The change of width for a partially coherent beam on paraxial propagation

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It is shown that for paraxially propagating beams of any state of coherence, the width changes along the propagation axis with the same law as that of a gaussian beam. A quality factor related to the rate of width change can then be defined. A formula expressing this quality factor in terms of the cross spectral density across the beam is derived.

1. Introduction

Let us consider the beam from a laser. Very often it is assumed that the laser oscillates on the fundamental TEM_{00} gaussian mode. In this case, the "width" of the beam at any plane orthogonal to the propagation axis can be measured by the spot size, whose rate of change on paraxial propagation is well known [1]. The inherent simplicity of this approach is due to the fact that the fundamental gaussian mode (and indeed any higher-order gaussian mode) has the property of being shape invariant on paraxial propagation. Unfortunately, many lasers produce beams that are only approximately gaussian, if not altogether different (e.g., beams from unstable cavities). This has two main consequences. First, the beam loses its shape invariance property so that the spot size is no longer well defined. Second, the angular divergence of the beam increases, roughly speaking, with respect to the case of a truly gaussian beam of equivalent waist width.

In order to characterize a real laser beam, one has first to define its width at a typical cross section. Among several possible definitions [2] it has been proposed [3–6] to use the one, well familiar from quantum mechanics, that identifies the squared width with the variance of the squared modulus of the transverse field distribution. We shall recall the exact definition in the next section.

It has been shown [3,5] that for any type of mode the variance changes along the propagation axis according to a law quite similar to the one holding for gaussian beams. More precisely, there exists a waist cross section of the beam where the variance has a minimum. On either sides of this cross section, the variance increases proportionally to the square of the distance from the waist. The proportionality factor accounting for the rate or increase of the variance can be taken as an indication of the beam quality in the sense that smaller values of the proportionality factor denote better-quality beams. For a given waist variance, the minimum value of the proportionality factor is attained by the lowest-order gaussian beam. Then, a dimensionless quality factor can be introduced by comparing the rate of increase of the variance of a given beam with the corresponding rate of a gaussian beam with the same waist width. This is the so called M^2 factor, whose precise definition will be seen later.

It has also been shown [3,5] that the M^2 factor can be defined for the beam produced by a laser oscillating on a superposition of modes (e.g. Hermite–Gauss or Laguerre–Gauss modes).

We can now observe that light beams of a more general nature than just laser beams are of interest, as, for example, beams produced by collimating the light emitted by thermal sources, and in fact partially coherent beams of several types have received a lot of attention in recent years [7–21].

With reference to partially coherent beams of any type, two questions arise. First, does the transverse variance of the beam increase in paraxial propagation with a quadratic law so that an M^2 factor can be defined? Second, what is the relationship between the increase of variance and the coherence properties of the field?

In this paper, we will show that the answer to the first question is in the affirmative. We will further derive an expression for the M^2 factor pertaining to a partially coherent beam of any state of coherence. The main results will be illustrated by means of a simple example.

2. Preliminaries

The spatial properties of a partially coherent field are described, at any temporal frequency, by the cross spectral density [22,23]. Let us refer to a field propagating in the half-space $z \geq 0$ of a certain system of cartesian coordinates x, y, z. For the sake of simplicity, we assume that the fields does not depend on the y-coordinate. Accordingly, we denote the $W_z(x_1, x_2)$ the cross spectral density between two typical points with coordinates x_1 and x_2 at a typical plane $z=$const. In particular, $W_0(x_1, x_2)$ refers to the plane $z=0$. We assume that the temporal frequency has a fixed value and we omit the dependence of the cross spectral density on it. The optical intensity across a plane $z=$const., say $I_z(x)$, is obtained from $W_z(x_1, x_2)$ by letting $x_1=x_2=x$,

$$I_z(x)=W_z(x,x)\,. \tag{2.1}$$

For sufficiently directional fields, the propagation phenomena can be studied in the paraxial approximation [#1], where the relationship between W_0 and W_z is given by [#2]

$$W_z(x_1,x_2)=\alpha\iint W_0(\xi_1,\xi_2) \times\exp\{\pi i\alpha[(x_1-\xi_1)^2-(x_2-\xi_2)^2]\}\,d\xi_1\,d\xi_2\,. \tag{2.2}$$

Here and in the following, infinite limits of integration are assumed. Furthermore, α is given by

$$\alpha=1/\lambda z\,, \tag{2.3}$$

where σ is the wavelength corresponding to the fixed temporal frequency. On inserting from eq. (2.2) into eq. (2.1) we obtain the optical intensity across the plane $z=$const.,

$$I_z(x)=\alpha\iint W_0(\xi_1,\xi_2) \times\exp\{\pi i\alpha[\xi_1^2-\xi_2^2-2x(\xi_1-\xi_2)]\}\,d\xi_1\,d\xi_2\,. \tag{2.4}$$

Let us introduce the two-dimensional Fourier transform of $W_z(x_1, x_2)$, say $\tilde{W}_z(p_1, p_2)$, through the usual formula

$$\tilde{W}_z(p_1,p_2)=\iint W_z(x_1,x_2) \times\exp[-2\pi i(p_1x_1+p_2x_2)]\,dx_1\,dx_2\,. \tag{2.5}$$

Using the convolution theorem for the Fourier transform, we easily obtain the following relation:

$$\tilde{W}_z(p_1,p_2) =\tilde{W}_0(p_1,p_2)\exp[-(\pi i/\alpha)(p_1^2-p_2^2)]\,. \tag{2.6}$$

As is well known [25], the optical intensity distribution in the far field is proportional to the anti-diagonal elements of $\tilde{W}_0$ or, because of eq. (2.6), of $\tilde{W}_z$. More precisely, the quantity $\mathcal{I}^{(\infty)}(p)$ defined as follows

$$\mathcal{I}^{(\infty)}(p)=\tilde{W}_0(p,-p)=\tilde{W}_z(p,-p)\,, \tag{2.7}$$

is proportional to the radiant intensity in a direction

[#1] Conditions under which paraxial propagation can be assumed are discussed in ref. [24].

[#2] The sign to be put in front of the exponent in the integral of eq. (2.2) is plus or minus depending on whether one defines the mutual coherence function in its original form [22] or in its complex conjugate form [23]. As far as classical (i.e. non-quantum) treatments are concerned, the two definitions are equivalent. Here, we use the first one.

specified by p. We are not interested in the proportionality factor because we are going to work with normalized quantities. Accordingly, we shall loosely refer to $\mathscr{I}^{(\infty)}(p)$ as the intensity in the far field.

In defining the width of the beam we shall need the following normalization factor:

$$N=\int I_0(x)\,\mathrm{d}x\,. \tag{2.8}$$

This factor can be thought of as the total power passing through the plane $z=0$ (at the fixed temporal frequency). It is easy to prove the physically obvious fact that such a power remains the same across any plane $z=$const., either in the near or in the far field. In other words, we have the identities

$$N=\int I_z(x)\,\mathrm{d}x=\int \mathscr{I}^{(\infty)}(p)\,\mathrm{d}p\,, \tag{2.9}$$

for any choice of z.

We now define the width Δx_z of the beam at any (finite) value of z as well as the far-field width Δp through the formulas

$$\Delta x_z^2=\frac{1}{N}\int (x-\bar{x}_z)^2 I_z(x)\,\mathrm{d}x\,, \tag{2.10}$$

$$\Delta p^2=\frac{1}{N}\int (p-\bar{p})^2 \mathscr{I}^{(\infty)}(p)\,\mathrm{d}p\,, \tag{2.11}$$

where the central values $\bar{x}_z$ and $\bar{p}$ are defined as follows:

$$\bar{x}_z=\frac{1}{N}\int xI_z(x)\,\mathrm{d}x\,, \tag{2.12}$$

$$\bar{p}=\frac{1}{N}\int p\mathscr{I}^{(\infty)}(p)\,\mathrm{d}p\,. \tag{2.13}$$

3. The change of width of the beam

In this section we inquire about the variation of Δx_z with respect to z. First of all, we observe that $\bar{x}_0$ can be made to vanish by a suitable choice of the origin of the x-axis. Similarly, $\bar{p}$ can also be made to vanish by suitably orienting the z-axis. On assuming $\bar{x}_0=\bar{p}=0$, it can be proved (appendix A) that

$$\bar{x}_z=0\,, \tag{3.1}$$

for any value of z. Accordingly, we shall use eqs. (2.10) and (2.11) in the simplified form

$$\Delta x_z^2=\frac{1}{N}\int x^2 I_z(x)\,\mathrm{d}x\,, \tag{3.2}$$

$$\Delta p^2=\frac{1}{N}\int p^2 \mathscr{I}^{(\infty)}(p)\,\mathrm{d}p\,. \tag{3.3}$$

It is not difficult to show (appendix B) that the following relation holds:

$$\int x^2 I_z(x)\,\mathrm{d}x = \frac{1}{4\pi^2}\int \left(\frac{\partial^2 \tilde{W}_z(p_1,-p_2)}{\partial p_1\,\partial p_2}\right)_{p,p}\mathrm{d}p\,, \tag{3.4}$$

where the double index p, p means that we let $p_1=p_2=p$ after taking the second derivative of $\tilde{W}_z(p_1, -p_2)$. Using eq. (2.6) we obtain

$$\left(\frac{\partial^2 \tilde{W}_z(p_1,-p_2)}{\partial p_1\,\partial p_2}\right)_{p,p}=\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_1\,\partial p_2}\right)_{p,p} + \frac{4\pi^2p^2}{\alpha^2}\tilde{W}_0(p,-p)+\frac{2\pi\mathrm{i}}{\alpha}p\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_1}\right)_{p,p} - \frac{2\pi\mathrm{i}}{\alpha}p\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_2}\right)_{p,p}. \tag{3.5}$$

It can be shown (appendix B) that

$$\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_2}\right)_{p,p}=\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_1}\right)^*_{p,p}, \tag{3.6}$$

where the asterisk denotes the complex conjugate. On inserting from eqs. (3.4), (3.5) and (3.6) into eq. (3.2) we have

$$\Delta x_z^2=\frac{1}{4\pi^2N}\int\left(\frac{\partial^2 \tilde{W}_0(p_1,-p_2)}{\partial p_1\,\partial p_2}\right)_{p,p}\mathrm{d}p + \frac{1}{\alpha^2N}\int p^2\tilde{W}_0(p,-p)\,\mathrm{d}p - \frac{1}{\pi\alpha N}\mathrm{Im}\left[\int p\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_1}\right)_{p,p}\mathrm{d}p\right], \tag{3.7}$$

where Im stands for imaginary part. Taking into account eqs. (3.4) and (3.2), written with $z=0$, as well as eqs. (2.7) and (3.3), we see that eq. (3.7) can be written in the form

$$\Delta x_z^2 = \Delta x_0^2 + \frac{1}{\alpha^2}\Delta p^2 - \frac{1}{\pi\alpha N}\mathrm{Im}\left[\int p\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_1}\right)_{p,p}\mathrm{d}p\right]. \tag{3.8}$$

We now recall eq. (2.3) and we define the quantity

$$\zeta = \frac{1}{2\pi\lambda N\,\Delta p^2}\mathrm{Im}\left[\int p\left(\frac{\partial \tilde{W}_0(p_1,-p_2)}{\partial p_1}\right)_{p,p}\mathrm{d}p\right]. \tag{3.9}$$

Then, eq. (3.8) can be transformed into the formula

$$\Delta x_z^2 = \Delta x_\zeta^2 + \lambda^2\,\Delta p^2\,(z-\zeta)^2\,, \tag{3.10}$$

where

$$\Delta x_\zeta^2 = \Delta x_0^2 - \lambda^2\,\Delta p^2\,\zeta^2\,. \tag{3.11}$$

Eq. (3.10) is the main result of this section. It proves that for any partially coherent beam that can be described in the paraxial approximation the squared width increases quadratically with respect to the distance from a certain waist plane $z=\zeta$, where the width has the minimum value Δx_ζ. The location of the waist plane is given by eq. (3.9) whereas the minimum width is obtained from eq. (3.11).

Another useful result that can be proved without difficulty (appendix C) is the following. If in the plane $z=0$ the cross spectral density is purely real then $\zeta=0$, i.e., the plane $z=0$ coincides with the waist plane.

4. The M_2 factor

Let us consider a spatially coherent TEM_{00} gaussian beam. The optical intensity distribution that it produces at any plane $z=$const. is of the form

$$I_z(x) = I_{\mathrm{M}z}\exp(-2x^2/w_z^2)\,, \tag{4.1}$$

where w_z is the usual spot size [1] and $I_{\mathrm{M}z}$ is the maximum value attained by $I_z(x)$. On inserting from eq. (4.1) into eq. (3.2) we easily find

$$\Delta x_z^2 = w_z^2/4\,. \tag{4.2}$$

The well-known law for the variation of the spot size [1] can then be expressed by means of the width and gives

$$\Delta x_z^2 = \Delta x_\zeta^2 + \frac{\lambda^2}{16\pi^2\,\Delta x_\zeta^2}(z-\zeta)^2\,, \tag{4.3}$$

where $z=\zeta$ is the plane of the waist of the gaussian beam. We can now recast eq. (3.10) in a form analogous to eq. (4.3). To this aim, we introduce the so called M^2 factor [3–5],

$$M^2 = 4\pi\,\Delta x_\zeta\,\Delta p. \tag{4.4}$$

Using this definition, eq. (3.10) can be written

$$\Delta x_z^2 = \Delta x_\zeta^2 + M^4\frac{\lambda^2}{16\pi^2\,\Delta x_\zeta^2}(z-\zeta)^2\,. \tag{4.5}$$

This equation holds for beams of any state of coherence, provided only that the paraxial approximation can be used. The comparison between eqs. (4.3) and (4.5) shows that for a spatially coherent, lowest-order gaussian mode the M^2 factor is unity. As a matter of fact, this is the case in which the M^2 factor has the minimum value among all possible spatially coherent distributions [3–5]. This could be expected from the analogy with similarly defined quantities of wave mechanics or indeed with similar considerations relating to the concept of coherence time [22]. For a laser oscillating on a superposition of gaussian modes of any order, the M^2 factor can also be evaluated.

For a partially coherent beam of any origin, the M^2 factor (4.4) can be evaluated by first computing ζ through eq. (3.9) and then using eq. (3.2) with $z=\zeta$ and eq. (3.3). To bring into evidence the role played by the coherence characteristics it is useful to give eq. (3.3) a different form. The following equality is easily proved (appendix B):

$$\int p^2\mathscr{I}^{(\infty)}(p)\,\mathrm{d}p = \frac{1}{4\pi^2}\int\left(\frac{\partial^2 W_z(x_1,x_2)}{\partial x_1\,\partial x_2}\right)_{x,x}\mathrm{d}x\,, \tag{4.6}$$

for any value of z. In particular, we let $z=\zeta$. With the aid of eqs. (2.9) and (4.6), the insertion of eqs. (3.2) and (3.3) into eq. (4.4) gives

$$M^2=2\left(\int I_\zeta(x)\,dx\right)^{-1}$$
$$\times\left[\int x^2 I_\zeta(x)\,dx \int \left(\frac{\partial^2 W_\zeta(x_1,x_2)}{\partial x_1\,\partial x_2}\right)_{x,x} dx\right]^{1/2}. \tag{4.7}$$

Eq. (4.7) is the main result of this section. It shows how the M^2 factor depends on the coherence properties of the beam across the waist plane. As a particular case, the field can be coherent. The cross spectral density is then of the form [26]

$$W_\zeta(x_1,x_2)=V_\zeta(x_1)V_\zeta^*(x_2)\,, \tag{4.8}$$

where $V_\zeta(x)$ is the coherent field distribution across the waist plane. Eq. (4.6) then reduces to

$$\int p^2 \mathscr{I}^\infty(p)\,dp=\frac{1}{4\pi^2}\int |dV_\zeta(x)/dx|^2 dx\,, \tag{4.9}$$

and eq. (4.7) is to be modified accordingly.

5. An example

Let us consider a partially coherent beam whose cross spectral density in the plane $z=0$ is given by

$$W_0(x_1,x_2)$$
$$=I_0\exp\left(-\frac{x_1^2+x_2^2}{w_0^2}\right)\mathrm{sinc}\left(\frac{x_1-x_2}{L}\right), \tag{5.1}$$

where I_0, w_0 and L are constants and $\mathrm{sinc}(t)$ stands for $\sin(\pi t)/(\pi t)$. The plane $z=0$ can be though of as a secondary Schell-model source [27] with a gaussian profile of optical intensity and a sinc-shaped degree of spectral coherence. Such a source could be easily synthesized starting from a primary spatially incoherent source in the form of a slit. By virtue of the van Cittert–Zernike theorem [22], the degree of spectral coherence in the far field of the slit source is a sinc function. Then, letting the far-field radiation pass through a gaussian transparency produces the required secondary source.

Accordingly to a result given in section 3, the waist plane is the plane $z=0$ itself, because the cross spectral density (5.1) is real. Letting $\zeta=0$ in eq. (4.7) we can easily evaluate the M^2 factor pertaining to the beam described by eq. (5.1). The result is as follows:

$$M^2=\sqrt{1+\pi^2 w_0^2/3L^2}\,. \tag{5.2}$$

It is seen that the M^2 factor is an increasing function of the ratio w_0/L, i.e., of the ratio between the spot size and the parameter L, whose physical meaning is that of a transverse coherence length. If L is much greater than w_0, the source is highly coherent and the M^2 factor is near to one just as in the case of an ideal gaussian beam. On the other hand, if L is of the order of w_0 or smaller than that, the reduced coherence gives rise to an increase of M^2.

Appendix A

Let us derive the law of variation of $\bar{x}_z$ with respect to z. First, we note that eq. (2.12) can be written in the form

$$\bar{x}_z=\frac{1}{N}\int [x_1 W_z(x_1,x_2)]_{x,x}\,dx\,, \tag{A1}$$

where the double index x, x means that we set $x_1=x_2=x$. We now take the derivative of both sides of eq. (2.5) with respect to p_1 and then we make a Fourier inversion. This gives

$$x_1 W_z(x_1,x_2)=-\frac{1}{2\pi i}\iint \frac{\partial \tilde{W}_z(p_1,p_2)}{\partial p_1}$$
$$\times\exp[2\pi i(p_1x_1+p_2x_2)]\,dp_1\,dp_2\,. \tag{A.2}$$

On substituting from (A.2) into (A.1), we obtain

$$\bar{x}_z=-\frac{1}{2\pi i N}\iint \frac{\partial \tilde{W}_z(p_1,p_2)}{\partial p_1}\,dp_1\,dp_2$$
$$\times\int \exp[2\pi i x(p_1+p_2)]\,dx\,. \tag{A.3}$$

The last integral equals the Dirac function $\delta(p_1+p_2)$. It follows at once that eq. (A.3) reduces to

$$\bar{x}_z=-\frac{1}{2\pi i N}\int \left(\frac{\partial \tilde{W}_z(p_1,-p_2)}{\partial p_1}\right)_{p,p} dp\,, \tag{A.4}$$

where the double index p, p means that we set $p_1=p_2=p$ after taking the derivative. On using eq. (2.6), we have

$$\left(\frac{\partial \tilde{W}_z(p_1, -p_2)}{\partial p_1}\right)_{p,p} = \left(\frac{\partial \tilde{W}_0(p_1, -p_2)}{\partial p_1}\right)_{p,p} - \frac{2\pi i p}{\alpha} \tilde{W}_0(p, -p). \quad (A.5)$$

Therefore, eq. (A.4) becomes

$$\bar{x}_z = -\frac{1}{2\pi i N} \int \left(\frac{\partial \tilde{W}_0(p_1, -p_2)}{\partial p_1}\right)_{p,p} \mathrm{d}p + \frac{1}{\alpha N} \int p \tilde{W}_0(p, -p)\, \mathrm{d}p. \quad (A.6)$$

Taking into account eqs. (2.3) and (2.13) as well as eq. (A.4) written for $z=0$, we see that eq. (A.6) reduces to

$$\bar{x}_z = \bar{x}_0 + \lambda z \bar{p}. \quad (A.7)$$

This is the required law of variation for $\bar{x}_z$. It follows at once that eq. (3.1) holds when $\bar{x}_0 = \bar{p} = 0$.

Appendix B

Here, we shall prove eqs. (3.4), (3.6) and (4.6). We first write eq. (2.5) with p_2 replaced by $-p_2$. Then, on taking the second derivative of $\tilde{W}_z(p_1, -p_2)$ with respect to p_1 and p_2 we obtain

$$\frac{\partial^2 \tilde{W}_z(p_1, -p_2)}{\partial p_1\, \partial p_2} = 4\pi^2 \iint x_1 x_2 W_z(x_1, x_2) \times \exp[-2\pi i(p_1 x_1 - p_2 x_2)]\, \mathrm{d}x_1\, \mathrm{d}x_2. \quad (B.1)$$

On letting $p_1 = p_2 = p$ and on integrating with respect to p, we have

$$\int \left(\frac{\partial^2 \tilde{W}_z(p_1, -p_2)}{\partial p_1\, \partial p_2}\right)_{p,p} \mathrm{d}p = 4\pi^2 \iint x_1 x_2 W_z(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \times \int \exp[-2\pi i p(x_1 - x_2)]\, \mathrm{d}p. \quad (B.2)$$

Eq. (3.4) then follows at once because the last integral equals $\delta(x_1 - x_2)$.

In order to prove eq. (3.6) we start again from eq. (2.5) written at $z=0$ and with p_2 replaced by $-p_2$. We then take the derivatives of $\tilde{W}_0(p_1, -p_2)$ with respect to p_1 and p_2 and set $p_1 = p_2 = p$. The resulting formulas are

$$\left(\frac{\partial \tilde{W}_0(p_1, -p_2)}{\partial p_1}\right)_{p,p} = 2\pi i \iint x_1 W_0(x_1, x_2) \times \exp[-2\pi i p(x_1 - x_2)]\, \mathrm{d}x_1\, \mathrm{d}x_2, \quad (B.3)$$

$$\left(\frac{\partial \tilde{W}_0(p_1, -p_2)}{\partial p_2}\right)_{p,p} = -2\pi i \iint x_2 W_0(x_1, x_2) \times \exp[-2\pi i p(x_1 - x_2)]\, \mathrm{d}x_1\, \mathrm{d}x_2. \quad (B.4)$$

Interchanging the variables x_1 and x_2 in eq. (B.4), we have

$$\left(\frac{\partial \tilde{W}_0(p_1, -p_2)}{\partial p_2}\right)_{p,p} = -2\pi i \iint x_1 W_0^*(x_1, x_2) \times \exp[2\pi i p(x_1 - x_2)]\, \mathrm{d}x_1\, \mathrm{d}x_2, \quad (B.5)$$

where the asterisk denotes the complex conjugate and where use has been made of the well-known hermiticity property of the cross spectral density, namely $W_0(x_2, x_1) = W_0^*(x_1, x_2)$. The comparison between eqs. (B.3) and (B.5) proves eq. (3.6).

Finally, eq. (4.6) can be derived in much the same way as eq. (3.4).

Appendix C

We want to prove that $z=0$ if $W_0(x_1, x_2)$ is real. To this aim, we shall prove the following equality:

$$\int p \left(\frac{\partial \tilde{W}_0(p_1, -p_2)}{\partial p_1}\right)_{p,p} \mathrm{d}p = \int x \left(\frac{\partial W_0(x_1, x_2)}{\partial x_2}\right)_{x,x} \mathrm{d}x. \quad (C.1)$$

Observe that the left-hand side of eq. (C.1) can be written

$$\int p \left(\frac{\partial \tilde{W}_0(p_1, -p_2)}{\partial p_1}\right)_{p,p} \mathrm{d}p = \int \left(\frac{\partial}{\partial p_1} p_2 \tilde{W}_0(p_1, -p_2)\right)_{p,p} \mathrm{d}p. \quad (C.2)$$

On expressing $W_0(x_1, x_2)$ through its Fourier transform, we have

$$W_0(x_1,x_2)=\iint \tilde{W}_0(p_1,-p_2)$$

$$\times\exp[2\pi i(p_1x_1-p_2x_2)]\,dp_1\,dp_2\,, \qquad (C.3)$$

where p_2 has been replaced by $-p_2$. Let us take the derivative of both sides of eq. (A.3) with respect to x_2. This gives

$$\frac{\partial W_0(x_1,x_2)}{\partial x_2}=-2\pi i\iint p_2\tilde{W}_0(p_1,-p_2)$$

$$\times\exp[2\pi i(p_1x_1-p_2x_2)]\,dp_1\,dp_2\,, \qquad (C.4)$$

so that by Fourier inversion we obtain

$$p_2\tilde{W}_0(p_1,-p_2)=-\frac{1}{2\pi i}\iint\frac{\partial W_0(x_1,x_2)}{\partial x_2}$$

$$\times\exp[-2\pi i(p_1x_1-p_2x_2)]\,dx_1\,dx_2\,. \qquad (C.5)$$

On differentiating with respect to p_1 and setting $p_1=p_2=p$, we have

$$\left(\frac{\partial}{\partial p_1}p_2\tilde{W}_0(p_1,-p_2)\right)_{p,p}=\iint x_1\frac{\partial W_0(x_1,x_2)}{\partial x_2}$$

$$\times\exp[-2\pi ip(x_1-x_2)]\,dx_1\,dx_2\,. \qquad (C.6)$$

Integration of both sides of eq. (A.6) with respect to p leads to

$$\int\left(\frac{\partial}{\partial p_1}p_2\tilde{W}_0(p_1,-p_2)\right)_{p,p}dp$$

$$=\int x\left(\frac{\partial W_0(x_1,x_2)}{\partial x_2}\right)_{x,x}dx\,, \qquad (C.7)$$

where again the integral expression of the Dirac function $\delta(x_1-x_2)$ has been used. By virtue of eq. (C.2), we see that eq. (C.7) proves eq. (C.1). It follows at once from eq. (C.1) that the right-hand side of eq. (3.9) vanishes if $W_0(x_1,x_2)$ is real. Hence, in this case we have $\zeta=0$.

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