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LETTER TO THE EDITOR

# Generalized self-Fourier functions

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**Abstract.** By generalizing the definition of self-Fourier functions recently introduced by Caola, we show that any Fourier transformable function is the linear combination of four self-Fourier functions whose explicit form can be found.

It is well known that certain functions are self-reproducing under Fourier transformation. As remarked by Caola in a recent letter [1], the stock examples are the Gaussian and the comb functions, so that one might suspect that these are the only functions endowed with such a property. Although other examples are sometimes quoted, such as $1/\sqrt{|x|}$ and $\operatorname{sech}(\pi x)$, the idea may survive that all these functions are somehow exceptional. Actually, Caola showed that it is extremely simple to generate functions that are self-reproducing under Fourier transformation, i.e. self-Fourier functions (SFFs), starting from any pair of Fourier transforms (FTs). The relevance of these results for optical applications has been pointed out by Caola [1] and Lohmann [2].

By a slight extension of Caola's definition, we show that not only are SFFs by no means exceptional but any Fourier transformable function is in itself the sum of four SFFs. Let us denote by $\tilde{f}(\nu)$ the FT of any function $f(x) \in L^2(R)$, namely

$$\tilde{f}(\nu) = \int_{-\infty}^{+\infty} f(x) \exp(-2\pi i \nu x)\, dx. \tag{1}$$

Caola defines a SFF as any function $f(x)$ such that

$$\tilde{f}(\nu) = f(\nu). \tag{2}$$

Here, we use the same name in a slightly generalized sense. More precisely, we say that $f(x)$ is a SFF if it satisfies the following condition

$$\tilde{f}(\nu) = a f(\nu) \tag{3}$$

where $a$ is a constant complex factor. By applying equation (1) twice on the function $f(x)$ and using equation (3) it is evident that $f(-x) = a^2 f(x)$. Hence $f(x)$ is to be symmetric (even or odd) and $a^2 = +1$ or $-1$, depending on the parity of $f(x)$. Once this definition is adopted, a whole set of SFFs can be given. This is the set of Hermite-Gauss functions, defined as

$$G_n(x) = \frac{(2)^{1/4}}{\sqrt{2_n n!}} H_n(\sqrt{2\pi}\, x) \exp(-\pi x^2) \qquad (n = 0, 1, 2, \ldots) \tag{4}$$

where $H_n$ is the $n$th Hermite polynomial. It is known [3], although not often quoted, that the $G_n$ are SFFs. As a matter of fact, the Fourier transformation rule holds:

$$\tilde{G}_n(\nu) = i^{-n} G_n(\nu) \qquad (n = 0, 1, 2, \ldots). \tag{5}$$

The Hermite–Gauss functions form an orthonormal set in the $L^2$ space [3]. Accordingly, any function $g(x)$ belonging to that space can be expanded into a series of the form

$$g(x)=\sum_{n=0}^{\infty} c_n G_n(x). \tag{6}$$

Let us now subdivide the series into four partial series and write

$$g(x)=\sum_{k=0}^{3}\sum_{m=0}^{\infty} c_{k+4m} G_{k+4m}(x). \tag{7}$$

Denote by $g_k(x)$ the partial series

$$g_k(x)=\sum_{m=0}^{\infty} c_{k+4m} G_{k+4m}(x) \qquad (k=0,1,2,3). \tag{8}$$

Let us assume for the moment that the above series are convergent. As we shall see in a moment, such a hypothesis is unnecessary because the function $g_k(x)$ can be evaluated in a different way. Thanks to equation (5), the FT of $g_k$ is given by

$$\tilde{g}_k(\nu)=\mathrm{i}^{-k} g_k(\nu) \qquad (k=0,1,2,3). \tag{9}$$

This proves that $g(x)$ is the sum of four SFFs. Conversely, any function $g(x)$ can be used to generate four SFFs. Actually, the series expression for $g_k(x)$ given by equation (8) could seldom be used due to the need of summing up the series. We shall prove, however, that the functions $g_k(x)$ can be determined in a simpler way.

To this end, let us define the following functions:

$$g^{(+)}(x)=\tfrac{1}{2}[g_0(x)+g_1(x)+g_2(x)+g_3(x)] \tag{10}$$

$$g^{(-)}(x)=\tfrac{1}{2}[g_0(x)-g_1(x)+g_2(x)-g_3(x)]. \tag{11}$$

Taking into account equations (7) and (8) as well as the parity properties of the $G_n$, we see that $g^{(+)}$ and $g^{(-)}$ are also given by

$$g^{(+)}(x)=\tfrac{1}{2}g(x) \qquad g^{(-)}(x)=\tfrac{1}{2}g(-x). \tag{12}$$

Because of the rule (5) the FTs of equations (10) and (11) give

$$\tilde{g}^{(+)}(\nu)=\tfrac{1}{2}[g_0(\nu)-\mathrm{i}g_1(\nu)-g_2(\nu)+\mathrm{i}g_3(\nu)] \tag{13}$$

$$\tilde{g}^{(-)}(\nu)=\tfrac{1}{2}[g_0(\nu)+\mathrm{i}g_1(\nu)-g_2(\nu)-\mathrm{i}g_3(\nu)]. \tag{14}$$

By virtue of equation (12), $\tilde{g}^{(+)}$ and $\tilde{g}^{(-)}$ can also be evaluated as follows:

$$\tilde{g}^{(+)}(\nu)=\tfrac{1}{2}\tilde{g}(\nu) \qquad \tilde{g}^{(-)}(\nu)=\tfrac{1}{2}\tilde{g}(-\nu). \tag{15}$$

We now replace the variable $\nu$ by the variable $x$ in equations (13) and (14) and synthesize equations (10), (11) and (13), (14) in the matrix form

$$\begin{pmatrix} g^{(+)} \\ g^{(-)} \\ \tilde{g}^{(+)} \\ \tilde{g}^{(-)} \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix} \tag{16}$$

where, for the sake of brevity, we have dropped the explicit dependence on $x$. It is easily seen that the matrix leading from the right-hand side column vector to the left-hand side one is unitary. As a consequence, the inverse matrix coincides with the Hermitian transpose. Accordingly, equation (16) can be inverted as follows:

$$\begin{pmatrix} g_0 \\ g_1 \\ g_3 \\ g_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & \mathrm{i} & -\mathrm{i} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -\mathrm{i} & \mathrm{i} \end{pmatrix} \begin{pmatrix} g^{(+)} \\ g^{(-)} \\ \tilde{g}^{(+)} \\ \tilde{g}^{(-)} \end{pmatrix}. \tag{17}$$

Equation (17) furnishes the SFFs $g_k$ as linear combinations of the functions $g^{(+)}$, $g^{(-)}$, $\tilde{g}^{(+)}$ and $\tilde{g}^{(-)}$. These latter, in turn, are immediately evaluated through equations (12) and (15) once $g(x)$ and $\tilde{g}(\nu)$ are known. It is easily seen that, except for a numerical factor, $g_0(x)$ has the form suggested by Caola. On the other hand, $g_1$, $g_2$ and $g_3$ are also SFFs *in the generalized sense of equation* (3).

As an example, in figure 1 we show the four self-Fourier functions obtained with the generating function

$$g(x) = (1+x)\exp(-|x|) \tag{18}$$

whose FT is given by

$$\tilde{g}(\nu) = \frac{2}{1+4\pi^2\nu^2} - \mathrm{i}\,\frac{8\pi\nu}{(1+4\pi^2\nu^2)^2}. \tag{19}$$

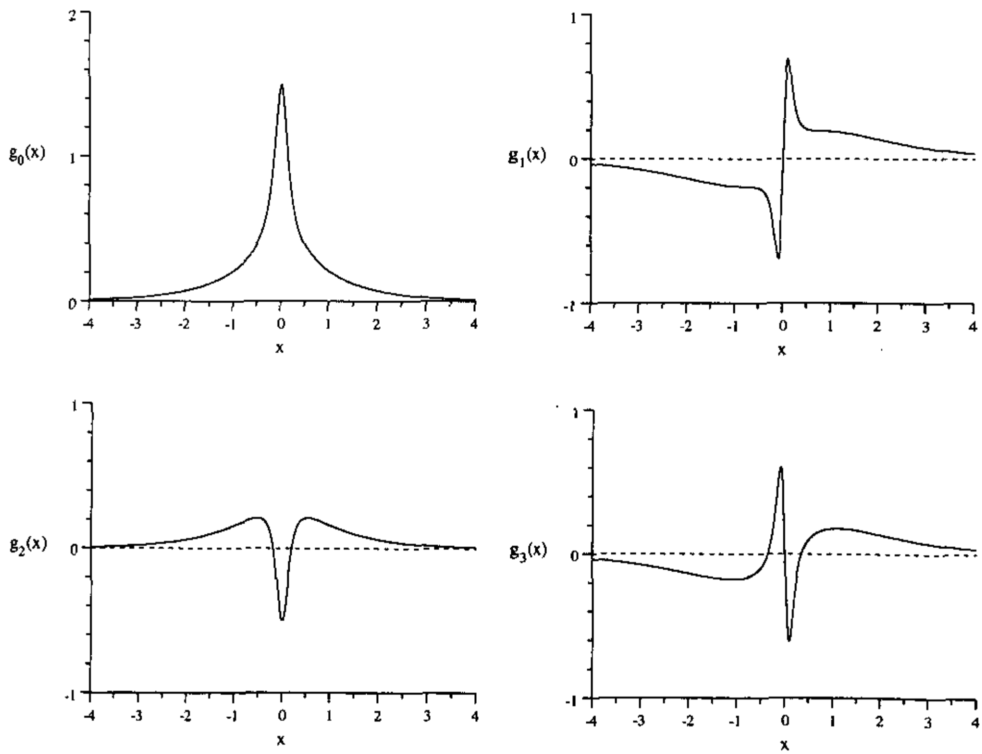


**Figure 1.** Self-Fourier functions $g_0(x)$, $g_1(x)$, $g_2(x)$ and $g_3(x)$.

The explicit form of the functions represented in figure 1 is then

$$g_0(x) = \tfrac{1}{2}\,\mathrm{e}^{-|x|} + \frac{1}{1+4\pi^2x^2} \tag{20$a$}$$

$$g_1(x) = x\left[\tfrac{1}{2}\,\mathrm{e}^{-|x|} + \frac{4\pi}{(1+4\pi^2x^2)^2}\right] \tag{20$b$}$$

$$g_2(x) = \tfrac{1}{2}\,\mathrm{e}^{-|x|} - \frac{1}{1+4\pi^2x^2} \tag{20$c$}$$

$$g_3(x) = x\left[\tfrac{1}{2}\,\mathrm{e}^{-|x|} - \frac{4\pi}{(1+4\pi^2x^2)^2}\right]. \tag{20$d$}$$

We finally note that the previous results can be easily extended to the multidimensional case.

## References

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[3] Arsac J 1966 *Fourier Transforms and the Theory of Distributions* (Englewood Cliffs, NJ: Prentice-Hall) p 63