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Generalized Fierz identities

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Abstract

Low energy weak interactions calculations with fermions frequently involve a superposition of quartic products of Dirac spinors, in which the order of the spinors is not the same in all the contributing terms. A common trick that is used to bring them to a uniform ordering is the Fierz transformation. We show that the standard Fierz rearrangement formula quoted in textbooks is one element of a class of transformations of a quartic product amplitude, under which the spinors are rearranged with different orderings and, in the general case, some or all of the spinors are transformed to their Lorentz-invariant complex conjugate form. We give a pedagogical derivation of the explicit forms of all such transformation matrices. In addition to the usual Lorentz scalar quartic products, we consider pseudoscalar ones as well. Such manipulations and formulas are useful, in particular, when some of the fermions involved are Majorana particles.

1 Introduction

1.1 What are the generalized Fierz identities

Dirac bilinears are matrix elements of the Dirac γ matrices and products of the γ matrices, taken between two spinors. Such bilinears appear in any calculation involving fermions and usually the spinors are the plane wave solutions of the Dirac equation. In general, the spinors correspond to different momenta and/or different particles, and therefore are different. Typical calculations involve the product of two such bilinears and, since they contain four spinors, we will call them quadrilinears for the sake of brevity. Although the order of the two bilinears in the quadrilinear product is not important, the order of appearance of the individual spinors is.

The Fierz identities [1] are relations between the quadrilinears, written with two different orders of the spinors. It is easy to count how many such different orderings are possible. Suppose we start with a quadrilinear in the form

$$\Big[\overline{w}_1 M w_2\Big]\Big[\overline{w}_3 M' w_4\Big]\,, \tag{1.1}$$

where M and M' are numerical matrices and the w_n denote four Dirac spinors. Each w_n can be either a positive energy u-spinor or a negative energy v-spinor and the numerical subscript n labels the combination of momentum (including the mass) and spin that defines the spinor. If we want to interchange the places of these spinors, the same product will be expressed by some different matrices between the spinors. Obviously, there can be $4! = 24$ different orderings. However, since the ordering of the bilinears is irrelevant as we have said, half as many of these orderings would give rise to new quadrilinears. There can therefore be 12 different orderings. One of these will be the same as the order that we start with, so there are 11 alternative orders left. The usual Fierz transformation refers to only one of them, in which w_2 and w_4, or equivalently w_1 and w_3, are interchanged. In addition, the standard identities involve combinations of products of bilinears such that the quadrilinears involved are Lorentz scalars. In this article, we give explicit formulas for all 11 orderings, and we also consider additional quadrilinear products besides the usual ones. To state this more precisely we introduce some notation.

1.2 Notation for Dirac matrices

As is well known, starting with the standard 4×4 Dirac matrices γ^μ and taking suitable products of them, a set of 16 matrices can be constructed which, among other properties, span the 16 dimensional space of all the 4×4 matrices. In the notation that we will use in the subsequent material, a conventional choice of the 16 matrices is

$$
\begin{array}{r}
\Gamma_S^1 \equiv \mathbb{1}\,, \\
\Gamma_V^1 \text{ through } \Gamma_V^4 \equiv \gamma^\mu\,, \\
\Gamma_T^1 \text{ through } \Gamma_T^6 \equiv \sigma^{\mu\nu}\,, \quad (\mu < \nu) \\
\Gamma_A^1 \text{ through } \Gamma_A^4 \equiv i\gamma^\mu\gamma_5\,, \\
\Gamma_P^1 \equiv \gamma_5\,,
\end{array}
\tag{1.2}
$$

where

$$
\sigma_{\mu\nu} = \frac{i}{2}\Big[\gamma^\mu, \gamma^\nu\Big]\,, \tag{1.3}
$$

$$
\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{1.4}
$$

Further, for definiteness, we adopt the convention that

$$
\Big\{\gamma^\mu, \gamma^\nu\Big\} = 2g^{\mu\nu}\mathbb{1}\,, \tag{1.5}
$$

where the curly braces denote the anticommutator, $\mathbb{1}$ is the unit matrix, and the metric tensor is given by

$$
g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)\,. \tag{1.6}
$$

Then it is easily verified that

$$
\mathrm{Tr}\,(\Gamma_I^r \Gamma_{Jq}) = 4\delta_{IJ}\delta_q^r\,, \tag{1.7}
$$

where the matrices Γ_{Ir} are defined by lowering the Lorentz indices in Eq (1.2). One more convention we adopt is the following. Whenever the Lorentz indices (e.g. r, q in the above formula) appear repeated in a given formula, the usual summation convention over repeated indices is assumed. On the other hand, we do not use that convention for the indices I, J, which take the *values* S, V, T, A, P. Any summation over these indices will always be indicated explicitly.

A useful set of relations are the generalizations of the formulas such as

$$
\begin{aligned}
\gamma^\mu\gamma^\alpha\gamma_\mu &= -2\gamma^\alpha \\
\gamma^\mu\sigma^{\alpha\beta}\gamma_\mu &= 0 \\
\sigma^{\mu\nu}\sigma_{\mu\nu} &= 12\,,
\end{aligned}
\tag{1.8}
$$

and similar ones which follow very simply by repeated application of Eq. (1.5). Thus, writing

$$
\Gamma_I^q \Gamma_J^s \Gamma_{Iq} = f_{IJ}\Gamma_J^s\,, \tag{1.9}
$$

they can be summarized in the form

$$
\boldsymbol{f} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} . \tag{1.10}
$$

Another useful relation that we quote for future purposes is the identity

$$
\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta} = \sigma^{\mu\nu}\gamma_5 \, , \tag{1.11}
$$

with the convention that

$$
\epsilon^{0123} = +1 \, . \tag{1.12}
$$

1.3 Dirac quadrilinears and Fierz identities

In correspondence with the 16 Dirac matrices Γ_I^r we then construct the following bilinears of two spinors w_1 and w_2,

$$
e_I^r(12) = \overline{w}_1 \Gamma_I^r w_2 \, . \tag{1.13}
$$

The simplest quadrilinears are products of the form $e_I^r(12)e_{Ir}(34)$. In anticipation of obtaining the final formulas in a convenient form we define

$$
e_I(1234) = n_I^2 e_I^r(12) e_{Ir}(34) \, , \tag{1.14}
$$

with the numerical coefficients n_I chosen to be

$$
n_I = \begin{cases} 1 & \text{if } I = S, V, P, \\ -i & \text{if } I = A, \\ \sqrt{2} & \text{if } I = T, \end{cases} \tag{1.15}
$$

so that the quadrilinears are given in terms of the Dirac matrices simply as

$$
\begin{aligned}
e_S(1234) &= \left[\overline{w}_1 w_2\right]\left[\overline{w}_3 w_4\right] , \\
e_V(1234) &= \left[\overline{w}_1 \gamma^\mu w_2\right]\left[\overline{w}_3 \gamma_\mu w_4\right] , \\
e_T(1234) &= \left[\overline{w}_1 \sigma_{\mu\nu} w_2\right]\left[\overline{w}_3 \sigma^{\mu\nu} w_4\right] , \\
e_A(1234) &= \left[\overline{w}_1 \gamma^\mu \gamma_5 w_2\right]\left[\overline{w}_3 \gamma_\mu \gamma_5 w_4\right] , \\
e_P(1234) &= \left[\overline{w}_1 \gamma_5 w_2\right]\left[\overline{w}_3 \gamma_5 w_4\right] .
\end{aligned} \tag{1.16}
$$

In this notation, the standard Fierz transformation gives the relation between the $e_I(1234)$ and the $e_J(1432)$.

As already mentioned, in this article, we generalize the standard Fierz transformation in two ways. Firstly, we give the explicit formulas for the relation between the $e_I(1234)$ and the $e_I(abcd)$ for all the 11 possible final orderings. Secondly, we consider more general quadrilinear products of the form

$$e_I^r(12) e_{Jr}(34)\,, \tag{1.17}$$

with $I \neq J$, but for those pairs of I, J for which the Lorentz indices match and are contracted. Thus, the possibilities are $(I, J) = (V, A)$ and $(I, J) = (S, P)$. In addition we consider another possible contraction

$$e_T^r(12) e_{\tilde{T}r}(34)\,, \tag{1.18}$$

where $e_{\tilde{T}}^r(ab)$ is defined as $e_T^r(ab)$ but with the gamma matrices

$$\Gamma_{\tilde{T}}^1 \text{ through } \Gamma_{\tilde{T}}^6 \equiv \tfrac{i}{2} \epsilon^{\mu\nu\alpha\beta} \sigma_{\alpha\beta}\,, \quad (\mu < \nu) \quad . \tag{1.19}$$

2 Fierz identities for scalar combinations

2.1 Derivation of the usual Fierz identity

The usual Fierz identities are relations of the type

$$e_I(1234) = \sum_J F_{IJ} e_J(1432)\,, \tag{2.1}$$

where the F_{IJ} are numerical coefficients. The fact that they must exist follows from two properties of the generalized gamma matrices. The first is the closure relation

$$\frac{1}{4} \sum_J \left(\Gamma_J^r\right)_{ad} \left(\Gamma_{Jr}\right)_{cb} = \delta_{ab} \delta_{cd}\,, \tag{2.2}$$

which is easily proven by noticing that any 4×4 matrix M must be expandable in terms of the gamma matrices in the form

$$M = \sum_I m_{Ir} \Gamma_I^r\,, \tag{2.3}$$

where the orthogonality relation in Eq. (1.7) implies that the coefficients are given by

$$m_{Ir} = \frac{1}{4} \operatorname{Tr} \Gamma_{Ir} M\,. \tag{2.4}$$

Therefore

$$M = \frac{1}{4} \sum_I \Gamma_I^r \left(\mathrm{Tr}\, \Gamma_{Ir} M \right) \, , \tag{2.5}$$

and since this holds for any matrix M, the closure relation follows.

The second property is that for any two gamma matrices Γ_I^q and Γ_J^r, there is a third matrix Γ_K^s such that

$$\Gamma_I^q \Gamma_J^r = \Gamma_K^s \times \text{number} \, . \tag{2.6}$$

In general, for any other choice of basis, the right-hand side in Eq. (2.6) would involve a linear combination of all the basis matrices. The distinguishing feature of Eq. (2.6) is that only one member of the basis appears on the right-hand side.

Multiplying Eq. (2.2) by

$$(\Gamma_I^q)_{a'a} \, (\Gamma_{Iq})_{c'c} \tag{2.7}$$

gives

$$\frac{1}{4} \sum_J \left(\Gamma_I^q \Gamma_J^r \right)_{ad} \left(\Gamma_{Iq} \Gamma_{Jr} \right)_{cb} = (\Gamma_I^q)_{ab} \, (\Gamma_{Iq})_{cd} \, , \tag{2.8}$$

and using Eq. (2.6) it implies that there is a relation of the form

$$(\Gamma_I^q)_{ab} \, (\Gamma_{Iq})_{cd} = \sum_K C_{IK} \, (\Gamma_K^r)_{ad} \, (\Gamma_{Kr})_{cb} \, , \tag{2.9}$$

which is in fact the Fierz identity in disguise. Notice that we have used the fact that, since the Lorentz indices in the quantity on the left-hand side are contracted such that the quantity is a scalar, the same must hold on the right-hand side. The coefficients C_{IJ} can be found by multiplying Eq. (2.9) by

$$(\Gamma_{Js})_{da} \, (\Gamma_J^s)_{bc} \tag{2.10}$$

and using Eq. (1.7) once more, which yields

$$C_{IJ} = \frac{1}{16 N_J} \, \mathrm{Tr}\, \Gamma_{Js} \Gamma_I^q \Gamma_J^s \Gamma_{Iq} \, , \tag{2.11}$$

where N_J is the number of gamma matrices in each group J, i.e., $N_S = N_P = 1$, $N_T = 6$ and $N_V = N_A = 4$. The traces in Eq. (2.12) can be evaluated simply by using Eqs. (1.9) and (1.7), and thus we obtain

$$C_{IJ} = \frac{1}{4} f_{IJ} \, , \tag{2.12}$$

with the f_{IJ} given in Eq. (1.10). Remembering the non-trivial factors of n_I in the definitions of e_T and e_A given in Eq. (1.15), the relation given in Eq. (2.9) implies Eq. (2.1), with the coefficients being

$$F_{IJ} = \frac{n_I^2}{n_J^2}\, C_{IJ}\,. \tag{2.13}$$

In other words, the matrix $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = \frac{1}{4}\begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 & 1 \\ 4 & -2 & 0 & -2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ -4 & -2 & 0 & -2 & 4 \\ 1 & -1 & \frac{1}{2} & 1 & 1 \end{pmatrix}, \tag{2.14}$$

which is the usual Fierz transformation matrix. We will use the matrix notation

$$\boldsymbol{e}(1234) = \boldsymbol{F}\boldsymbol{e}(1432) \tag{2.15}$$

as a short hand for Eq. (2.1).

2.2 Other orderings

In the relation appearing in Eq. (2.1), the nature of the exchanged spinors remains the same. In other words, a u-spinor remains a u-spinor, and the same applies for v-spinors. In fact, if we restrict the transformations to those for which this property holds, there are no other possible rearrangements of the spinors. Here we contemplate other transformations which do not have that property.

For illustration, we consider the particularly simple example where we want to interchange the positions of the spinors in the first and second place. Clearly this is not possible unless we change a u-spinor to a v-spinor and vice versa. Therefore, the appropriate quantities to consider are

$$e_I(2^c1^c34) = \left[\overline{w^c}_2 \Gamma_I^r w_1^c\right]\left[\overline{w}_3 \Gamma_{Ir} w_4\right], \tag{2.16}$$

were we have introduced the notation w^c. If w is a u-spinor, w^c denotes the corresponding v-spinor, and vice versa. In either case, w and w^c are related by

$$w^c = \gamma_0 C w^*\,, \tag{2.17}$$

or equivalently

$$\overline{w^c} = w^\top C^{-1}\,, \tag{2.18}$$

where C is the conjugation matrix which satisfies the relations

$$C^\top = -C\,, \tag{2.19}$$

and

$$C^{-1}\Gamma_I^r C = \eta_I \Gamma_I^{r\top}\,, \tag{2.20}$$

with the values η_I are summarized in the following table:

I	S	V	T	A	P
η_I	$+1$	-1	-1	$+1$	$+1$

(2.21)

Therefore, using the fact that the w's are numerical spinors, by simple matrix algebra we can write

$$\overline{w_2^c} M w_1^c = -\,\overline{w}_1 C M^\top C^{-1} w_2\,, \tag{2.22}$$

for any 4×4 matrix M. In particular, using Eqs. (2.19) and (2.20),

$$\overline{w}_1 \Gamma_I^r w_2 = -\,\eta_I\,\overline{w_2^c}\Gamma_I^r w_1^c\,. \tag{2.23}$$

If the w's were to denote anticommuting fields, the relations analogous to Eqs. (2.22) and (2.23) would not have the overall minus sign on the right-hand side.

It follows from Eq. (2.23) that the Fierz transformation matrix $\boldsymbol{S}$ defined by the relation

$$e_I(1234) = \sum_J S_{IJ} e_J(2^c 1^c 34)\,, \tag{2.24}$$

is simply

$$\boldsymbol{S} = \mathrm{diag}\Big(-1, +1, +1, -1, -1\Big)\,, \tag{2.25}$$

and it is clear that the same matrix appears in the relation

$$e_I(1234) = \sum_J S_{IJ} e_J(124^c 3^c)\,. \tag{2.26}$$

Combining the standard Fierz transformations with Eqs. (2.24) and (2.26), the transformations for the other permutations are simple to obtain.

Table 1: Fierz matrices for all scalar combinations. The initial order of the spinors is taken to be (1234), as expressed in the first entry. We are omitting those final orderings that, by using the rule $e_I(abcd) = e_I(cdab)$, reduce to the ones already included.

Final order	$\boldsymbol{K}$
(1234)	$\mathbb{1}$
(1432)	$\boldsymbol{F}$
(2^c1^c34)	$\boldsymbol{S}$
(124^c3^c)	$\boldsymbol{S}$
(13^c2^c4)	$\boldsymbol{SFS}$
(13^c4^c2)	$\boldsymbol{SF}$
(142^c3^c)	$\boldsymbol{FS}$
$(2^c1^c4^c3^c)$	$\boldsymbol{SS} = \mathbb{1}$
(31^c2^c4)	$\boldsymbol{SF}$
(31^c4^c2)	$\boldsymbol{SFS}$
$(4^c1^c2^c3^c)$	$\boldsymbol{F}$
(4^c1^c32)	$\boldsymbol{FS}$

For example, consider the interchange $2 \leftrightarrow 3$. Using Eq. (2.14) followed by Eq. (2.24), we obtain

$$\boldsymbol{e}(1234) = \boldsymbol{Fe}(1432) = \boldsymbol{FSe}(142^c3^c) = \boldsymbol{FSFe}(13^c2^c4)\,. \tag{2.27}$$

Alternatively, we can write

$$\boldsymbol{e}(1234) = \boldsymbol{Se}(2^c1^c34) = \boldsymbol{SFe}(2^c431^c) = \boldsymbol{SFSe}(13^c2^c4)\,, \tag{2.28}$$

which shows that

$$\boldsymbol{SFS} = \boldsymbol{FSF}\,. \tag{2.29}$$

Proceeding in similar fashion, the Fierz matrices for all the interchanges are obtained. We denote the initial ordering always by (1234), and any of the final re-orderings, such as those we have considered explicitly, by $(abcd)$. If we write the Fierz relations in the form

$$\boldsymbol{e}(1234) = \boldsymbol{K}^{(abcd)}\boldsymbol{e}(abcd)\,, \tag{2.30}$$

then the matrices $\boldsymbol{K}$ are summarized in Table 1. The matrices $\boldsymbol{F}$ and $\boldsymbol{S}$

have already been given explicitly in Eqs. (2.14) and (2.25), respectively. For a ready reference, we give the explicit forms of their combinations that occur in this list.

$$\boldsymbol{FS} = \frac{1}{4}\begin{pmatrix} -1 & 1 & \frac{1}{2} & 1 & -1 \\ -4 & -2 & 0 & 2 & 4 \\ -12 & 0 & -2 & 0 & -12 \\ 4 & -2 & 0 & 2 & -4 \\ -1 & -1 & \frac{1}{2} & -1 & -1 \end{pmatrix}, \tag{2.31}$$

$$\boldsymbol{SF} = \frac{1}{4}\begin{pmatrix} -1 & -1 & -\frac{1}{2} & 1 & -1 \\ 4 & -2 & 0 & -2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ 4 & 2 & 0 & 2 & -4 \\ -1 & 1 & -\frac{1}{2} & -1 & -1 \end{pmatrix}, \tag{2.32}$$

$$\boldsymbol{SFS} = \frac{1}{4}\begin{pmatrix} 1 & -1 & -\frac{1}{2} & -1 & 1 \\ -4 & -2 & 0 & 2 & 4 \\ -12 & 0 & -2 & 0 & -12 \\ -4 & 2 & 0 & -2 & 4 \\ 1 & 1 & -\frac{1}{2} & 1 & 1 \end{pmatrix}. \tag{2.33}$$

3 Fierz identities for pseudoscalar combinations

In Section 2, we have dealt with scalar quadrilinears only. The Fierz transformations are most useful for weak interaction calculations [2], where parity violating pseudoscalar quadrilinears appear as often as the scalar ones. Therefore the transformation matrices for such combinations are also useful. We derive them in this section.

We want to consider the quadrilinears of the form indicated in Eqs. (1.17) and (1.18), and we thus define

$$e'_I(1234) = n_I n_{\tilde{I}}\, e_I^r(12) e_{\tilde{I}r}(34)\,, \tag{3.1}$$

where $\tilde{I}$ denotes the parity transform of the index I, i.e., if $I = S, V, A, P$, then $\tilde{I} = P, A, V, S$ respectively. In addition, the matrices $\Gamma^r_{\tilde{T}}$ have been defined in Eq. (1.19), while $n_{\tilde{T}} \equiv n_T$. From the definition, we can write down the results for some interchanges immediately. Denoting by $(abcd)$ any ordering of the indices, we have, for example,

$$\begin{aligned} e'_S(abcd) &= e'_P(cdab)\,, \\ e'_V(abcd) &= e'_A(cdab)\,, \\ e'_T(abcd) &= e'_T(cdab)\,. \end{aligned} \tag{3.2}$$

In order to summarize them in a convenient form, we introduce the notation

$$\overleftarrow{\mathrm{diag}}\left(d_1, d_2, d_3, d_4, d_5\right) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & d_1 \\ 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & d_3 & 0 & 0 \\ 0 & d_4 & 0 & 0 & 0 \\ d_5 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{3.3}$$

We can then write

$$\boldsymbol{e}'(abcd) = \boldsymbol{X} e'(cdab) \, , \tag{3.4}$$

where

$$\boldsymbol{X} = \overleftarrow{\mathrm{diag}}\left(1,1,1,1,1\right) . \tag{3.5}$$

The other interchanges can be worked out as follows. From the definitions in Eqs. (1.2) and (1.19), and remembering Eq. (1.11), we find

$$\begin{aligned} \Gamma^r_A &= i\Gamma^r_V \Gamma_P \, , \\ \Gamma^r_{\tilde{T}} &= \Gamma^r_T \Gamma_P \, . \end{aligned} \tag{3.6}$$

Then, using also the fact that $\Gamma_P^2 = \mathbb{1}$, we can write

$$e'_I(1234) = e_I(1234') \, , \tag{3.7}$$

where the notation $4'$ indicates that the spinor that appears in the fourth position is

$$w'_4 \equiv \Gamma_P w_4 \, . \tag{3.8}$$

It is clear that, for any reordering in which the last spinor is not moved, the same manipulations of Section 2 lead to the same the Fierz transformation. In other words, if we denote by $(abc4)$ any such reordering of (1234), Eqs. (2.30) and (3.7) imply that

$$\boldsymbol{e}'(1234) = \boldsymbol{K}^{(abc4)} \boldsymbol{e}'(abc4) \, , \tag{3.9}$$

with the same matrices $\boldsymbol{K}$ tabulated in Table 1.

With the help of Eqs. (2.30), (3.4) and (3.9) we can find all the remaining transformations in terms of the matrices $\boldsymbol{F}$, $\boldsymbol{S}$ and $\boldsymbol{X}$, and their products. For example, the analog of Eq. (2.15) can be obtained through the following steps,

$$\boldsymbol{e}'(1234) = \boldsymbol{X}\boldsymbol{e}'(3412) = \boldsymbol{X}\boldsymbol{F}\boldsymbol{e}'(1432) \, , \tag{3.10}$$

Table 2: Fierz matrices for all pseudoscalar combinations. The initial order of spinors is taken to be (1234).

Final order	Fierz matrix	Final order	Fierz matrix
(1234)	$\mathbb{1}$	(3412)	$\boldsymbol{X}$
(3214)	$\boldsymbol{F}$	(1432)	$\boldsymbol{FX}$
(2^c1^c34)	$\boldsymbol{S}$	(342^c1^c)	$\boldsymbol{SX}$
(2^c3^c14)	$\boldsymbol{FS}$	(142^c3^c)	$\boldsymbol{FSX}$
(31^c2^c4)	$\boldsymbol{SF}$	(2^c431^c)	$\boldsymbol{SFX}$
(13^c2^c4)	$\boldsymbol{SFS}$	(2^c413^c)	$\boldsymbol{SFSX}$
(124^c3^c)	$\boldsymbol{XSX}$	(4^c3^c12)	$\boldsymbol{XS}$
(13^c4^c2)	$\boldsymbol{XSF}$	(4^c213^c)	$\boldsymbol{XSFX}$
$(2^c1^c4^c3^c)$	$\boldsymbol{SXSX}$	$(4^c3^c2^c1^c)$	$\boldsymbol{SXS}$
(324^c1^c)	$\boldsymbol{XFSX}$	(4^c1^c32)	$\boldsymbol{XFS}$
(31^c4^c2)	$\boldsymbol{XSFS}$	(4^c231^c)	$\boldsymbol{SFXS}$
$(2^c3^c4^c1^c)$	$\boldsymbol{SXSF}$	$(4^c1^c2^c3^c)$	$\boldsymbol{FSXS}$

and the analog of Eq. (2.26) would be obtained from

$$\boldsymbol{e}'(1234) = \boldsymbol{X}\boldsymbol{e}'(3412) = \boldsymbol{X}\boldsymbol{S}\boldsymbol{e}'(4^c3^c12) = \boldsymbol{X}\boldsymbol{S}\boldsymbol{X}\boldsymbol{e}'(124^c3^c)\,. \tag{3.11}$$

In analogy with Eq. (2.30), we summarize these transformations by writing

$$\boldsymbol{e}'(1234) = \boldsymbol{K}'^{(abcd)}\boldsymbol{e}'(abcd)\,, \tag{3.12}$$

where the complete list of the matrices $\boldsymbol{K}'$ is tabulated in Table 2. Apart from those entries in which only one of the matrices $\boldsymbol{F}$, $\boldsymbol{S}$ or $\boldsymbol{X}$ appear, the product matrices can be split in two groups according to whether or not the product contains the matrix $\boldsymbol{X}$. The explicit expressions of the products that do not contain any factor of X have already been given in Eqs. (2.31), (2.32) and (2.33). Those that contain one or more factors of $\boldsymbol{X}$ are given by

$$\boldsymbol{XSX} = \mathrm{diag}\Big(-1,-1,+1,+1,-1\Big)\,, \tag{3.13}$$

$$\boldsymbol{SXS} = \overleftarrow{\mathrm{diag}}\Big(+1,-1,+1,-1,+1\Big)\,, \tag{3.14}$$

$$\boldsymbol{XFS} = \frac{1}{4}\begin{pmatrix} -1 & -1 & \frac{1}{2} & -1 & -1 \\ 4 & -2 & 0 & 2 & -4 \\ -12 & 0 & -2 & 0 & -12 \\ -4 & -2 & 0 & 2 & 4 \\ -1 & 1 & \frac{1}{2} & 1 & -1 \end{pmatrix}\,, \tag{3.15}$$

$$\boldsymbol{XSF} = \frac{1}{4}\begin{pmatrix} -1 & 1 & -\frac{1}{2} & -1 & -1 \\ 4 & 2 & 0 & 2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ 4 & -2 & 0 & -2 & -4 \\ -1 & -1 & -\frac{1}{2} & 1 & -1 \end{pmatrix}, \tag{3.16}$$

$$\boldsymbol{FSXS} = \frac{1}{4}\begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 & 1 \\ -4 & 2 & 0 & 2 & 4 \\ 12 & 0 & -2 & 0 & 12 \\ 4 & 2 & 0 & 2 & -4 \\ 1 & -1 & \frac{1}{2} & 1 & 1 \end{pmatrix}, \tag{3.17}$$

$$\boldsymbol{SXSF} = \frac{1}{4}\begin{pmatrix} 1 & -1 & \frac{1}{2} & 1 & 1 \\ 4 & 2 & 0 & 2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ -4 & 2 & 0 & 2 & 4 \\ 1 & 1 & \frac{1}{2} & -1 & 1 \end{pmatrix}, \tag{3.18}$$

$$\boldsymbol{SFXS} = \frac{1}{4}\begin{pmatrix} 1 & 1 & -\frac{1}{2} & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ -12 & 0 & -2 & 0 & -12 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & -\frac{1}{2} & -1 & 1 \end{pmatrix}. \tag{3.19}$$

The remaining ones are of the form of one of the matrices already given, multiplied by the matrix $\boldsymbol{X}$ to the left or to the right. Multiplying a matrix by $\boldsymbol{X}$ to the right is equivalent to reading its columns backwards and, similarly, multiplying a matrix by $\boldsymbol{X}$ to the left amounts to reading its rows backwards. In either case, the result can be easily read off the explicit expressions given above and in Section 2. In addition, it is helpful to note that the Fierz matrices that appear in Table 2 may be rewritten in alternative forms, using the relations

$$\begin{aligned} \boldsymbol{FX} &= \boldsymbol{XF}, \\ \boldsymbol{SXSX} &= \boldsymbol{XSXS}, \\ \boldsymbol{XFSXS} &= \boldsymbol{SXSF}, \\ \boldsymbol{XSFSX} &= \boldsymbol{SFXS}, \end{aligned} \tag{3.20}$$

and similar ones. They can be derived in much the same way as Eq. (2.29).

4 Examples and Summary

The manipulations involved in the use of the Fierz transformations are useful in practical calculations, but the question of when and how to use them can only be answered in the context of the specific situation at hand. Thus, while it is not possible to give a single answer to those questions, we illustrate, with some examples, how we can judge if it is prudent to use them, and how.

4.1 Invariants

A well known result is that the combinations of quadrilinears

$$Q_\eta(1234) \equiv \Big[\overline{w}_1\gamma^\mu(1+\eta\gamma_5)w_2\Big]\Big[\overline{w}_3\gamma_\mu(1+\eta\gamma_5)w_4\Big]\,, \tag{4.1}$$

where $\eta = \pm 1$, are form invariant under the standard Fierz transformation. In our notation, they can be expressed in the form

$$Q_\eta(1234) = e_V(1234) + e_A(1234) + \eta\left(e'_V(1234) + e'_A(1234)\right)\,, \tag{4.2}$$

and the form invariance statement is the property that

$$Q_\eta(1234) = -\,Q_\eta(1432)\,. \tag{4.3}$$

(If we were dealing with field operators instead of numerical spinors, the overall minus sign in Eq. (4.3) would have been absent.) The obvious question is whether there are other combinations that share this property, and not just under the standard transformation, but also under the generalized Fierz transformations that we have considered. The answer to this question is affirmative. We illustrate this below with a few examples, in a way that shows how we can make more general statements, in a systematic fashion.

Let us denote an arbitrary combination of the scalar quadrilinears by

$$Q_{\{a\}} \equiv \sum_I a_I e_I(1234)\,, \tag{4.4}$$

where the a_I are numerical coefficients. Using Eq. (2.30), it can be expressed with a different spinor ordering as

$$Q_{\{a\}} = \sum_{I,J} a_I K^{(abcd)}_{IJ} e_J(abcd)\,. \tag{4.5}$$

This will be of the same form as in Eq. (4.4) if

$$\sum_I a_I K^{(abcd)}_{IJ} \quad = \quad \lambda a_J\,, \tag{4.6}$$

where λ is a numerical factor. That is, the row matrix composed of the elements a_I is a left eigenvector of the matrix $\boldsymbol{K}^{(abcd)}$ with the eigenvalue λ or, equivalently, the column matrix composed of the elements a_I is a right eigenvector of the matrix $[\boldsymbol{K}^{(abcd)}]^\top$ with the eigenvalue λ. Since the transformation matrices are not symmetric (strictly speaking, they are not normal matrices), the distinction between right and left eigenvectors is important.

Similarly, we define an arbitrary pseudoscalar combination by

$$Q'_{\{a'\}} \equiv \sum_I a'_I e'_I(1234)\,. \tag{4.7}$$

Then using Eq. (3.12),

$$Q'_{\{a'\}} = \sum_{I,J} a'_I K'^{(abcd)}_{IJ} e'_J(abcd)\,, \tag{4.8}$$

so that $Q'_{\{a'\}}$ will retain its form if

$$\sum_I a'_I K'^{(abcd)}_{IJ} \quad = \quad \lambda' a'_J\,, \tag{4.9}$$

which indicates that the column matrix composed of the elements a'_I is a right eigenvector of the matrix $[\boldsymbol{K}'^{(abcd)}]^\top$ with the eigenvalue λ'. Notice that, for those invariant combinations $Q_{\{a\}}$ and $Q'_{\{a'\}}$ such that $\lambda = \lambda'$, then any linear combination

$$\alpha Q_{\{a\}} + \beta Q'_{\{a'\}} \tag{4.10}$$

is form invariant as well.

As an example, we consider the final ordering $(abcd) = (1432)$, for which the transformation matrices $\boldsymbol{K}$ and $\boldsymbol{K}'$ are equal to $\boldsymbol{F}$. To find the eigenvectors of $\boldsymbol{F}^\top$, we observe that $\boldsymbol{F}^2 = 1$, which implies that the eigenvalues are ± 1. Further, since $\mathrm{Tr}\,\boldsymbol{F} = -1$, it follows that the eigenvalue $+1(-1)$ is doubly (triply) degenerate. A particular choice of the eigenvectors is listed below, along with the corresponding eigenvalues:

Eigenvector	Eigenvalue
$(0, 1, 0, 1, 0)$	-1
$(2, 0, -1, 0, 2)$	-1
$(-2, 1, 0, -1, 2)$	-1
$(1, 0, \frac{1}{6}, 0, 1)$	$+1$
$(2, 1, 0, -1, -2)$	$+1$

(4.11)

Thus, for example, the first line indicates that the combination

$$e_V + e_A \tag{4.12}$$

is invariant, and so is, separately, the combination

$$e'_V + e'_A \,. \tag{4.13}$$

By Eq. (4.10), any linear combination

$$\alpha(e_V + e_A) + \beta(e'_V + e'_A) \tag{4.14}$$

is also invariant, which includes the chiral combinations given in Eq. (4.1) as special cases.

The others show other invariants. For example, the last line of the table shows that

$$\begin{aligned} &2e_S + e_V - e_A - 2e_P \,, \\ &2e'_S + e'_V - e'_A - 2e'_P \,, \end{aligned} \tag{4.15}$$

and any linear combination of them are invariant as well.

We can apply the same method to the other transformation matrices. For example, consider the final ordering $(abcd) = (1423)$, for which $\boldsymbol{K} = \boldsymbol{FS}$ while $\boldsymbol{K}' = \boldsymbol{FSX}$. Since this permutation, performed thrice, brings back the original ordering, we must have $(\boldsymbol{FS})^3 = 1$, which can also be checked explicitly from the matrix given in Eq. (2.31). Consequently, the eigenvalues are the cube roots of 1; i.e., the only (non-degenerate) real eigenvalue is 1, with the corresponding eigenvector $(1, 0, 0, 1, -1)$. Thus, in particular, $(e_S + e_A - e_P)$ is invariant under the interchange being considered. For the pseudoscalars $([\boldsymbol{FSX}]^T)$, also there is only one real eigenvalue $(+1)$ with the eigenvector $(1, 1, 0, 0, -1)$. Therefore $(e'_S + e'_V - e'_P)$ is invariant, and by Eq. (4.10) so is any combination of the form

$$\alpha(e_S + e_A - e_P) + \beta(e'_S + e'_V - e'_P) \,. \tag{4.16}$$

4.2 Chiral quadrilinears

By chiral quadrilinears we mean those in which only one chiral component of each spinor appears. Consider, for example,

$$Q_{RL} \equiv \Big[\overline{u}_1 R v_2\Big]\Big[\overline{v}_3 L u_4\Big] \,. \tag{4.17}$$

where, as usual,

$$\begin{aligned} R &= \frac{1}{2}(1 + \gamma_5) \\ L &= \frac{1}{2}(1 - \gamma_5) \,. \end{aligned} \tag{4.18}$$

This can be brought to a form in which only u-type spinors appear by combining the transformations given in Tables 1 and 2, but in this case it is more economical to proceed in the following way. Decomposing each spinor w in terms of its chiral components,

$$w = w_L + w_R\,, \tag{4.19}$$

with $w_L = Lw$ and $w_R = Rw$, we can write

$$\begin{aligned} Q_{RL} &= \Big[\overline{u}_{L1} v_{R2}\Big]\Big[\overline{v}_{R3} u_{L4}\Big] \\ &\equiv e_S(1234)\,, \end{aligned} \tag{4.20}$$

where the index 1 is associated with th spinor u_{L1}, the index 2 with v_{R2}, and so on. This can be rewritten with the spinors in the order (1423) by means of the matrix $\boldsymbol{FS}$ from Table 1; i.e.,

$$\begin{aligned} e_S(1234) &= \frac{1}{4}\Big[-e_S(142^c3^c) + e_V(142^c3^c) + \frac{1}{2}e_T(142^c3^c) \\ &\quad + e_A(142^c3^c) - e_P(142^c3^c)\Big]\,. \end{aligned} \tag{4.21}$$

The definition Eq. (2.17) implies that, for the chiral components,

$$\begin{aligned} (u_{L,R})^c &= v_{R,L}\,, \\ (v_{L,R})^c &= u_{R,L}\,, \end{aligned} \tag{4.22}$$

and using the fact that

$$\begin{aligned} \overline{u}_L \Gamma_{S,T,P}\, u_L &= 0 \\ \overline{u}_L \Gamma_A\, u_L &= -\overline{u}_L \Gamma_V\, u_L\,, \end{aligned} \tag{4.23}$$

we finally arrive at

$$\Big[\overline{u}_1 R v_2\Big]\Big[\overline{v}_3 L u_4\Big] = \frac{1}{2}\Big[\overline{u}_1 \gamma^\mu L u_4\Big]\Big[\overline{u}_2 \gamma_\mu L u_3\Big]\,. \tag{4.24}$$

In practice however, the amplitudes generally consist of a superposition of terms so that, instead of the single term Q_{RL}, the relevant quantity to consider would likely be of the form

$$\Big[\overline{u}_1 (a + b\gamma_5) v_2\Big]\Big[\overline{v}_3 (c + d\gamma_5) u_4\Big]\,. \tag{4.25}$$

This could be manipulated by using the chiral decomposition of the spinors, which would generate several chiral quadrilinears (Q_{RL}, Q_{LL} and so on), and then proceeding as above with each of them. While such manipulations can of course be readily mastered, the transformations given in Tables 1 and 2 provide a ready-to-use recipe for cases such as this, and more general ones.

References

[1] M. Fierz, Z. Physik **104**, 553 (1937).

[2] The Fierz theorem is discussed in many textbooks on quantum field theory and/or weak interactions. See, e.g., R. E. Marshak, C. P. Ryan and Riazuddin, *Theory of Weak Interactions in Particle Physics*, (John Wiley & Sons, 1969), p. 82; C. Itzykson and J. B. Zuber, *Quantum Field Theory*, (Mc Graw Hill, 1980), p. 160; Ta-Pei Cheng and Ling-Fong Li, *Gauge Theory of elementary particle physics*, (Clarendon Press, Oxford 1984), p. 495; P. Renton, *Electroweak interactions*, (Cambridge University Press 1990), p. 578; R. N. Mohapatra and P. B. Pal, *Massive neutrinos in physics and astrophysics*, (World Scientific, Singapore, second edition 1998), p. 7.