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Sources with spatially sinusoidal modes

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Optical sources whose cross-spectral density can be expanded into spatially sinusoidal modes are investigated. It is shown that the corresponding cross-spectral density can be constructed by means of a function of a single variable, called the mother function. The coherence properties of the source are then derived from the mother function characteristics.

1. Introduction

The Wolf's theory of partial coherence in the space-frequency domain [1,2] is now a well established tool for characterizing the coherence properties of an optical source [3–5]. The theory is based on the representation of the cross-spectral density through a superposition of coherent source modes. However, as the integral equations defining the modes are not always solvable in a closed form, the modes are explicitly known only for a relatively small number of cases and they are usually expressed through special functions, like Bessel [6] or Hermite-Gauss [7,8] functions. In addition, the eigenfunctions (i.e. the modes) and the eigenvalues (i.e. the weights of the modes in the superposition) of the integral equation are to be evaluated for each particular cross-spectral density. Obviously enough, once the modes have been found, a whole class of partially coherent sources possessing the same modes could in principle be found by simply altering the eigenvalues. However, except for special cases [9], one is not able to express the cross-spectral density of these other sources in a closed form.

In this paper, we investigate the existence of sources whose cross-spectral density can be expanded in the particularly simple form of spatially sinusoidal modes. The reason for this investigation is twofold. On one hand, the elementary character of sinusoidal functions should offer a class of cases in which the mathematics of the modal expansion is as simple as possible. This, in turn, should be of help for a full appreciation of the role played by the modes and by their weights in determining the coherence properties of a source. On the other hand, a superposition of sinusoidal modes can be realized experimentally and this is of interest for the synthesis of partially coherent fields [10,11].

We demonstrate that sources with spatially sinusoidal modes do exist. We find that their cross-spectral density has the peculiar property of being completely determined by one function of a single (real) variable, that will be called the mother function. Among commonly used model sources, such a property is shared only by strictly homogeneous sources whose cross-spectral density is shift-invariant. As we shall see, this property is at the root of the simple character of the modal expansion of the sources under investigation, because it traces back the model expansion to the Fourier series (of a single variable). The class of possible mother functions is easily found. Depending on the choice of the mother

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function, sources with largely different coherence properties can be obtained, ranging from the coherent to the incoherent limit and including cases with peculiar coherence features such as specular and antispecular cross-spectral densities.

2. Statement of the problem

Let us consider a partially coherent one-dimensional source, symmetrically located around the origin of a coordinate axis (say the x-axis). We denote by L the finite extension of the source and by $W(x_1, x_2)$ its cross-spectral density function, where x_1 and x_2 are the coordinates of two typical points of the source. The temporal frequency is assumed to be fixed and the dependence on it of W is not explicitly shown. Under very general conditions, the following expansion holds [1]

$$W(x_1, x_2) = \sum_n \gamma_n \, \Phi_n(x_1) \, \Phi_n^*(x_2) \,, \quad (-L/2 \leqslant x_1, x_2 \leqslant L/2) \,, \tag{2.1}$$

where the asterisk denotes the complex conjugate. In eq. (2.1), the modes Φ_n are orthonormal inside the interval $(-L/2, L/2)$ and the eigenvalues γ_n are nonnegative. Throughout this paper, the summation index n is assumed to be an integer number running from one to infinity.

Starting from $W(x_1, x_2)$, the optical intensity distribution $I(x)$ [12] can be immediately written as

$$I(x) = W(x, x) = \sum_n \gamma_n \, |\Phi_n(x)|^2 \,. \tag{2.2}$$

We make the physically plausible hypothesis of zero intensity at the edges of the source, viz.

$$I(\pm L/2) = \sum_n \gamma_n \, |\Phi_n(\pm L/2)|^2 = 0 \,, \tag{2.3}$$

where upper and lower signs have to be taken simultaneously.

As a consequence, because of the nonnegativeness of the γ_n's, every Φ_n is required to satisfy the following conditions

$$\Phi_n(\pm L/2) = 0 \,. \tag{2.4}$$

The aim of this paper is the characterization of the sources whose modes are sinusoidal functions satisfying eqs. (2.4).

3. Solution of the problem

For convenience, let us consider the following dimensionless coordinate

$$y = \pi(x/L + 1/2) \,. \tag{3.1}$$

With reference to the y-axis, the source is included inside the interval $(0, \pi)$ and its cross-spectral density, say $W_y(y_1, y_2)$, can be written as

$$W_y(y_1, y_2) = W[L(y_1/\pi - 1/2), L(y_2/\pi - 1/2)] \,, \quad (0 \leqslant y_1, y_2 \leqslant \pi) \,. \tag{3.2}$$

The modal expansion (2.1) can now be written in the form

$$W_y(y_1, y_2) = \sum_n \gamma_n \, \Psi_n(y_1) \, \Psi_n^*(y_2) \,, \tag{3.3}$$

where

$$\Psi_n(y) = \Phi_n[L(y/\pi - 1/2)] \,. \tag{3.4}$$

Because of eqs. (2.4), every Ψ_n is required to satisfy the following conditions

$$\Psi_n(0) = \Psi_n(\pi) = 0 \,. \tag{3.5}$$

Let us consider the following sinusoidal functions which vanish at the edges of the interval $(0, \pi)$

$$f_n(y) = C_n \sin(ny) \,, \tag{3.6}$$

where the normalization factors C_n are given by

$$C_n = (2/\pi)^{1/2} \,. \tag{3.7}$$

The functions defined by (3.6) constitute an orthonormal set over the interval $(0, \pi)$ and, consequently, they can be thought of as a set of Ψ_n's in eq. (3.3). The resulting series can be considered as the modal expansion of a cross-spectral density, under the only condition that the series converges.

In order to prove that sources corresponding to this kind of cross-spectral density exist, let us write explicitly the kernels corresponding to the sinusoidal modes (3.6) as

$$W_y(y_1, y_2) = \frac{2}{\pi} \sum_n \gamma_n \sin(ny_1) \sin(ny_2) \,, \tag{3.8}$$

where the coefficients γ_n are nonnegative. By making use of elementary trigonometric formulas, eq. (3.8) can also be written

$$W_y(y_1, y_2) = M(y_1 - y_2) - M(y_1 + y_2) , \qquad (3.9)$$

where

$$M(t) = \frac{1}{\pi} \sum_n \gamma_n \cos(nt) + G . \qquad (3.10)$$

In eq. (3.10), G is an arbitrary constant.

We see from eq. (3.9) that the convergence of eq. (3.8) is assured by that of eq. (3.10). This, in turn, can be assessed through the convergence criteria for a Fourier series.

In conclusion, the existence of sources whose cross-spectral density has the modal expansion given by eq. (3.8) is directly connected with the existence of functions M as defined by eq. (3.10). This has to be consistent with the only constraint of nonnegativeness for the coefficients γ_n. Such a constraint can be easily satisfied. In fact, every function M that can be expressed as the autocorrelation of another function, say m, falls within the category of functions that we are looking for. Indeed, in this case, the Fourier coefficients of M are nonnegative being proportional to the squared moduli of the Fourier coefficients of m.

Once M is fixed, the cross-spectral density is completely specified by eq. (3.9). In this sense, M can be called a mother function for W_y. Obviously, the opposite is also true. Indeed, let us consider a cross-spectral density W_y for which an expansion of the form of eq. (3.8) is known to hold. It is immediately verified that the corresponding mother function is completely determined by the equation

$$M(t) = M(0) - W_y(t/2, t/2) . \qquad (3.11)$$

It is not difficult to show that $M(0)$ is a finite quantity so that the convergence of $M(t)$ is proved by eq. (3.11).

4. Source characteristics

Let us now examine the characteristics of the sources belonging to the class specified in the previous section. As far as the intensity distribution across the source is concerned, we note that eq. (3.11) implies that

$$I(y) = M(0) - M(2y) . \qquad (4.1)$$

From eq. (4.1), we see that $I(y)$ is obtained by subtracting the compressed version $M(2y)$ of $M(y)$ from its maximum value $M(0)$. As a consequence, the mother function can be interpreted as a complementary, expanded version of the intensity profile and, therefore, it can be used to control the intensity distribution.

For later use, we note that, by making use of eq. (3.10), eq. (4.1) can be explicitly written as

$$I(y) = \frac{1}{\pi} \sum_n \gamma_n - \frac{1}{\pi} \sum_n \gamma_n \cos(2ny) . \qquad (4.2)$$

We further note that, by making use of eqs. (3.9) and (4.1), the cross-spectral density function W_y can be expressed as a function of the intensity distribution I through the relation

$$W_y(y_1, y_2) = I\left(\frac{y_1 + y_2}{2}\right) - I\left(\frac{y_1 - y_2}{2}\right) . \qquad (4.3)$$

The peculiar result expressed by eq. (4.3) is a direct consequence of the correspondence established between the cross-spectral density function W_y and the single-variable mother function M.

The degree of spectral coherence [12] can be deduced from eqs. (3.9) and (4.1) as

$$\mu(y_1, y_2) = \frac{W_y(y_1, y_2)}{\sqrt{I(y_1)}\sqrt{I(y_2)}} = \frac{M(y_1 - y_2) - M(y_1 + y_2)}{\sqrt{M(0) - M(2y_1)}\sqrt{M(0) - M(2y_2)}} . \qquad (4.4)$$

It is interesting to note that, when $M(t)$ is substantially different from zero only for t near to zero (or near to integer multiples of 2π), a source region exists where the cross-spectral density function is approximately shift-invariant. In fact, it is seen from eq. (3.9) that in this case the following approximate relation holds

$$W_y(y_1, y_2) \cong M(y_1 - y_2) , \qquad (4.5)$$

provided that we consider two source points which are not both close to the same edge. In addition, eq. (4.1) shows that, in the same approximation, the intensity distribution is almost uniform across the greater part of the source area (except near the edges). As a consequence, for source points far from the edges, the degree of spectral coherence is approximately shift-invariant. Because of the small width of the mother function, this is the limit case of a coherence area small with respect to the source area. Obviously, the opposite completely coherent limit corresponds to a source expansion consisting of a single mode [1].

Finally, the above discussed sources can be investigated regarding their specularity properties [13]. To do this, let us consider mother functions M whose Fourier expansion (3.10) contains only terms with odd indices, viz.

$$M(t)=\frac{1}{\pi}\sum_n \gamma_{2n-1}\cos[(2n-1)t] . \tag{4.6}$$

It is immediately seen that the function M as defined by eq. (4.6) is antisymmetric with respect to the center of the source interval $(0, \pi)$, i.e. that the following relation holds

$$M(\pi-t)=-M(t) . \tag{4.7}$$

Let us now consider the cross-spectral density (3.9) and let us calculate it in correspondence to the pair of symmetric points y_1 and $\pi-y_1$, for a given y_2. Eqs. (3.9) and (4.7) imply that

$$\begin{aligned} W_y(\pi-y_1, y_2) &= M[\pi-(y_1+y_2)]-M[\pi-(y_1-y_2)] \\ &= W_y(y_1, y_2) , \end{aligned} \tag{4.8}$$

so that we conclude that the cross-spectral density function is specular. This result agrees with that obtained in ref. [13] concerning the role of the even modes in the modal expansion of a specular cross-spectral density function.

Opposite results are obtained if we consider mother functions M whose Fourier expansion (3.10) contains only terms with even indices, viz.

$$M(t)=\frac{1}{\pi}\sum_n \gamma_{2n}\cos(2nt) . \tag{4.9}$$

In this case, we have

$$M(\pi-t)=M(t) . \tag{4.10}$$

By making use of eq. (4.10), we obtain

$$\begin{aligned} W_y(\pi-y_1, y_2) &= M[\pi-(y_1+y_2)]-M[\pi-(y_1-y_2)] \\ &= -W_y(y_1, y_2) . \end{aligned} \tag{4.11}$$

A cross-spectral density function satisfying eq. (4.11) can be called antispecular.

The above outlined source characteristics will be evidentiated in the next section, with reference to a specific example of mother function.

5. Example

Let us consider the case in which the γ_n coefficients of eq. (3.10) assume the explicit form

$$\gamma_n=\pi q^n , \tag{5.1}$$

where q is a positive parameter less than one.

By taking into account eq. (5.1), the expansion given by eq. (3.10) can be written in the following closed form [14]

$$\begin{aligned} M(t) &= \sum_n q^n\cos(nt)+G \\ &= \frac{q\cos(t)-q^2}{1+q^2-2q\cos(t)}+G . \end{aligned} \tag{5.2}$$

It is easy to recognize that the function M as given by eq. (5.2) has the form of the transmitted intensity in a Fabry-Perot interferometer [15]. As it is well known, by increasing q, the M-values become very small except in the immediate neighbourhood of the limits of the interval $(0, 2\pi)$. As a consequence, for two source points which are not both close to the same edge, eq. (4.5) holds and the cross-spectral density function is approximately shift-invariant.

The intensity distribution is given by eq. (4.2). This, by making use of eq. (5.1), becomes

$$\begin{aligned} I(y) &= \sum_n q^n-\sum_n q^n\cos(ny) \\ &= \frac{q}{1-q}-\frac{q\cos(y)-q^2}{1+q^2-2q\cos(y)} . \end{aligned} \tag{5.3}$$

In fig. 1, the intensity distribution $I(y)$, as given by eq. (5.3), is drawn as a function of y, in correspondence to several values of q. It can be easily seen from fig. 1 that, by increasing q, the intensity distribution becomes approximately uniform in the central part of the source area.

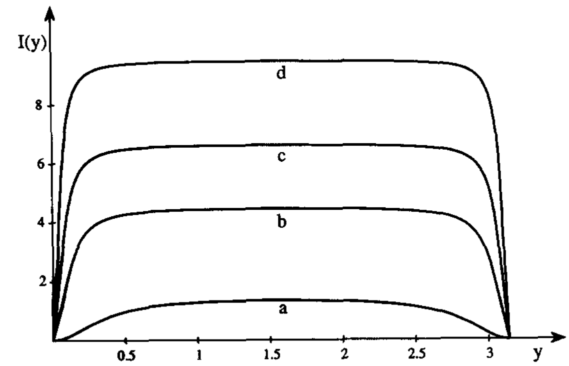

Fig. 1. Intensity distribution (as represented by eq. (5.3)) drawn as a function of the coordinate y in correspondence to (a) $q=0.50$, (b) $q=0.80$, (c) $q=0.86$ and (d) $q=0.90$.

The degree of spectral coherence is given by eq. (4.4). By taking into account eqs. (5.2) and (5.3), eq. (4.4) can be written

$$\mu(y_1;y_2)=\left(\frac{q\cos(y_1-y_2)-q^2}{1+q^2-2q\cos(y_1-y_2)}\right.$$
$$\left.-\frac{q\cos(y_1+y_2)-q^2}{1+q^2-2q\cos(y_1+y_2)}\right)$$
$$\times\left(\frac{q}{1-q}-\frac{q\cos(2y_1)-q^2}{1+q^2-2q\cos(2y_1)}\right)^{-1/2}$$
$$\times\left(\frac{q}{1-q}-\frac{q\cos(2y_2)-q^2}{1+q^2-2q\cos(2y_2)}\right)^{-1/2}. \qquad (5.4)$$

In fig. 2, the degree of spectral coherence is drawn as a function of y_1, in correspondence to several values of y_2, for $q=0.1$ (fig. 2a) and $q=0.9$ (fig. 2b). As it can be seen, as a consequence of the uniformity of the intensity distribution, the shift-invariance property exhibited by the cross-spectral density holds also for the degree of spectral coherence, within the same limits. On the contrary, for y_2-values near the source edges, due to the constraint of eq. (2.3), the intensity vanishes and the shift-invariance property is destroyed.

For what specularity and antispecularity are concerned, only terms with indices of well defined parity have to be considered in the mother function Fourier expansion. When only the Fourier terms with odd indices are considered, the mother function becomes

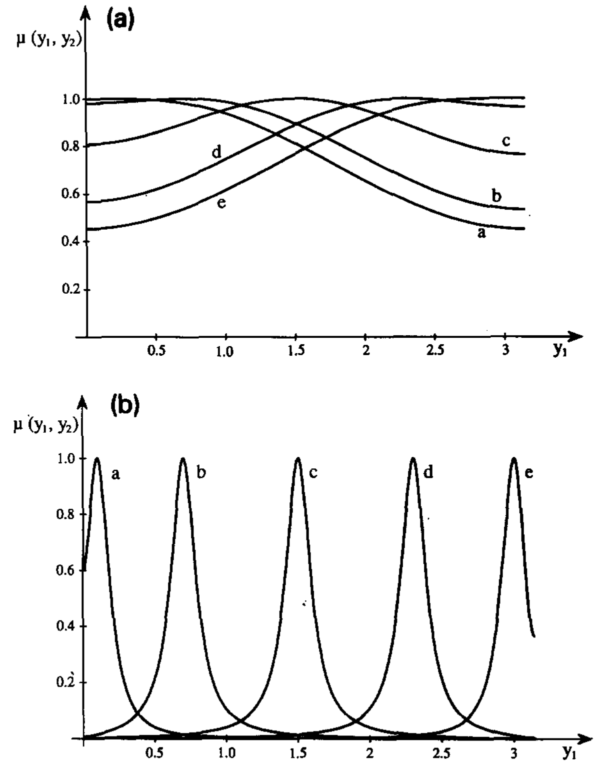

Fig. 2. Degree of spectral coherence (as represented by eq. (5.4)) drawn as a function of y_1 for $q=0.1$ (fig. 2a) and $q=0.9$ (fig. 2b), in correspondence to (a) $y_2=0.1$, (b) $y_2=0.7$, (c) $y_2=1.5$, (d) $y_2=2.3$ and (e) $y_2=3.0$.

$$M(t)=\sum_n q^{2n-1}\cos[(2n-1)t]$$
$$=\frac{q(1-q^2)\cos(t)}{(1+q^2)^2-4q^2\cos^2(t)}. \qquad (5.5)$$

The corresponding symmetric intensity distribution is shown in fig. 3a, for $q=0.9$. The specular degree of spectral coherence is drawn in fig. 3b, as a function of y_1, in correspondence to $q=0.9$ and $y_2=0.7\ \pi$.

Finally, when only the Fourier terms with even indices are considered, the mother function becomes

$$M(t)=\sum_n q^{2n}\cos(2nt)$$
$$=\frac{2q^2\cos^2(t)-q^2(1+q^2)}{(1+q^2)^2-4q^2\cos^2(t)}. \qquad (5.6)$$

The corresponding symmetric intensity distribution

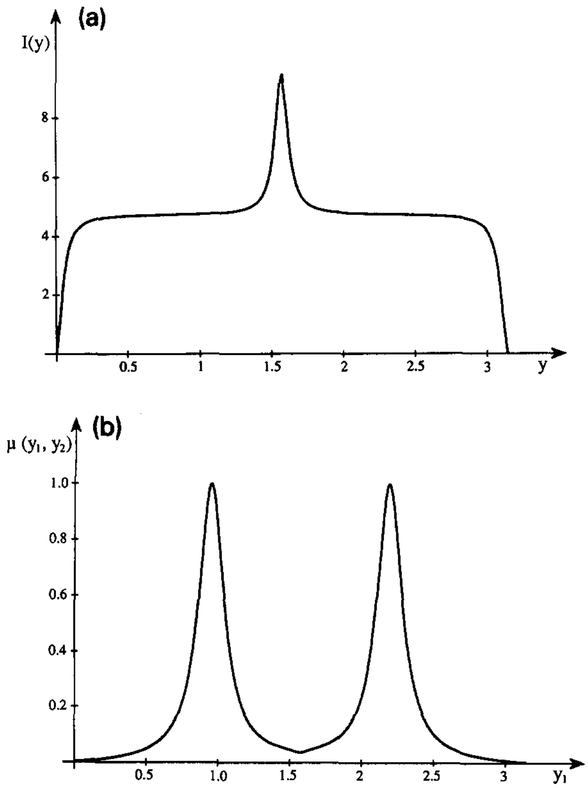

Fig. 3. Symmetric intensity distribution (a) and specular degree of spectral coherence (b) corresponding to a mother function Fourier expansion containing only terms with odd indices, with $q=0.9$ and $y_2=0.7\ \pi$.

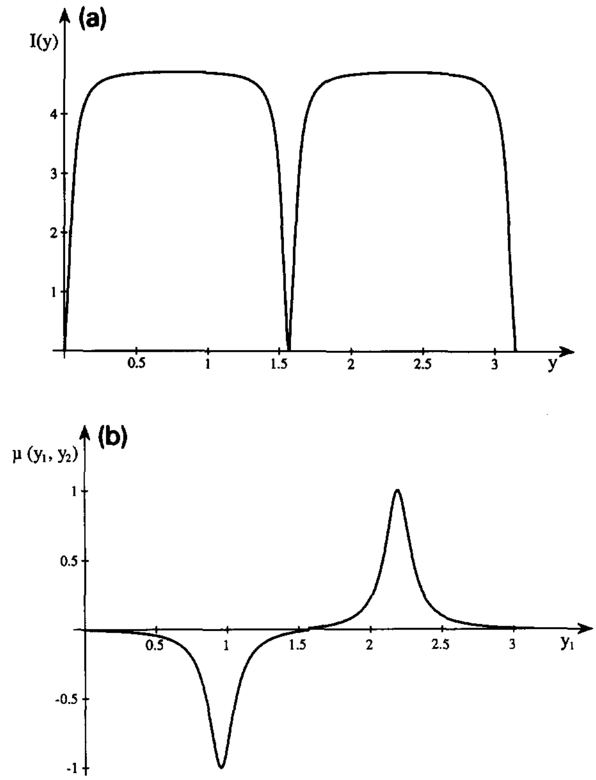

Fig. 4. Symmetric intensity distribution (a) and antispecular degree of spectral coherence (b) corresponding to a mother function Fourier expansion containing only terms with even indices, with $q=0.9$ and $y_2=0.7\ \pi$.

is shown in fig. 4a, for $q=0.9$. The antispecular degree of spectral coherence is drawn in fig. 4b, as a function of y_1, in correspondence to $q=0.9$ and $y_2=0.7\ \pi$.

6. Conclusions

In this paper, we have demonstrated the existence of sources whose cross-spectral density can be expanded in spatially sinusoidal modes. For such sources, the cross-spectral density as well as the intensity distribution and the degree of spectral coherence can be constructed starting from even, periodic mother functions with nonnegative Fourier coefficients. In the general case, neither the cross-spectral density nor the degree of spectral coherence are shift-invariant. However, we have shown that for a suitable choice of the mother function the shift-invariance property can be obtained across the greater part of the source area. Within the same limits, the intensity distribution results to be approximately uniform.

We have also shown that, when only terms with indices of well defined parity are considered in the mother function Fourier expansion, the resulting cross-spectral density functions exhibit peculiar behaviours regarding their specularity properties.

The whole class of autocorrelation functions can act as mother functions. As a simple example, we explicitly considered the case of a Fourier series whose coefficients decrease with a geometrical progression law. By slowing down the decreasing law of the coefficients, the cross-spectral density becomes progressively shift-invariant and the intensity distribution becomes approximately uniform across the greater part of the source area.

The mode structure based on spatially sinusoidal functions directly leads to an easy synthetic source implementation. This can be simply obtained by superposing many independent sinusoidal distributions. Furthermore, the weights of the superposition can be adjusted in such a way that a predetermined degree of shift-invariance and uniformity can also be obtained.

We limited ourselves to sources whose intensity vanishes at the edges and this led to the consideration of sine functions. However, also cosines could be used in the modal expansion of the cross-spectral density. In this case, eq. (2.4) is no longer valid and also sources with nonvanishing intensities at the edges can be represented.

For the sake of simplicity, one-dimensional sources have been considered throughout this paper. However, all the considerations we developed can be extended to the two-dimensional case.

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