdot\boldsymbol{\tau})\, d^2\sigma , \quad (3.3)$$

where

$$L_0(\boldsymbol{s}, \boldsymbol{t}) = W_0(\boldsymbol{s}-\boldsymbol{t}, \boldsymbol{s}+\boldsymbol{t}) . \quad (3.4)$$

We denote by $\tilde{I}(\boldsymbol{p}, z)$ and $\tilde{L}_0(\boldsymbol{p}, \boldsymbol{t})$ the Fourier transforms, with respect to the first variable, of $I(\boldsymbol{r}, z)$ and $L_0(\boldsymbol{s}, \boldsymbol{t})$, respectively

$$\tilde{I}(\boldsymbol{p}, z) = \int I(\boldsymbol{r}, z) \exp(-2\pi i \boldsymbol{r}\cdot\boldsymbol{p})\, d^2 r\,, \tag{3.5}$$

$$\tilde{L}_0(\boldsymbol{p}, \boldsymbol{t}) = \int L_0(\boldsymbol{s}, \boldsymbol{t}) \exp(-2\pi i \boldsymbol{p}\cdot\boldsymbol{s})\, d^2 s\,. \tag{3.6}$$

Using eqs. (3.5) and (3.6), the following relation is easily derived from eq. (3.3)

$$\tilde{I}(\boldsymbol{p}, z) = \tilde{L}_0\left(\boldsymbol{p}, \frac{\lambda z}{2}\boldsymbol{p}\right). \tag{3.7}$$

It is not difficult to show that eq. (3.7) holds for $z=0$ too. The one-dimensional version of eq. (3.7) was already established elsewhere [3]. We now discuss the meaning of this equation. If we knew $\tilde{L}_0(\boldsymbol{p}, \boldsymbol{t})$ for all possible pairs of vectors $\boldsymbol{p}$ and $\boldsymbol{t}$ then by Fourier inversion of eq. (3.6) we could recover the function $L_0(\boldsymbol{s}, \boldsymbol{t})$ or, which is the same, the CSD across the plane $z=0$ (see eq. (3.4)). Nevertheless, the knowledge of the optical intensity throughout the space, from which $\tilde{I}(\boldsymbol{p}, z)$ can be derived, gives only partial information about $\tilde{L}_0$. In fact, although the vectorial arguments of $\tilde{L}_0$ in eq. (3.7) can have arbitrary lengths (thanks to the possibility of changing z) they are forced to be parallel to one another. We want to prove now that such a limited information is sufficient to recover the projections of the CSD.

We shall begin our proof by writing the Fourier inverse of eq. (3.6). As we need to use cartesian coordinates, we write such an inverse as

$$L_0(s_x, s_y; t_x, t_y) = \iint \tilde{L}_0(p_x, p_y; t_x, t_y) \exp[2\pi i (p_x s_x + p_y s_y)]\, dp_x\, dp_y\,, \tag{3.8}$$

where we use the same symbols L_0 and $\tilde{L}_0$ both in the compact notation with two vectorial arguments and in the extended one with four scalar arguments.

Now we write the projection (2.2) on the plane $z=0$ in terms of L_0:

$$P_0(d_1, d_2, \phi) = \int L_0(CD - Sy', SD + Cy'; C\Delta, S\Delta)\, dy'\,, \tag{3.9}$$

where the lengths D and Δ are defined by

$$D = (d_1 + d_2)/2\,, \quad \Delta = (d_2 - d_1)/2\,, \tag{3.10}$$

and where, as in sect. 2, we used the notations: $C=\cos\phi$, $S=\sin\phi$.

On inserting from eq. (3.8) into eq. (3.9) we get

$$P_0(d_1, d_2, \phi) = \iint \tilde{L}_0(p_x, p_y; C\Delta, S\Delta) \exp[2\pi i D(p_x C + p_y S)]\, \delta(C p_y - S p_x)\, dp_x\, dp_y\,, \tag{3.11}$$

where the Fourier expansion of the Dirac δ-function has been used. It is not difficult to show that, for every value of ϕ, eq. (3.11) is equivalent to

$$P_0(d_1, d_2, \phi) = \int \tilde{L}_0(Cp, Sp; C\Delta, S\Delta) \exp(2\pi i p D)\, dp\,. \tag{3.12}$$

Indeed, this formula shows that the projections are determined by the values that $\tilde{L}_0$ takes on not at any pair of vectors, but only at the ones that are parallel to each other. These values are just the information one obtains from optical intensity, as shown by eq. (3.7).

We now insert eq. (3.7) into eq. (3.12), letting

$$z = 2\Delta/\lambda p\,. \tag{3.13}$$

Then, using the definitions of C, S, D, and Δ given before, we obtain a formula by which the projections are determined in terms of the Fourier transform of the spatial intensity distribution:

$$P_0(d_1,d_2,\phi)=\int \tilde{I}[p\cos\phi, p\sin\phi;\ (d_2-d_1)/\lambda p]\exp[i\pi(d_1+d_2)p]\,dp\,. \tag{3.14}$$

It will be noted that eq. (3.14) requires the knowledge of the optical intensity throughout the space. In particular, when p crosses the zero, the values of the intensity at $z=\pm\infty$ are required. Disregarding the possible difficulties of obtaining such a complete information, eq. (3.14) shows how, in principle, the projections of the CSD along any pair of lines can be recovered starting from knowledge of the optical intensity.

Next, we shall show that, conversely, the set of projections of the CSD at $z=0$ determines the optical intensity everywhere. Let us write eq. (3.3) in the more extended form

$$I(x,y,z)=\iint \exp[2\pi i(x\tau_x+y\tau_y)]\,d\tau_x\,d\tau_y \iint L_0(\sigma_x,\sigma_y;\tfrac{\lambda z}{2}\tau_x,\tfrac{\lambda z}{2}\tau_y)\exp[-2\pi i(\sigma_x\tau_x+\sigma_y\tau_y)]\,d\sigma_x\,d\sigma_y\,. \tag{3.15}$$

Using the changes of variables

$$x=r\cos\vartheta\,,\quad y=r\sin\vartheta\,,\quad \tau_x=\tau\cos\phi\,,\quad \tau_y=\tau\sin\phi\,, \tag{3.16}$$

eq. (3.15) becomes

$$I(r\cos\vartheta, r\sin\vartheta, z)=\int_0^\infty\int_0^{2\pi} \exp[2\pi irt\cos(\vartheta-\phi)]\,\tau\,d\tau\,d\phi \iint L_0\left(\sigma_x,\sigma_y;\frac{\lambda z}{2}\tau\cos\phi,\frac{\lambda z}{2}\tau\sin\phi\right)$$
$$\times\exp[-2\pi i\tau(\sigma_x\cos\phi+\sigma_y\sin\phi)]\,d\sigma_x\,d\sigma_y\,. \tag{3.17}$$

In the last integral, we pass from the variables σ_x, σ_y to the new variables σ'_x, σ'_y, corresponding to a rotation of the reference axes by an angle ϕ. More precisely, as in eq. (2.1), we let

$$\sigma_x=\sigma'_x\cos\phi-\sigma'_y\sin\phi\,,\quad \sigma_y=\sigma'_x\sin\phi+\sigma'_y\cos\phi\,. \tag{3.18}$$

On inserting from eq. (3.18) into eq. (3.17) and using the relation (3.9) we easily find

$$I(r\cos\vartheta, r\sin\vartheta, z)=\int_0^\infty\int_0^{2\pi} \exp[2\pi ir\tau\cos(\vartheta-\phi)]\,\tau\,d\tau\,d\phi\int\exp(-2\pi i\tau\sigma'_x)\,d\sigma'_x$$
$$\times\int L_0\left(\sigma'_x\cos\phi-\sigma'_y\sin\phi,\sigma'_x\sin\phi+\sigma'_y\cos\phi;\frac{\lambda z}{2}\tau\cos\phi,\frac{\lambda z}{2}\tau\sin\phi\right)d\sigma'_y$$
$$=\int_0^\infty\int_0^{2\pi}\exp[2\pi ir\tau\cos(\vartheta-\phi)]\,\tau\,d\tau\,d\phi\int P_0\left(\sigma'_x-\frac{\lambda z}{2}\tau,\sigma'_x+\frac{\lambda z}{2}\tau,\phi\right)\exp(-2\pi i\tau\sigma'_x)\,d\sigma'_x\,. \tag{3.19}$$

In conclusion, eq. (3.19) proves that the complete spatial intensity distribution can be determined once the projections of W_0 are known for all possible pairs of lines on $z=0$.

Before ending this section, we add two remarks. First, we tacitly excluded in the above considerations the one-dimensional case. This case has been dealt with already [3]. If we look at it as a particular case, we easily see that the projections of W_0 are now equivalent to W_0 itself. Second, we note that the previous results justify the following equivalence statement: "A necessary and sufficient condition for two fields to produce the same optical intensity everywhere is that their CSD's possess the same projections along any pair of lines at a certain plane $z=\text{const.}$" The choice of the plane is completely arbitrary because, as we have shown at the end of sect. 2, the equality of the projections is conserved in paraxial propagation.

4. Examples

The requirement that two fields have CSD's with identical projections along any pair of lines at $z=0$ is a strong one. So strong, in fact, that one might wonder whether it implies a complete coincidence of the two CSD's. In order to show that this is not the case, it may be useful to examine some examples.

In a previous paper [3] it was shown that there exists a whole class of partially coherent fields, characterized by different CSD's on the plane $z=0$, and yet producing the same optical intensity everywhere in the space. Such fields are of the form

$$V(\boldsymbol{r})=c^{(+)}V_n^{(+)}(r,\vartheta)+c^{(-)}V_n^{(-)}(r,\vartheta)\,, \tag{4.1}$$

where r and ϑ are polar coordinates in the plane $z=0$, and $V_n^{(+)}(r,\vartheta)$ and $V_n^{(-)}(r,\vartheta)$ are given by

$$V_n^{(\pm)}(\boldsymbol{r})=F(r)\exp(\pm\mathrm{i}n\vartheta)\,, \tag{4.2}$$

where $F(r)$ is an arbitrary function of the modulus of $\boldsymbol{r}$ and n is an integer. The coefficients $c^{(+)}$ and $c^{(-)}$ are uncorrelated random variables, whose mean square values give a fixed sum, say T:

$$T=\langle|c^{(+)}|^2\rangle+\langle|c^{(-)}|^2\rangle=\text{const}\,, \tag{4.3}$$

where the angular brackets denote ensemble average. For each field in this class we introduce a second parameter, namely,

$$\epsilon=\langle|c^{(+)}|^2\rangle-\langle|c^{(-)}|^2\rangle\,. \tag{4.4}$$

According to the coherence theory in the space–frequency domain [1], the CSD in the plane $z=0$ is

$$W_0(\boldsymbol{r}_1,\boldsymbol{r}_2)=\langle V(\boldsymbol{r}_1)\,V^*(\boldsymbol{r}_2)\rangle=F(r_1)\,F^*(r_2)\exp[\mathrm{i}\psi_n(\vartheta_1-\vartheta_2)]$$
$$\times\{T^2\cos^2[n(\vartheta_1-\vartheta_2)]+\epsilon^2\sin^2[n(\vartheta_1-\vartheta_2)]\}^{1/2}\,, \tag{4.5}$$

where

$$\psi_n(\vartheta_1-\vartheta_2)=\tan^{-1}\{(\epsilon/T)\tan[n(\vartheta_1-\vartheta_2)]\}\,. \tag{4.6}$$

In particular, letting $r_1=r_2=r$; $\vartheta_1=\vartheta_2=\vartheta$, we find that the optical intensity is circularly symmetric and has the expression

$$I(\boldsymbol{r},0)=T|F(r)|^2\,. \tag{4.7}$$

On inserting eqs. (4.5) and (4.7) into the definition of the degree of spectral coherence [1],

$$\mu_0(\boldsymbol{r}_1,\boldsymbol{r}_2)=W_0(\boldsymbol{r}_1,\boldsymbol{r}_2)/[I(\boldsymbol{r}_1,0)I(\boldsymbol{r}_2,0)]^{1/2} \tag{4.8}$$

we find

$$\mu_0(\boldsymbol{r}_1,\boldsymbol{r}_2)=\exp\{\mathrm{i}[\phi_1-\phi_2+\psi_n(\vartheta_1-\vartheta_2)]\}\{\cos^2[n(\vartheta_1-\vartheta_2)]+(\epsilon/T)^2\sin^2[n(\vartheta_1-\vartheta_2)]\}^{1/2}\,, \tag{4.9}$$

where ϕ_j is the argument of the function $F(r_j)$, i.e.,

$$\phi_j=\arg[F(r_j)]\,,\quad j=1,2\,. \tag{4.10}$$

The meaning of the present results can be seen from eqs. (4.7) and (4.9). The former shows that the optical intensity at $z=0$ does not depend on ϵ, whereas the latter shows that μ_0 is a function of ϵ. In this way, we have a one-parameter family of fields all of which possess the same intensity while possessing different coherence properties. The parameter ϵ can vary from $-T$ to T. The extreme values ($|\epsilon|=T$) correspond to the two coherent cases in which only one of the two fields $V_n^{(+)}$ and $V_n^{(-)}$ is present. The modulus of μ_0 is then equal to one. Any other value of ϵ gives a partially coherent field.

In a previous paper [3], we proved by explicit evaluation that all such fields produce the same intensity everywhere in space. Now we are going to give an alternative approach, showing that all the fields of this class lead to identical projections of the CSD along any pair of lines.

Using the cartesian coordinates $x=r\cos\vartheta$, $y=r\sin\vartheta$ and defining the function $G(r)$ so that

$$F(r)=G(r)\,r^n\,, \tag{4.11}$$

we write the fields of eq. (4.2) in the form

$$V_n^{(\pm)}(\boldsymbol{r})=G(r)\,(x\pm \mathrm{i}y)^n\,. \tag{4.12}$$

Starting from the field (4.1), the CSD in the plane $z=0$ can be evaluated, yielding

$$W_0(\boldsymbol{r}_1,\boldsymbol{r}_2)=G(r_1)\,G^*(r_2)\,\{\langle|c^{(+)}|^2\rangle[x_1x_2+y_1y_2+\mathrm{i}(x_2y_1-x_1y_2)]^n + \langle|c^{(-)}|^2\rangle[x_1x_2+y_1y_2-\mathrm{i}(x_2y_1-x_1y_2)]^n\}\,. \tag{4.13}$$

By means of the definitions (4.3) and (4.4) and letting

$$\alpha=x_1x_2+y_1y_2\,,\quad \beta=x_2y_1-x_1y_2\,,\quad \zeta=\alpha+\mathrm{i}\beta\,, \tag{4.14a,b,c}$$

the CSD takes the form

$$W_0(\boldsymbol{r}_1,\boldsymbol{r}_2)=G(r_1)\,G^*(r_2)\left(T\frac{\zeta^n+\zeta^{*n}}{2}+\epsilon\frac{\zeta^n-\zeta^{*n}}{2}\right), \tag{4.15}$$

because α and β are real quantities. Since $(\zeta^*)^n=(\zeta^n)^*$, the coefficients of T and ϵ in eq. (4.15) coincide with the real part of ζ^n and with i times the imaginary part of ζ^n, respectively. By the expansion of ζ^n,

$$\zeta^n=(\alpha+\mathrm{i}\beta)^n=\sum_{k=0}^{n}\binom{n}{k}(\mathrm{i}\beta)^k\,\alpha^{n-k}\,, \tag{4.16}$$

it can be seen that only even powers of β contribute to the real part of ζ^n, and, conversely, only odd powers to the imaginary one. As we shall see in a moment, this fact implies that all the projections of W_0 are independent from ϵ.

Because of the definition of the projection (eq. (2.2)), we have to evaluate the CSD at the points whose coordinates are given by eq. (2.1). By substituting those values into α and β (eqs. (4.14a) and (4.14b)), we obtain

$$\alpha=d_1d_2+y'^2\,,\quad \beta=(d_1-d_2)y'\,, \tag{4.17a,b}$$

so that α is an even function of y', while β is an odd one. The functions G and G^* are even too, because

$$r_j=(d_j^2+y'^2)^{1/2}\,,\quad j=1,2\,. \tag{4.18}$$

Thanks to the remarks following eq. (4.16), it is easy to conclude that the coefficient of ϵ in eq. (4.15) is an odd function of the integration variable y'. Hence, after integration, this coefficient vanishes. This proves that the projections depend on T, whereas they are insensitive to the coherence parameter ϵ.

It will be noted that eq. (4.2) encompasses as particular cases the well known Laguerre–Gauss beams [5]. As a consequence, the previously discussed properties can be exhibited by rather common light fields.

In the previous example we dealt with fields having radially symmetric intensity. It is worthwhile to give a different example in which this property does not hold. Let us consider an ensemble of fields of the form

$$V(\mathbf{r})=[c^{(+)}(ax+iby)+c^{(-)}(ax-iby)]\,F(r)\,, \tag{4.19}$$

where V is the disturbance at the point $\mathbf{r}$ of the plane $z=0$, $c^{(+)}$ and $c^{(-)}$ are random uncorrelated coefficients and $F(r)$ an arbitrary function of the modulus of $\mathbf{r}$, while a and b are real constants. The CSD pertaining to the fields (4.19) is

$$W_0(\mathbf{r}_1,\mathbf{r}_2)=F(r_1)\,F^*(r_2)\,[T(a^2x_1x_2+b^2y_1y_2)-\mathrm{i}\epsilon ab(x_1y_2-x_2y_1)]\,, \tag{4.20}$$

where T and ϵ are defined in eqs. (4.3) and (4.4). The corresponding optical intensity at $z=0$, say $I(\mathbf{r},0)$, is

$$I(\mathbf{r},0)\equiv W_0(\mathbf{r};\mathbf{r})=T(a^2x^2+b^2y^2)\,|F(r)|^2\,. \tag{4.21}$$

Furthermore, on evaluating the degree of spectral coherence given by eq. (4.8), we find

$$\mu_0(\mathbf{r}_1;\mathbf{r}_2)=\frac{(a^2x_1x_2+b^2y_1y_2)-\mathrm{i}(\epsilon/T)ab(x_1y_2-x_2y_1)}{[(a^2x_1^2+b^2y_1^2)(a^2x_2^2+b^2y_2^2)]^{1/2}}\exp[\mathrm{i}(\phi_1-\phi_2)]\,, \tag{4.22}$$

where we used the notation (4.10).

Let us now discuss the main features of these fields. Even in this case, once T has been fixed, we have a class of fields depending on a single real parameter ϵ. All these fields produce the same intensity across the plane $z=0$ while the degree of spectral coherence depends on ϵ. In particular, for $\epsilon=\pm T$ complete coherence is obtained ($|\mu_0|=1$) whereas this is not true for $|\epsilon|<T$.

We maintain that all the fields of the present class give rise to identical projections of W_0 along any pair of lines. Inasmuch as ϵ only appears in the second term on the right-hand side of eq. (4.20), it is enough to show that such a part of W_0 does not contribute to the projections. On inserting the specified term into eq. (2.2) and taking into account eq. (2.1), the equation to be proved is

$$\int[(d_1C-y'S)(d_2S+y'C)-(d_2C-y'S)(d_1S+y'C)]\,F(\sqrt{d_1^2+y'^2})\,F^*(\sqrt{d_2^2+y'^2})\,\mathrm{d}y'=0\,, \tag{4.23}$$

where $C=\cos\phi$, $S=\sin\phi$. By simple algebra, eq. (4.23) becomes

$$(d_1-d_2)\int y'\,F(\sqrt{d_1^2+y'^2})\,F^*(\sqrt{d_2^2+y'^2})\,\mathrm{d}y'=0\,, \tag{4.24}$$

and this is obviously true because an odd function of y' appears under the integral sign.

Because of the equivalence statement given at the end of sect. 3, we can conclude that, for a fixed value of T, all the fields described by eq. (4.20) give rise to the same intensity throughout the space regardless of their coherence properties. This conclusion could also be tested by computing explicitly the optical intensity in space. However, it would be seen that, for the present example, the projection approach is simpler to use than the direct evaluation of the optical intensity.

5. Conclusions

It is a widely held opinion that the spatial features of a light field are completely specified once its optical intensity is known throughout the space (at each temporal frequency). In particular, it tends to be taken for granted that fields in different states of coherence cannot give rise to the same intensity everywhere in space. This is perhaps to be ascribed to the idea that a reduction of the spatial coherence of a field implies necessarily an increase of its angular width. It was shown, however, that in certain cases [3] two fields with different coherence properties can be indistinguishable as far as the space distribution of intensity is concerned. On the other hand, the different state of coherence of two such fields would be revealed in most diffraction and interference experiments.

A question that arises quite naturally is the following. Which are the constraints that the space distribution of the optical intensity imposes on the cross spectral density of the field? In other words, which coherence features can be uniquely retrieved starting from knowledge of the optical intensity through the space? In this paper, we found that a possible answer is based on the projections of the cross spectral density. These projections, which are suitable integrals of the cross spectral density, determine uniquely the optical intensity in space and vice versa. Furthermore, we showed that, passing from a plane $z=\text{const}$ to another, the projections obey a propagation law whose dimensionality is smaller than the one pertaining to the cross spectral density. This might be a key for explaining the role of projections in the above problems. Basically, the projection operation gives a function with fewer degrees of freedom than the cross spectral density and this is why projections are connected in an invertible way to the spatial distribution of the optical intensity.

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