Classification and Identification of Lie Algebras

Lecture 1: Introduction and Essential Notions

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The purpose of these lectures

The purpose of these lectures is

- to present the mathematical objects and methods useful in the identification of Lie algebras,
- to develop the essential computational skills necessary for that purpose,
- to review some known facts about the structure of complex and real finite dimensional Lie algebras.

The main reference

The individual lectures will closely follow selected chapters in

Title: Classification and Identification of Lie Algebras

Authors: Libor Šnobl and Pavel Winternitz

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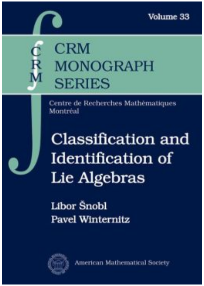

The topics to be covered

- motivation, basic notions, structure of semisimple Lie algebras - lectures I & II,
- invariants of the coadjoint representation, a.k.a. Casimir invariants - lectures III & IV,
- basis independent properties of Lie algebras and their use in the identification - lecture V,
- explicit decomposition into a direct sum - lecture VI,
- Levi decomposition - lecture VII,
- computation of the nilradical - lecture VIII,
- nilpotent Lie algebras, solvable Lie algebras with a given nilradical - lectures IX & X

Why to be interested in the identification and classification of Lie algebras?

Example: Consider two systems of PDEs, namely the well–known shallow water equations in a flat infinite basin

$$U_T + UU_X + VU_Y + H_X = 0, \; V_T + UV_X + VV_Y + H_Y = 0, \quad H_T + (UH)_X + (VH)_Y = 0 \tag{1}$$

and in a circular paraboloidal basin subjected to a Coriolis force due to the rotation of the fluid inside the basin (together with the Earth)

$$u_t + uu_x + vu_y + (Z + h)_x = fv, \; Z(x, y) = \frac{\omega^2 - f^2}{8}(x^2 + y^2), \quad v_t + uv_x + vv_y + (Z + h)_y = -fu, \quad h_t + (uh)_x + (vh)_y = 0 \tag{2}$$

and compute their algebras of infinitesimal point symmetries.

Symmetry algebras of shallow water equations, flat basin

One finds two seemingly different 9–dimensional symmetry algebras $\mathfrak{g}_A$ and $\mathfrak{g}_B$ spanned by the following vector fields, respectively

$$\begin{gathered} P_T = \partial_T,\ P_X = \partial_X,\ P_Y = \partial_Y,\ G_X = T\partial_X + \partial_U, \\ G_Y = T\partial_Y + \partial_V, D_1 = T\partial_T + X\partial_X + Y\partial_Y, \\ D_2 = -T\partial_T + U\partial_U + V\partial_V + 2H\partial_H, \\ L_1 = -Y\partial_X + X\partial_Y - V\partial_U + U\partial_V, \\ \Pi = T^2\partial_T + TX\partial_X + TY\partial_Y + (X - TU)\partial_U \\ +(Y - TV)\partial_V - 2TH\partial_H \end{gathered} \tag{3}$$

Symmetry algebra of shallow water equations, flat basin, $\mathfrak{g}_A$

	P_T	P_X	P_Y	G_X	G_Y	D_1	D_2	L_1	Π
P_T	0	0	0	P_X	P_Y	P_T	$-P_T$	0	$D_1 - D_2$
P_X	0	0	0	0	0	P_X	0	P_Y	G_X
P_Y	0	0	0	0	0	P_Y	0	$-P_X$	G_Y
G_X	$-P_X$	0	0	0	0	0	G_X	G_Y	0
G_Y	$-P_Y$	0	0	0	0	0	G_Y	$-G_X$	0
D_1	$-P_T$	$-P_X$	$-P_Y$	0	0	0	0	0	Π
D_2	P_T	0	0	$-G_X$	$-G_Y$	0	0	0	$-\Pi$
L_1	0	$-P_Y$	P_X	$-G_Y$	G_X	0	0	0	0
Π	$-D_1 + D_2$	$-G_X$	$-G_Y$	0	0	$-\Pi$	Π	0	0

Symmetry algebra of shallow water equations, paraboloidal basin, Coriolis force

$$P_0 = \partial_t, \quad D = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h,$$
$$Y_1 = \cos(R_1 t)\partial_x - \sin(R_1 t)\partial_y - R_1 \sin(R_1 t)\partial_u - R_1 \cos(R_1 t)\partial_v,$$
$$Y_2 = \sin(R_1 t)\partial_x + \cos(R_1 t)\partial_y + R_1 \cos(R_1 t)\partial_u - R_1 \sin(R_1 t)\partial_v,$$
$$Y_3 = \cos(R_2 t)\partial_x + \sin(R_2 t)\partial_y - R_2 \sin(R_2 t)\partial_u + R_2 \cos(R_2 t)\partial_v,$$
$$Y_4 = \sin(R_2 t)\partial_x - \cos(R_2 t)\partial_y + R_2 \cos(R_2 t)\partial_u + R_2 \sin(R_2 t)\partial_v,$$
$$R = y\partial_x - x\partial_y + v\partial_u - u\partial_v,$$
$$K_1 = \frac{1}{2}\cos(\omega t)\left(x\partial_x + y\partial_y - u\partial_u - v\partial_v + f(y\partial_u - x\partial_v) - 2h\partial_h\right) +$$
$$+\frac{1}{2\omega}\sin(\omega t)\left(f(y\partial_x - x\partial_y + v\partial_u - u\partial_v) - \omega^2(x\partial_u + y\partial_v) + 2\partial_t\right),$$
$$K_2 = -\frac{1}{2}\sin(\omega t)\left(x\partial_x + y\partial_y - u\partial_u - v\partial_v + f(y\partial_u - x\partial_v) - 2h\partial_h\right)$$
$$+\frac{1}{2\omega}\cos(\omega t)\left(f(y\partial_x - x\partial_y + v\partial_u - u\partial_v) - \omega^2(x\partial_u + y\partial_v) + 2\partial_t\right) \tag{4}$$

The Lie algebra $\mathfrak{g}_B$ of (4)

where[1]

$$R_1 = \frac{1}{2}(\omega + f), \qquad R_2 = \frac{1}{2}(\omega - f).$$

The Lie algebra $\mathfrak{g}_B$ of (4) has the radical (max. solvable ideal)

$$R(\mathfrak{g}_B) = span\{D, R, Y_1, Y_2, Y_3, Y_4\},$$

the nilradical (maximal nilpotent ideal)

$$NR(\mathfrak{g}_B) = span\{Y_1, Y_2, Y_3, Y_4\},$$

and the Levi factor (semisimple subalgebra complementing the radical)

$$\mathfrak{p} = span\{P_0 + \frac{f}{2}R, K_1, K_2\}.$$

[1] D. Levi, M.C. Nucci, C. Rogers, P. Winternitz 1989 J. Phys. A **22** 4743–4767

Symmetry algebra of shallow water equations, paraboloidal basin, Coriolis force, $\mathfrak{g}_B$

	P_0	K_1	K_2	D	R	Y_1	Y_2	Y_3	Y_4
P_0	0	ωK_2	$-\omega K_1$	0	0	$-\frac{f+\omega}{2} Y_2$	$\frac{f+\omega}{2} Y_1$	$\frac{f-\omega}{2} Y_4$	$\frac{\omega-f}{2} Y_3$
K_1	$-\omega K_2$	0	$\frac{-1}{\omega}(P_0 + \frac{f}{2} R)$	0	0	$-\frac{1}{2} Y_3$	$\frac{1}{2} Y_4$	$-\frac{1}{2} Y_1$	$\frac{1}{2} Y_2$
K_2	ωK_1	$\frac{1}{\omega}(P_0 + \frac{f}{2} R)$	0	0	0	$\frac{1}{2} Y_4$	$\frac{1}{2} Y_3$	$\frac{1}{2} Y_2$	$\frac{1}{2} Y_1$
D	0	0	0	0	0	$-Y_1$	$-Y_2$	$-Y_3$	$-Y_4$
R	0	0	0	0	0	Y_2	$-Y_1$	$-Y_4$	Y_3
Y_1	$\frac{f+\omega}{2} Y_2$	$\frac{1}{2} Y_3$	$-\frac{1}{2} Y_4$	Y_1	$-Y_2$	0	0	0	0
Y_2	$-\frac{f+\omega}{2} Y_1$	$-\frac{1}{2} Y_4$	$-\frac{1}{2} Y_3$	Y_2	Y_1	0	0	0	0
Y_3	$\frac{\omega-f}{2} Y_4$	$\frac{1}{2} Y_1$	$-\frac{1}{2} Y_2$	Y_3	Y_4	0	0	0	0
Y_4	$\frac{f-\omega}{2} Y_3$	$-\frac{1}{2} Y_2$	$-\frac{1}{2} Y_1$	Y_4	$-Y_3$	0	0	0	0

The Lie algebra $\mathfrak{g}_B$ of (4), continued

In the adjoint representation of $\mathfrak{g}_B$ the element D acts on the nilradical $NR(\mathfrak{g}_B)$ diagonally as a multiple of a unit matrix whereas R acts on it as a rotation. Both elements commute with the Levi factor.

From the indefinite signature of the Killing form of the Levi factor $\mathfrak{p}$ it follows that $\mathfrak{p}$ is isomorphic to the simple algebra $\mathfrak{sl}(2,\mathbb{R})$. The adjoint action of the Levi factor $\mathfrak{p}$ on the nilradical $NR(\mathfrak{g}_B)$ corresponds to a direct sum of two 2–dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$.

The Lie algebra $\mathfrak{g}_A$ of (3)

Somewhat surprisingly, the Lie algebra $\mathfrak{g}_A$ of (3) has the same structure. When expressed in suitable bases which make the structure transparent, the two algebras $\mathfrak{g}_A$ and $\mathfrak{g}_B$ turn out to be isomorphic as Lie algebras. Namely, the Lie brackets expressed in the following two bases of $\mathfrak{g}_A$ and $\mathfrak{g}_B$, respectively, imply the same structure constants

$$e_1 = P_T,\ e_2 = D_1 - D_2,\ e_3 = -\Pi,\ e_4 = -(D_1 + D_2),$$
$$e_5 = L_1,\ e_6 = P_Y,\ e_7 = P_X,\ e_8 = G_Y,\ e_9 = G_X,$$

$$\tilde{e}_1 = -\frac{1}{\omega}\left(P_0 + \frac{f}{2}R\right) + K_2,\ \tilde{e}_2 = -2K_1,$$
$$\tilde{e}_3 = \frac{1}{\omega}\left(P_0 + \frac{f}{2}R\right) + K_2,\ \tilde{e}_4 = -D,\ \tilde{e}_5 = R,\ \tilde{e}_6 = Y_1 - Y_3,$$
$$\tilde{e}_7 = Y_2 + Y_4,\ \tilde{e}_8 = -Y_2 + Y_4,\ \tilde{e}_9 = Y_1 + Y_3.$$

The Lie algebras $\mathfrak{g}_A \simeq \mathfrak{g}_B$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
e_1	0	$2e_1$	$-e_2$	0	0	0	0	e_6	e_7
e_2	$-2e_1$	0	$2e_3$	0	0	$-e_6$	$-e_7$	e_8	e_9
e_3	e_2	$-2e_3$	0	0	0	e_8	e_9	0	0
e_4	0	0	0	0	0	e_6	e_7	e_8	e_9
e_5	0	0	0	0	0	e_7	$-e_6$	e_9	$-e_8$
e_6	0	e_6	$-e_8$	$-e_6$	$-e_7$	0	0	0	0
e_7	0	e_7	$-e_9$	$-e_7$	e_6	0	0	0	0
e_8	$-e_6$	$-e_8$	0	$-e_8$	$-e_9$	0	0	0	0
e_9	$-e_7$	$-e_9$	0	$-e_9$	e_8	0	0	0	0

Isomorphisms between vector field realizations

That does not by itself imply that the two sets of vector fields (3) and (4) are related to each other by a point transformation but it is a necessary condition for it and a hint that such a transformation may exist.

Indeed, using computer algebra we find a locally invertible map

$$\Phi : \mathbb{R}^6[t, x, y, u, v, h] \to \mathbb{R}^6[T, X, Y, U, V, H]$$

which transforms the algebra of vector fields (3) into (4).

Explicitly, the transformation Φ reads

Mapping the algebra of vector fields $\mathfrak{g}_A$ to $\mathfrak{g}_B$

$$
\begin{aligned}
T &= \cot\left(\frac{\omega}{2}t\right), \qquad X = \frac{1}{2\sin\left(\frac{\omega}{2}t\right)}\left(\cos\left(\frac{f}{2}t\right)x - \sin\left(\frac{f}{2}t\right)y\right), \\
H &= C h \sin\left(\frac{\omega}{2}t\right)^2, \quad Y = -\frac{1}{2\sin\left(\frac{\omega}{2}t\right)}\left(\sin\left(\frac{f}{2}t\right)x + \cos\left(\frac{f}{2}t\right)y\right), \\
U &= \frac{1}{2\omega}\left(-2\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)u + 2\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)v+\right. \\
&\quad + \left(\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)f + \cos\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)x+ \\
&\quad \left.+ \left(\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)f - \sin\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)y\right), \\
V &= \frac{1}{2\omega}\left(2\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)u + 2\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)v+\right. \\
&\quad + \left(\sin\left(\tfrac{\omega}{2}t\right)\cos\left(\tfrac{f}{2}t\right)f - \sin\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)x- \\
&\quad \left.- \left(\sin\left(\tfrac{\omega}{2}t\right)\sin\left(\tfrac{f}{2}t\right)f + \cos\left(\tfrac{f}{2}t\right)\cos\left(\tfrac{\omega}{2}t\right)\omega\right)y\right),
\end{aligned} \tag{5}
$$

where C is an integration constant.

Equivalence of the two shallow water equations

What is more, for a particular choice of the parameter C, namely

$$C = \frac{1}{\omega^2},$$

the two shallow water equations (1) and (2) are mapped one into the other by the change of dependent and independent variables (5). Thus, mathematically they are locally equivalent although their physical interpretation is different; any solution of one of them gives rise to a (local) solution of the other. This equivalence of equations (1) and (2) would be very difficult, if not impossible, to discover without understanding the structure of the two Lie algebras involved.

Basic concepts and notation

Lie algebra $\mathfrak{g}$ is a vector space V equipped by an antisymmetric bilinear bracket

$$[\ ,\ ] : V \times V \to V$$

such that Jacobi identity holds

$$\big[x,[y,z]\big] + \big[y,[z,x]\big] + \big[z,[x,y]\big] = 0, \qquad \forall x,y,z \in V. \tag{6}$$

From now on we shall assume that the underlying vector space V is over the field of real or complex numbers. In addition, we shall identify the vector space and the algebra, $V \simeq \mathfrak{g}$.

Basic concepts and notation, cont'd

A subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is closed under the bracket,

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}. \tag{7}$$

An ideal $\mathfrak{i}$ of the Lie algebra $\mathfrak{g}$ is a subalgebra such that

$$[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}. \tag{8}$$

The Lie algebra $\mathfrak{g}$ itself and $\{0\}$ are trivial ideals. A Lie algebra which does not possess any nontrivial ideal and has dimension strictly greater than 1 is called simple.

Characteristic series

Three series of ideals – characteristic series of $\mathfrak{g}$:

- derived series $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined
$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}], \ \mathfrak{g}^{(0)} = \mathfrak{g}.$$
If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is solvable.
- lower central series $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined
$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}], \ \mathfrak{g}^1 = \mathfrak{g}.$$
If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ nilpotent. The largest value of K s.t. $\mathfrak{g}^K \neq 0$ is the degree of nilpotency.
- upper central series $\mathfrak{z}_1 \subseteq \ldots \subseteq \mathfrak{z}_k \subseteq \ldots \subseteq \mathfrak{g}$ where $\mathfrak{z}_1$ is the center of $\mathfrak{g}$, $\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0, \ \forall y \in \mathfrak{g}\}$ and $\mathfrak{z}_k$ are the higher centers defined recursively through
$$\mathfrak{z}_{k+1}/\mathfrak{z}_k = C(\mathfrak{g}/\mathfrak{z}_k).$$
For nilpotent Lie algebras exists a number l such that $\mathfrak{z}_l = \mathfrak{g}$.

Centralizer, normalizer

Any Lie algebra $\mathfrak{g}$ has a uniquely defined radical $\mathrm{R}(\mathfrak{g})$, i.e. the maximal solvable ideal, and nilradical $\mathrm{NR}(\mathfrak{g})$, i.e. the maximal nilpotent ideal. Their uniqueness follows from the observation that sum of two solvable (nilpotent) ideals is again a solvable (nilpotent) ideal, respectively.

The centralizer $cent_{\mathfrak{g}}(\mathfrak{h})$ of a given subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements in $\mathfrak{g}$ commuting with all elements in $\mathfrak{h}$, i.e.,

$$\mathrm{cent}_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, y] = 0, \ \forall y \in \mathfrak{h}\}. \tag{9}$$

The normalizer $norm_{\mathfrak{g}}(\mathfrak{h})$ of a given subspace $\mathfrak{h} \subseteq \mathfrak{g}$ in $\mathfrak{g}$ is the set of all elements x in $\mathfrak{g}$ such that $[x, h]$ is in the subspace $\mathfrak{h}$ for any $h \in \mathfrak{h}$, i.e.,

$$norm_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, y] \in \mathfrak{h}, \ \forall y \in \mathfrak{h}\}. \tag{10}$$

The normalizer of an ideal in $\mathfrak{g}$ is the whole algebra $\mathfrak{g}$.

Representations

A representation ρ of a given Lie algebra $\mathfrak{g}$ on a vector space V is a linear map of $\mathfrak{g}$ into the space $\mathscr{L}(V)$ of linear operators acting on V

$$\rho\colon \mathfrak{g} \to \mathscr{L}(V)\colon x \to \rho(x)$$

such that for any pair x, y of elements of $\mathfrak{g}$

$$\rho([x,y]) = \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) \tag{11}$$

holds. When the map ρ is injective, the representation is called *faithful*.

Representations, cont'd

A subspace W of V is called *invariant* if

$$\rho(\mathfrak{g})W = \{\rho(x)w \,|\, x \in \mathfrak{g}, w \in W\} \subseteq W.$$

A representation ρ of $\mathfrak{g}$ on V is

- *reducible* if a proper nonvanishing invariant subspace W of V exists,
- *irreducible* if no nontrivial invariant subspace of V exists,
- *fully reducible* when every invariant subspace W of V has an invariant complement $\widetilde{W}$, i.e.,

$$V = W \oplus \widetilde{W}, \quad \rho(\mathfrak{g})\widetilde{W} \subseteq \widetilde{W}. \tag{12}$$

Representations, Schur lemma

An important criterion for irreducibility of a given representation is

Theorem 1 (Schur lemma)

Let $\mathfrak{g}$ be a complex Lie algebra and ρ its representation on a finite-dimensional vector space V.

- *Let ρ be irreducible. Then any operator A on V which commutes with all $\rho(x)$,*

  $$[A, \rho(x)] = 0, \quad \forall x \in \mathfrak{g},$$

  has the form $A = \lambda \mathbf{1}$ for some complex number λ.
- *Let ρ be fully reducible and such that every operator A on V which commutes with all $\rho(x)$ has the form $A = \lambda \mathbf{1}$ for some complex number λ. Then ρ is irreducible.*

Adjoint representation

A particular representation is defined for any Lie algebra $\mathfrak{g}$, namely the *adjoint representation* of a given Lie algebra $\mathfrak{g}$ is a linear map of $\mathfrak{g}$ into the space of linear operators acting on $\mathfrak{g}$

$$ad\colon \mathfrak{g} \to \mathscr{L}(\mathfrak{g})\colon x \to ad(x)$$

defined for any pair x, y of elements of $\mathfrak{g}$ via

$$ad(x)\ y = [x, y]. \tag{13}$$

The image of ad is denoted by $ad\mathfrak{g}$.
The following theorem allows us to express nilpotency of a given algebra in terms of operators in the adjoint representation.

Theorem 2 (Engel theorem)

A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $k \in \mathbb{N}$ exists such that $\left(ad(x)\right)^k = 0$ for all $x \in \mathfrak{g}$, i.e., if all $ad(x)$ are nilpotent operators on $\mathfrak{g}$.

Lie theorem

Similarly, the solvability of a given complex algebra can be formulated in terms of operators representing it in any faithful representation.

Theorem 3 (Lie theorem)

Let $\mathfrak{g}$ be a Lie algebra and ρ its faithful representation on a complex vector space V, $n = \dim V$. The algebra $\mathfrak{g}$ is solvable if and only if a filtration of V by codimension 1 subspaces $(V_k)_{k=1}^{n}$ invariant with respect to the representation ρ exists, i.e.,

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n \equiv V, \qquad (14)$$
$$\rho(\mathfrak{g}) V_k \subset V_k.$$

When the representation ρ is chosen to be the adjoint representation, the statement of the Lie theorem holds even when it is not faithful.

Relation between radical and nilradical

As a direct consequence of the Engel and Lie theorems we find that we have

$$D\big(R(\mathfrak{g})\big) = [R(\mathfrak{g}), R(\mathfrak{g})] \subseteq NR(\mathfrak{g}). \tag{15}$$

and moreover

$$[R(\mathfrak{g}), \mathfrak{g}] \subseteq NR(\mathfrak{g}).$$

Derivation

A derivation D of a given Lie algebra $\mathfrak{g}$ is a linear map

$$D: \ \mathfrak{g} \to \mathfrak{g}$$

such that for any pair x, y of elements of $\mathfrak{g}$

$$D([x, y]) = [D(x), y] + [x, D(y)]. \tag{16}$$

If an element $z \in \mathfrak{g}$ exists, such that

$$D = \mathrm{ad}_z, \ \text{ i.e. } D(x) = [z, x], \ \forall x \in G,$$

the derivation is inner, any other one is outer.

Automorphism

An automorphism Φ of $\mathfrak{g}$ is a regular linear map

$$\Phi : \ \mathfrak{g} \to \mathfrak{g}$$

such that for any pair x, y of elements of $\mathfrak{g}$

$$\Phi([x,y]) = [\Phi(x), \Phi(y)]. \tag{17}$$

Ideals invariant with respect to all automorphisms (and thus also w.r.t. derivations) are called characteristic. E.g. ideals in the characteristic series and their centralizers are characteristic ideals.

Summary

- We have introduced the intended goal of this lecture series,
- we have given a detailed example demonstrating the relevance of Lie algebra theory to practical computations in mathematical physics,
- and we have introduced some of the basic notions that will be used throughout the lectures.

Classification and Identification of Lie Algebras

Lecture 2: Root systems and classification of complex semisimple Lie algebras

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Semisimple vs. simple Lie algebras

A Lie algebra $\mathfrak{g}$ is semisimple if its radical $R(\mathfrak{g})$ vanishes. Equivalently, $\mathfrak{g}$ possesses no nonvanishing Abelian ideal. Using

Theorem 1 (Cartan's criteria)

A Lie algebra $\mathfrak{g}$ is

- *semisimple if and only if its Killing form K*

$$K(x,y) = Tr\big(ad(x)\,ad(y)\big).$$

  is nondegenerate;
- *solvable if and only if the restriction of its Killing form to the derived algebra vanishes.*

one finds that any semisimple Lie algebra is a direct sum of its simple ideals. Thus, the classification of semisimple Lie algebras follows immediately from the classification of simple ones.

Cartan subalgebra

Let us briefly review the standard results concerning the classification of simple and semisimple Lie algebras. This classification belongs to the greatest achievements in the theory of Lie algebras. These results were originally obtained by W. Killing and É. Cartan.

Let $\mathfrak{g}$ be a complex Lie algebra. Any nilpotent subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ coinciding with its normalizer $norm_{\mathfrak{g}}(\mathfrak{g}_0)$ is called a Cartan subalgebra. It can be constructed in the following way.

Let $x \in \mathfrak{g}$. Consider the linear operator $ad(x) \in \mathscr{L}(\mathfrak{g})$ and find its generalized nullspace

$$\mathfrak{g}_0(x) = \lim_{k \to \infty} \ker\big(ad(x)\big)^k. \tag{1}$$

When $\dim \mathfrak{g}_0(x)$ is minimal, i.e., $\dim \mathfrak{g}_0(x) = \min_{y \in \mathfrak{g}} \dim \mathfrak{g}_0(y)$, we call the element $x \in \mathfrak{g}$ regular.

Cartan subalgebra, cont'd

Theorem 2

Let $x \in \mathfrak{g}$ be a regular element of the complex Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}_0(x)$ is a Cartan subalgebra of $\mathfrak{g}$. Any other Cartan subalgebra of $\mathfrak{g}$ is related to $\mathfrak{g}_0(x)$ by an automorphism of $\mathfrak{g}$.

Consequently, the dimension of the Cartan subalgebra $\mathfrak{g}_0(x)$ is independent of the choice of the regular element x and is called the rank of the Lie algebra $\mathfrak{g}$. We point out that the proposition holds whether or not $\mathfrak{g}$ is semisimple, i.e., any complex Lie algebra has a Cartan subalgebra unique up to automorphisms. The uniqueness is lost for real algebras; e.g., a finite number of distinct Cartan subalgebras exists for each finite-dimensional semisimple Lie algebra over $\mathbb{R}$.

Roots

Cartan subalgebras of semisimple algebras have special properties. The Cartan subalgebra $\mathfrak{g}_0$ of a semisimple Lie algebra is Abelian. In addition, all elements of the Cartan subalgebra are *ad*-diagonalizable or semisimple, meaning that $ad(h) \in \mathfrak{gl}(\mathfrak{g})$ is diagonalizable for every $h \in \mathfrak{g}_0$. Therefore, there exist common eigenspaces $\mathfrak{g}_\lambda \subset \mathfrak{g}$ of all operators $ad(h)$, $h \in \mathfrak{g}_0$ and nonvanishing functionals $\lambda \in \mathfrak{g}_0^*$ such that

$$ad(h)e_\lambda = \lambda(h) \cdot e_\lambda, \quad h \in \mathfrak{g}_0, e_\lambda \in \mathfrak{g}_\lambda \tag{2}$$

(where $\mathfrak{g}_0^*$ is the dual space of the vector space $\mathfrak{g}_0$.) These functionals λ are called roots of the semisimple Lie algebra $\mathfrak{g}$. The collection of all roots is called the root system of the algebra $\mathfrak{g}$ and denoted by Δ. The diagonalizability of $ad(h)$ implies that

$$\mathfrak{g} = \mathfrak{g}_0 \dot{+} \left(\dot{+}\{\mathfrak{g}_\lambda \mid \lambda \in \Delta\}\right)$$

where $\dot{+}$ stands for a direct sum of vector spaces.

Positive and simple roots

It is always possible to introduce an ordering among the roots via a choice of $h_0 \in \mathfrak{g}_0$ such that $\lambda(h_0) \neq 0$ and $\lambda(h_0) \in \mathbb{R}$ for all roots λ. This ordering is not unique but different choices give results equivalent up to automorphism of $\mathfrak{g}$. For any pair of roots λ, κ one writes $\lambda > \kappa$ if and only if $\lambda(h_0) > \kappa(h_0)$. Similarly one defines positive roots $\lambda > 0$, i.e., $\lambda(h_0) > 0$ and negative roots $\lambda < 0$, i.e., $\lambda(h_0) < 0$. The set of all positive roots is denoted Δ^+, the set of negative roots Δ^-. We have $\Delta = \Delta^+ \cup \Delta^-$. Simple roots are positive roots which cannot be written as a sum of two positive roots. We denote the set of all simple roots by Δ^S.

Properties of the root systems of semisimple algebras

We list the most important properties of the root system Δ and root subspaces $\mathfrak{g}_\lambda$ of a semisimple complex Lie algebra $\mathfrak{g}$ as a review. Their derivation can be found in any standard introductory course on Lie algebras.

1. The Killing form K of $\mathfrak{g}$ when restricted to $\mathfrak{g}_0 \times \mathfrak{g}_0$ is nondegenerate.
2. To any functional $\lambda \in \mathfrak{g}_0^*$ we can associate a unique element $h_\lambda \in \mathfrak{g}_0$ such that

$$\lambda(h) = K(h_\lambda, h), \quad \forall h \in \mathfrak{g}_0 \tag{3}$$

   and we can define a nondegenerate bilinear symmetric form $\langle \ , \ \rangle$ on $\mathfrak{g}_0^*$ so that

$$\langle \lambda, \kappa \rangle = K(h_\lambda, h_\kappa), \quad \forall \lambda, \kappa \in \mathfrak{g}_0^*.$$

Properties of the root systems of semisimple algebras, cont'd

3. If λ is a root then so is $-\lambda$ and no other multiple of λ is a root.
4. All root subspaces $\mathfrak{g}_\lambda$ are 1-dimensional.
5. $[\mathfrak{g}_\lambda, \mathfrak{g}_\kappa] = \mathfrak{g}_{\lambda+\kappa}$ whenever λ, κ and $\lambda + \kappa$ are roots.
6. When $\lambda + \kappa$ is neither 0 nor a root we have $[\mathfrak{g}_\lambda, \mathfrak{g}_\kappa] = 0$.
7. $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subset \mathfrak{g}_0$.
8. There is a basis of $\mathfrak{g}$ consisting of elements of the Cartan subalgebra $\mathfrak{g}_0$ and of the root subspaces $\mathfrak{g}_\lambda$ such that the structure constants of $\mathfrak{g}$ in this basis are integers; such a basis is called the Weyl–Chevalley basis of $\mathfrak{g}$ and the real form of the Lie algebra $\mathfrak{g}$ corresponding to this choice of basis is called the split real form of $\mathfrak{g}$.
9. Simple roots are linearly independent.

Properties of the root systems of semisimple algebras, cont'd

10. Any positive root is a linear combination of simple roots with nonnegative integer coefficients; therefore, the root system Δ is contained in the real subspace of $\mathfrak{g}_0^*$ spanned by the simple roots, we denote this subspace by $\mathfrak{h}^*$.
11. $\langle\ ,\ \rangle$ defines a real scalar product on $\mathfrak{h}^*$.
12. The whole Lie algebra $\mathfrak{g}$ is obtained by multiple Lie brackets of root vectors e_α where $\alpha \in \Delta^S$ or $-\alpha \in \Delta^S$.
13. The root system Δ is invariant under all reflections

$$S_\lambda(\alpha) = \alpha - 2\frac{\langle \alpha, \lambda\rangle}{\langle \lambda, \lambda\rangle}\lambda, \quad \lambda, \alpha \in \Delta; \tag{4}$$

all such reflections generate a finite group called the Weyl group of the root system Δ.
14. Any root is an image of some simple root under the action of an element of the Weyl group; thus, it has the same length.

Cartan matrix

It turns out that the structure of a semisimple complex Lie algebra is fully determined up to isomorphism by angles and relative lengths of its simple roots in the Euclidean space $\mathfrak{h}^*$. This information is usually encoded either in the Cartan matrix $A = (a_{\kappa\lambda})$

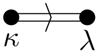
$$a_{\kappa\lambda} = 2\frac{\langle\kappa,\lambda\rangle}{\langle\lambda,\lambda\rangle}, \quad \kappa,\lambda \in \Delta^S \tag{5}$$

or equivalently in Dynkin diagrams. The Cartan matrix has only integer entries: 2 on the diagonal, $0, -1, -2, -3$ off the diagonal since

$$\langle\kappa,\lambda\rangle \leq 0, \quad a_{\kappa\lambda}a_{\lambda\kappa} = 4\frac{|\langle\kappa,\lambda\rangle|^2}{\langle\lambda,\lambda\rangle\langle\kappa,\kappa\rangle} = 4\cos\sphericalangle(\kappa,\lambda), \quad \kappa \neq \lambda.$$

Dynkin diagram

The Dynkin diagram associated with the Cartan matrix is a graph with vertices corresponding to the simple roots where the number of edges connecting the vertices labelled by κ and λ is equal to $a_{\kappa\lambda}a_{\lambda\kappa} \in \{0, 1, 2, 3\}$. Further, one distinguishes graphically between shorter and longer roots either by different symbols for vertices or different types of arrows connecting vertices. We shall use a convention that the arrow goes from the longer root to the shorter one, e.g., a subdiagram of the form



implies the following values of the elements of the Cartan matrix

$$a_{\kappa\lambda} = 2\frac{\langle\kappa, \lambda\rangle}{\langle\lambda, \lambda\rangle} = -2, \quad a_{\lambda\kappa} = 2\frac{\langle\lambda, \kappa\rangle}{\langle\kappa, \kappa\rangle} = -1.$$

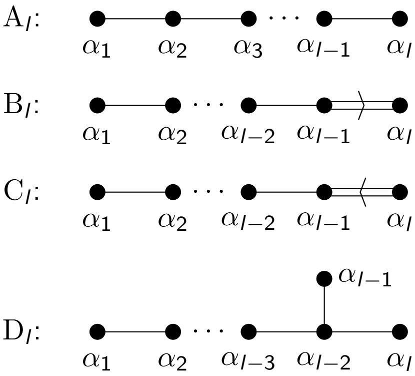
Classification of root systems

The structure of any root system can be shown to be such that:

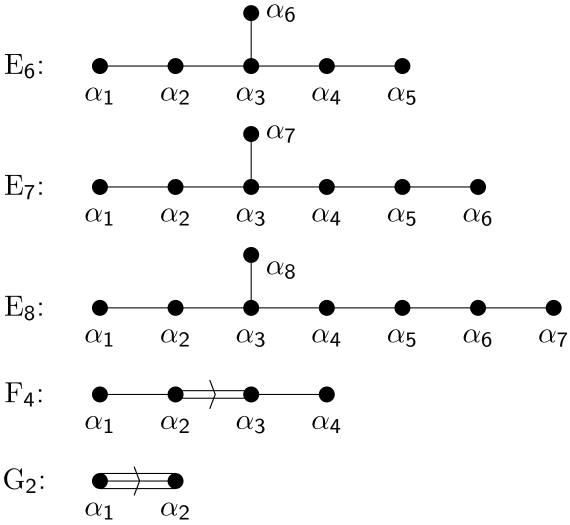
1. Simple components $\mathfrak{g}_k$ of a semisimple Lie algebra $\mathfrak{g}$ correspond to connected subdiagrams of the Dynkin diagram of $\mathfrak{g}$.
2. There are no closed loops in Dynkin diagrams.
3. A connected Dynkin diagram is either simply laced meaning that it contains only simple edges and consequently all roots are of the same length, or the corresponding root system contains roots of precisely two different lengths.

The fundamental classification result is due to W. Killing and É. Cartan whose computations were later significantly simplified by E. Dynkin using his diagrammatic approach. The result is the following list of possible connected diagrams and their corresponding simple algebras.

Classification of root systems - Dynkin diagrams of simple Lie algebras



Classification of root systems - Dynkin diagrams of simple Lie algebras, cont'd



Classification of complex simple Lie algebras

Any finite-dimensional complex simple Lie algebra $\mathfrak{g}$ either belongs to one of the classical series of simple Lie algebras, or is one of the exceptional simple Lie algebras. The classical Lie algebras are

- $\mathfrak{sl}(l+1,\mathbb{C})$ of rank $l \geq 1$, also denoted A_l, the algebra of traceless $(l+1)\times(l+1)$ matrices,
- $\mathfrak{so}(2l+1,\mathbb{C})$ of rank $l \geq 2$, also denoted B_l, the algebra of skew-symmetric $(2l+1)\times(2l+1)$ matrices,
- $\mathfrak{sp}(2l,\mathbb{C})$ of rank $l \geq 3$, also denoted C_l, the algebra of $2l \times 2l$ matrices skew-symmetric with respect to a nondegenerate antisymmetric form on $\mathbb{C}^{2l}$,
- $\mathfrak{so}(2l,\mathbb{C})$ of rank $l \geq 4$, also denoted D_l, the algebra of skew-symmetric $2l \times 2l$ matrices.

The five exceptional algebras are denoted by E_6, E_7, E_8, F_4, G_2. Out of these algebras, the algebras A_l, D_l, E_6, E_7, E_8 are simply laced.

Summary

- We have introduced the notions of Cartan subalgebra and root system, and
- employed them in the complete classification of semisimple complex Lie algebras.

Classification and Identification of Lie Algebras

Lecture 3: Invariants of the Coadjoint Representation of a Lie Algebra

August 3, 2015

Casimir operators and generalized Casimir invariants

The term *Casimir operator*, or *Casimir invariant*, of a Lie algebra $\mathfrak{g}$ is usually reserved for elements of the center of the enveloping algebra of the Lie algebra $\mathfrak{g}$. These elements should be algebraically independent so that all other elements of the center of the enveloping algebra are appropriately symmetrized polynomials in the given ones. The Casimir operators are in 1-to-1 correspondence with polynomial invariants characterizing orbits of the coadjoint representation of $\mathfrak{g}$.

In general, invariants of the coadjoint representation of a Lie algebra are not necessarily polynomials and we shall call these nonpolynomial invariants *generalized Casimir invariants*. On the other hand, for certain classes of Lie algebras their invariants can be expressed in a particular form. E.g., for perfect Lie algebras including semisimple ones and for nilpotent Lie algebras, all invariants of the coadjoint representation are functions of polynomial ones.

Casimir operators and generalized Casimir invariants, cont'd

Casimir invariants are of primordial importance in physics. They represent such important quantities as angular momentum, elementary particle mass and spin, Hamiltonians of various physical systems, etc. The same can be said of generalized Casimir invariants. Indeed, Hamiltonians and integrals of motion for physical systems are not necessarily polynomials in the momenta, though typically they are invariants of some group action.

Essentially two methods of calculating Casimir and generalized Casimir invariants exist and both of them are based on the calculation of invariants of the coadjoint representation. The first method is an infinitesimal one and amounts to solving a system of first order linear partial differential equations. The second method is more global in nature; it uses the (local) action of the Lie group G corresponding to the Lie algebra $\mathfrak{g}$. It is an application of Cartan's method of moving frames and its modern formulation is due to M. Fels and P. Olver.

Formulation of the problem

In order to calculate the (generalized) Casimir invariants we consider some basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$, in which the structure constants are ${c_{ij}}^k$. A basis for the coadjoint representation is given by the vector fields,

$$\widehat{X}_k = \sum_{a,b=1}^{n} x_b {c_{ka}}^b \frac{\partial}{\partial x_a}, \quad 1 \leq k \leq n. \tag{1}$$

In (1) the quantities x_a are commuting independent variables which can be identified with coordinates in the dual basis of the space $\mathfrak{g}^*$, dual to the algebra $\mathfrak{g}$. (Recall that there is a canonical isomorphism ι between any finite dimensional vector space V and its double dual $(V^*)^*$, defined through the relation

$$\iota(v)(\alpha) = \alpha(v)$$

for any $v \in V$ and all $\alpha \in V^*$).

Generalized Casimir invariants

The invariants of the coadjoint representation, i.e., the *generalized Casimir invariants*, are solutions of the following system of partial differential equations

$$\widehat{X}_k I(x_1, \ldots, x_n) = 0, \quad k = 1, \ldots, n. \tag{2}$$

Let us first determine the number of functionally independent solutions of the system (2). We can rewrite this system as

$$C \cdot \nabla I = 0 \tag{3}$$

where C is the antisymmetric matrix

$$C = \begin{pmatrix} 0 & {c_{12}}^b x_b & \ldots & {c_{1n}}^b x_b \\ -{c_{12}}^b x_b & 0 & \ldots & {c_{2n}}^b x_b \\ \vdots & & & \vdots \\ -{c_{1,n-1}}^b x_b & \ldots & 0 & {c_{n-1,n}}^b x_b \\ -{c_{1n}}^b x_b & \ldots & -{c_{n-1,n}}^b x_b & 0 \end{pmatrix} \tag{4}$$

in which summation over the repeated index b is to be understood in each term and ∇ is the gradient operator $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})^T$.

Number of generalized Casimir invariants

The number of independent equations in the system (2) is $r(C)$, the generic rank of the matrix C. The number of functionally independent solutions of the system (2) is hence

$$n_I = n - r(C). \tag{5}$$

Since C is antisymmetric, its rank is even. It follows that n_I has the same parity as n. Equation (5) gives the number of functionally independent generalized Casimir invariants. The individual equations in the system of partial differential equations (PDEs) (2) can be solved by the method of characteristics, or equivalently by integration of the vector fields (1).

Method of characteristics

The *method of characteristics* is applicable to a linear homogeneous first-order PDE

$$\sum_{k=0}^{n} f^k(x_1, \dots, x_n) \frac{\partial}{\partial x_k} u(x_1, \dots, x_n) = 0 \tag{6}$$

for an unknown function u. Instead of attempting to solve (6) directly, we can consider an associated systems of ODEs

$$\frac{\mathrm{d}\tilde{x}_k(t)}{\mathrm{d}t} = f^k\big(\tilde{x}_1(t), \dots, \tilde{x}_n(t)\big), \quad 1 \leq k \leq n \tag{7}$$

and find its solution satisfying a generic initial condition

$$\tilde{x}_k(0) = x_k, \quad 1 \leq k \leq n. \tag{8}$$

Method of characteristics, cont'd

In the language of differential geometry that means that we are constructing the flow, i.e., the collection of all integral curves, of the vector field

$$\widehat{F} = \sum_{j=0}^{n} f^a(x_1, \ldots, x_n) \frac{\partial}{\partial x_a}. \tag{9}$$

Once the integral curves, i.e., solutions of (7), are known, we construct functionally independent functions which are constant along the integral curves in the following way. We choose a hypersurface in $\mathbb{R}^n$ such that it is transversal to all integral curves (this is often done only locally). We associate to every integral curve its intersection with the chosen hypersurface. The coordinates of that point of intersection are invariants of the vector field $\widehat{F}$, i.e., solutions of (6), because they are by construction the same for any pair of points connected by an integral curve of $\widehat{F}$ and consequently are annihilated by the vector field.

Method of characteristics, cont'd

For the sake of the argument let us assume that the hypersurface is expressed in our coordinates as the hyperplane $x_1 = 1$. Let us take $(x_1, \ldots, x_n)$ as the initial condition (8). We determine the value of the curve parameter $t(x_1, \ldots, x_n)$ such that $\tilde{x}\big(t(x_1, \ldots, x_n)\big)$ lies on the hyperplane $x_1 = 1$, i.e., $\tilde{x}_1\big(t(x_1, \ldots, x_n)\big) = 1$. The remaining $n-1$ coordinates $\tilde{x}_k\big(t(x_1, \ldots, x_n)\big)$, $2 \leq k \leq n$ of the intersection of the integral curve with the hyperplane $x_1 = 1$ are invariants of the vector field (9)

$$I_k(x_1, \ldots, x_n) = \tilde{x}_{k+1}\big(t(x_1, \ldots, x_n)\big), \quad 1 \leq k \leq n-1.$$

They are by construction functionally independent.

Method of characteristics, cont'd

We remark that any invariant of the vector field $\widehat{F}$ is obviously also an invariant of the vector field $\widehat{G} = f\widehat{F}$ for any smooth function $f : \mathbb{R}^n \to \mathbb{R}$. On the other hand, the integral curves of $\widehat{G}$ differ from those of $\widehat{F}$ by a reparameterization, i.e., the differential equations (7) are different for $\widehat{G}$ and for $\widehat{F}$. Consequently, the solution of the system of ODEs (7) can be often significantly simplified through a suitable choice of the function f. This independence of the invariants on the reparameterization of the integral curves is symbolically depicted by rewriting of the system (7) in the form

$$\frac{\mathrm{d}x_1}{f^1(x_1,\dots,x_n)} = \frac{\mathrm{d}x_2}{f^2(x_1,\dots,x_n)} = \dots = \frac{\mathrm{d}x_n}{f^n(x_1,\dots,x_n)}. \qquad (10)$$

Method of characteristics, example

Consider the vector field

$$\widehat{F}_0 = x_1\,\partial_{x_2} + \cdots + x_{n-1}\,\partial_{x_n}. \tag{11}$$

Equation (7) becomes

$$\frac{\mathrm{d}\tilde{x}_1(t)}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\tilde{x}_k(t)}{\mathrm{d}t} = \tilde{x}_{k-1}(t).$$

The integral curves are given by the formula

$$\tilde{x}_1(t) = x_1, \quad \tilde{x}_k(t) = \sum_{j=0}^{k-1} \frac{t^j}{j!}\,x_{k-j}. \tag{12}$$

Method of characteristics, example

In this case we cannot choose our hyperplane as $x_1 = 1$ because x_1 is constant along the integral curves (12). A convenient choice of the hyperplane is $x_2 = 0$ which implies

$$t = -\frac{x_2}{x_1}. \tag{13}$$

Substituting (13) into (12) we obtain the invariants

$$I_1 = x_1, \qquad \tilde{I}_k = \sum_{j=0}^{k-1} \frac{x_{k-j}}{j!}(-1)^j \left(\frac{x_2}{x_1}\right)^j, \quad k = 3, \dots, n.$$

Multiplying $\tilde{I}_k$ by the invariant $(x_1)^{k-1}$ and shifting the label k we obtain the invariants in the form of homogeneous polynomials

$$I_1 = x_1, \qquad I_k = \sum_{j=0}^{k} \frac{(-1)^j}{j!} x_1^{k-1-j} x_2^j x_{k+1-j}, \quad 2 \leq k \leq n-1. \tag{14}$$

Solution of the system of PDEs (2) for generalized Casimir invariants

Let us now assume that one of the equations of the system (2) has been solved by the method of characteristics, i.e., we already know the $n-1$ invariants I_k of the first vector field $\widehat{X}_1$. Next, we transform all remaining vector fields to a new set of coordinates

$$(x_1, \ldots, x_n) \to (I_1, \ldots, I_{n-1}, s) \tag{15}$$

where s is a coordinate along the integral curves of $\widehat{X}_1$. We obtain

$$\begin{aligned}
\widehat{X}_1 &= \frac{\partial}{\partial s}, \\
\widehat{X}_k &= \sum_{c=1}^{n-1} \phi_k^c(I_1, \ldots, I_{n-1}, s)\frac{\partial}{\partial I_c} + \phi_k^s(I_1, \ldots, I_{n-1}, s)\frac{\partial}{\partial s}, \\
\phi_k^c &= \sum_{a,b=1}^{n} x_b c_{ka}^b \frac{\partial I_c}{\partial x_a}, \quad 2 \le k \le n.
\end{aligned} \tag{16}$$

Solution of the system of PDEs (2), cont'd

Any function J of $I_1, \ldots, I_{n-1}$ is an invariant of the vector field $\widehat{X}_1$. For J to be an invariant of the entire Lie algebra it must be a solution of the system of equations

$$\sum_{c=1}^{n-1} \phi_k^c(I_1, \ldots, I_{n-1}, s) \frac{\partial J}{\partial I_c} = 0, \quad 2 \leq k \leq n \tag{17}$$

for all values of the noninvariant parameter s. Since the vector fields $\widehat{X}_k$, $1 \leq k \leq n$ span a Lie algebra, that is an integrable distribution in the sense of the Frobenius theorem, the system (17) is compatible. It will have precisely n_I functionally independent solutions, as stated in (5).

Solution of the system of PDEs (2), cont'd

We can continue by solving another chosen equation of the system (17) using the method of characteristics. In this way we may be able to fully solve the system (2) equation by equation. However after the first step, i.e., the substitution of invariants of the first vector field $\widehat{X}_1$ into the system, the vector fields no longer have linear coefficients. Consequently, it may be difficult or indeed impossible to find the solution in closed form.

Calculation of generalized Casimir invariants by the method of moving frames

An alternative method of calculation is the method of moving frames. It can be roughly divided into the following steps.

Step 1

Integration of the coadjoint action of the Lie algebra $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ as given by the vector fields (1) *to the (local) action of the group G.*

This is usually realized by choosing a convenient (local) parameterization of G in terms of one-parameter subgroups, e.g.,

$$g(\vec{\alpha}) = \exp(\alpha_N x_N)\cdot\dots\cdot\exp(\alpha_2 x_2)\cdot\exp(\alpha_1 x_1) \in G, \quad \vec{\alpha} = (\alpha_1, \dots, \alpha_N) \tag{18}$$

and correspondingly composing the flows $\Psi^{\alpha_k}_{\widehat{X}_k}$ of $\widehat{X}_k$ defined in (1)

$$\frac{\mathrm{d}\Psi^{\alpha_k}_{\widehat{X}_k}(p)}{\mathrm{d}\alpha_k} = \widehat{X}_k\big(\Psi^{\alpha_k}_{\widehat{X}_k}(p)\big), \quad p \in \mathfrak{g}^*. \tag{19}$$

Method of moving frames, cont'd

Thus, we have

$$\Psi(g(\vec{\alpha})) = \Psi^{\alpha_N}_{\widehat{X}_N} \circ \cdots \circ \Psi^{\alpha_2}_{\widehat{X}_2} \circ \Psi^{\alpha_1}_{\widehat{X}_1}. \tag{20}$$

For a given point $p \in \mathfrak{g}^*$ with coordinates $x_k = x_k(p)$, $\vec{x} = (x_1, \ldots, x_N)$ we denote the coordinates of the transformed point $\Psi(g(\vec{\alpha}))p$ by $\tilde{x}_k$

$$\tilde{x}_k \equiv \Psi_k(\vec{\alpha})(\vec{x}) = x_k\big(\Psi\big(g(\vec{\alpha})\big)p\big). \tag{21}$$

We consider $\tilde{x}_k$ to be a function of both the group parameters $\vec{\alpha}$ and the coordinates $\vec{x}$ of the original point p.

Method of moving frames, cont'd

Step 2

Choice of a section cutting through the orbits of the action Ψ.

We choose in a smooth way a single point on each of the (generic) orbits of the action of the group G. Typically this is done as follows: we find a subset of r coordinates, say $(x_{\pi(i)})_{i=1}^{r}$, on which the group G acts transitively, at least locally in an open neighborhood of chosen values $(x^0_{\pi(i)})_{i=1}^{r}$. Here π denotes a suitable injection $\pi\colon \{1,\dots,r\} \to \{1,\dots,N\}$ and r is the rank of the matrix C in (4). Points whose coordinates satisfy

$$x_{\pi(i)} = x^0_{\pi(i)}, \quad 1 \leq i \leq r \tag{22}$$

form our section Σ, intersecting each generic orbit once.

Method of moving frames, cont'd

Step 3

Construction of invariants.

For a given point $p \in \mathfrak{g}^*$ we find group elements transforming p into $\tilde{p} \in \Sigma$ by the action Ψ. We express as many of their parameters as possible (i.e., r of them) in terms of the original coordinates $\vec{x}$ and substitute them back into (21). This gives us $\tilde{x}_k$ as functions of $\vec{x}$ only. Out of them, $\tilde{x}_{\pi(i)}$, $i = 1, \ldots, r$ have the prescribed fixed values $x^0_{\pi(i)}$. The remaining $N - r$ functions $\tilde{x}_k$ are by construction invariant under the coadjoint action of G, i.e., define the invariants of the coadjoint representation.

Technically, it may not be necessary to evaluate all the functions $\tilde{x}_k$ so that a suitable choice of the basis in $\mathfrak{g}$ can substantially simplify the whole procedure.

Method of moving frames, cont'd

This happens when only a smaller subset of say r_0 group parameters α_k enters into the computation of $N-r+r_0$ functions $\tilde{x}_k$, $k=1,\ldots,N-r+r_0$ (possibly after a rearrangement of x_k's). In this case the remaining parameters can be ignored throughout the computation since they do not enter into the expressions for $\tilde{x}_k$, $1 \leq k \leq N-r+r_0$ which define the invariants.

The method of moving frames exploits the fact that the flows can be computed easily due to the linear dependence on the coordinates in the coefficients of the vector fields (1). Thus the problem is reduced to a choice of the section and the elimination of the group parameters, i.e., to a system of algebraic equations. Unfortunately, these equations may be difficult or impossible to solve (and strongly depend on the choice of the section).

Example

Let us consider the Lie algebra $\mathfrak{g}$ with the nonvanishing Lie brackets

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = e_3, \quad [e_3, e_4] = -e_2. \tag{23}$$

The vector fields (1) are

$$\begin{aligned} \widehat{E}_1 &= 0, & \widehat{E}_2 &= e_1\partial_{e_3} + e_3\partial_{e_4}, \\ \widehat{E}_3 &= -e_1\partial_{e_2} - e_2\partial_{e_4}, & \widehat{E}_4 &= -e_3\partial_{e_2} + e_2\partial_{e_3}. \end{aligned} \tag{24}$$

Consequently, the matrix C takes the form

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_3 \\ 0 & -e_1 & 0 & -e_2 \\ 0 & -e_3 & e_2 & 0 \end{pmatrix}. \tag{25}$$

Example, cont'd

The generic rank of C is 2 and the number (5) of functionally independent Casimir invariants is $n_I = 4 - 2 = 2$. Since the first column of C consists of zeros, e_1 is a solution. We take $\widehat{E}_2$ as the first vector field to which we apply the method of characteristics. We have

$$\frac{\mathrm{d}e_3}{e_1} = \frac{\mathrm{d}e_4}{e_3},$$

the invariants of $\widehat{E}_2$ are e_1, e_2 and $\xi = e_3^2 - 2e_1 e_4$. Any invariant must take the form $J = J(e_1, e_2, \xi)$. Applying $\widehat{E}_3$ to J we find

$$\widehat{E}_3 J = e_1 \left(2e_2 \frac{\partial J}{\partial \xi} - \frac{\partial J}{\partial e_2} \right) \tag{26}$$

and we obtain two independent solution of $\widehat{E}_3 J = 0$ in the form $\eta = e_2^2 + e_3^2 - 2e_1 e_4$ and e_1. Both e_1 and η are annihilated by $\widehat{E}_4$. Altogether, we have found that our algebra has 2 Casimir invariants

$$I_1 = e_1, \quad I_2 = e_2^2 + e_3^2 - 2e_1 e_4. \tag{27}$$

Example, using the method of moving frames

The flows of the vector fields $\widehat{E}_1, \ldots, \widehat{E}_4$ are

$$\Psi^{\alpha_1}_{\widehat{E}_1}(e_1, e_2, e_3, e_4) = (e_1, e_2, e_3, e_4),$$

$$\Psi^{\alpha_2}_{\widehat{E}_2}(e_1, e_2, e_3, e_4) = \left(e_1, e_2, \alpha_2 e_1 + e_3, \frac{\alpha_2^2}{2} e_1 + \alpha_2 e_3 + e_4\right),$$

$$\Psi^{\alpha_3}_{\widehat{E}_3}(e_1, e_2, e_3, e_4) = \left(e_1, -\alpha_3 e_1 + e_2, e_3, \frac{\alpha_3^2}{2} e_1 - \alpha_3 e_2 + e_4\right),$$

$$\Psi^{\alpha_4}_{\widehat{E}_4}(e_1, e_2, e_3, e_4) = (e_1, e_2 \cos\alpha_4 - e_3 \sin\alpha_4, e_2 \sin\alpha_4 + e_3 \cos\alpha_4, e_4).$$

We compose the flows as in (20) and obtain

$$\begin{aligned}
\Psi\big(g(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\big) &= \Psi^{\alpha_4}_{\widehat{E}_4} \circ \Psi^{\alpha_3}_{\widehat{E}_3} \circ \Psi^{\alpha_2}_{\widehat{E}_2} \circ \Psi^{\alpha_1}_{\widehat{E}_1}, \\
\Psi\big(g(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\big)(e_1, e_2, e_3, e_4) &= \Big(e_1, \cos\alpha_4(-\alpha_3 e_1 + e_2) - \sin\alpha_4(\alpha_2 e_1 + e_3), \\
&\quad \sin\alpha_4(-\alpha_3 e_1 + e_2) + \cos\alpha_4(\alpha_2 e_1 + e_3), \frac{\alpha_2^2 + \alpha_3^2}{2} e_1 - \alpha_3 e_2 + \alpha_2 e_3 + e_4\Big).
\end{aligned}$$

Example, using the method of moving frames, cont'd

We choose a section Σ given by the equations

$$e_2 = 0, \quad e_3 = 1. \tag{28}$$

The intersection of our section Σ with the orbit $\Psi(g(\alpha_1, \alpha_2, \alpha_3, \alpha_4))(e_1, e_2, e_3, e_4)$ starting from the point (e_1, e_2, e_3, e_4) has the following values of α_2, α_3

$$\alpha_2 = \frac{\cos\alpha_4 - e_3}{e_1}, \quad \alpha_3 = \frac{e_2 - \sin\alpha_4}{e_1} \tag{29}$$

(generically, i.e., when $e_1 \neq 0$).

Example, using the method of moving frames, cont'd

The coordinates of the intersection

$$\left(e_1, 0, 1, \frac{2e_1 e_4 - e_2^2 - e_3^2 + 1}{2e_1}\right). \tag{30}$$

are independent of the remaining two parameters α_1, α_4. That means that we have found using the method of moving frames that two functionally independent functions

$$e_1, \qquad (2e_1 e_4 - e_2^2 - e_3^2 + 1)/(2e_1)$$

are generalized Casimir invariants. Equivalently, taking functional combinations we find that

$$I_1 = e_1, \quad I_2 = e_2^2 + e_3^2 - 2e_1 e_4$$

of (27) are Casimir invariants.

Summary

- We have defined invariants of the coadjoint representation, a.k.a. generalized Casimir invariants,
- we have presented two methods of their determination, namely the direct method using the method of characteristics repeatedly, and the method of moving frames, and
- we have illustrated both methods on an example.

Classification and Identification of Lie Algebras

Lecture 4: Invariants of the Coadjoint Representation of a Lie Algebra, cont'd

August 3, 2015

Reminder: Casimir invariants

In order to calculate the (generalized) Casimir invariants we consider some basis $(x_1, \ldots, x_n)$ of $\mathfrak{g}$, in which the structure constants are ${c_{ij}}^k$. A basis for the coadjoint representation is given by the vector fields,

$$\widehat{X}_k = \sum_{a,b=1}^{n} x_b {c_{ka}}^b \frac{\partial}{\partial x_a}, \quad 1 \leq k \leq n. \tag{1}$$

In (1) the quantities x_a are commuting independent variables which can be identified with coordinates in the dual basis of the space $\mathfrak{g}^*$, dual to the algebra $\mathfrak{g}$.
The invariants of the coadjoint representation, i.e., the *generalized Casimir invariants*, are solutions of the following system of partial differential equations

$$\widehat{X}_k I(x_1, \ldots, x_n) = 0, \quad k = 1, \ldots, n. \tag{2}$$

Solution of the system of PDEs (2) for generalized Casimir invariants

Let us now assume that one of the equations of the system (2) has been solved by the method of characteristics, i.e., we already know the $n-1$ invariants I_k of the first vector field $\widehat{X}_1$. Next, we transform all remaining vector fields to a new set of coordinates

$$(x_1, \ldots, x_n) \to (I_1, \ldots, I_{n-1}, s) \tag{3}$$

where s is a coordinate along the integral curves of $\widehat{X}_1$. We obtain

$$\begin{aligned} \widehat{X}_1 &= \frac{\partial}{\partial s}, \\ \widehat{X}_k &= \sum_{c=1}^{n-1} \phi_k^c(I_1, \ldots, I_{n-1}, s) \frac{\partial}{\partial I_c} + \phi_k^s(I_1, \ldots, I_{n-1}, s) \frac{\partial}{\partial s}, \\ \phi_k^c &= \sum_{a,b=1}^{n} x_b c_{ka}^b \frac{\partial I_c}{\partial x_a}, \quad 2 \le k \le n. \end{aligned} \tag{4}$$

Solution of the system of PDEs (2), cont'd

Any function J of $I_1, \ldots, I_{n-1}$ is an invariant of the vector field $\widehat{X}_1$. For J to be an invariant of the entire Lie algebra it must be a solution of the system of equations

$$\sum_{c=1}^{n-1} \phi_k^c(I_1, \ldots, I_{n-1}, s) \frac{\partial J}{\partial I_c} = 0, \quad 2 \leq k \leq n \tag{5}$$

for all values of the noninvariant parameter s. Since the vector fields $\widehat{X}_k$, $1 \leq k \leq n$ span a Lie algebra, that is an integrable distribution in the sense of the Frobenius theorem, the system (5) is compatible. It will have precisely n_I functionally independent solutions, as stated in (**??**).

Solution of the system of PDEs (2), cont'd

We can continue by solving another chosen equation of the system (5) using the method of characteristics. In this way we may be able to fully solve the system (2) equation by equation. However after the first step, i.e., the substitution of invariants of the first vector field $\widehat{X}_1$ into the system, the vector fields no longer have linear coefficients. Consequently, it may be difficult or indeed impossible to find the solution in closed form.

Calculation of generalized Casimir invariants by the method of moving frames

An alternative method of calculation is the method of moving frames. It can be roughly divided into the following steps.

Step 1

Integration of the coadjoint action of the Lie algebra $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ as given by the vector fields (1) to the (local) action of the group G.

This is usually realized by choosing a convenient (local) parameterization of G in terms of one-parameter subgroups, e.g.,

$$g(\vec{\alpha}) = \exp(\alpha_N x_N) \cdot \dots \cdot \exp(\alpha_2 x_2) \cdot \exp(\alpha_1 x_1) \in G, \quad \vec{\alpha} = (\alpha_1, \dots, \alpha_N) \tag{6}$$

and correspondingly composing the flows $\Psi^{\alpha_k}_{\widehat{X}_k}$ of $\widehat{X}_k$ defined in (1)

$$\frac{\mathrm{d}\Psi^{\alpha_k}_{\widehat{X}_k}(p)}{\mathrm{d}\alpha_k} = \widehat{X}_k\big(\Psi^{\alpha_k}_{\widehat{X}_k}(p)\big), \quad p \in \mathfrak{g}^*. \tag{7}$$

Method of moving frames, cont'd

Thus, we have

$$\Psi(g(\vec{\alpha})) = \Psi^{\alpha_N}_{\widehat{X}_N} \circ \cdots \circ \Psi^{\alpha_2}_{\widehat{X}_2} \circ \Psi^{\alpha_1}_{\widehat{X}_1}. \tag{8}$$

For a given point $p \in \mathfrak{g}^*$ with coordinates $x_k = x_k(p)$, $\vec{x} = (x_1, \ldots, x_N)$ we denote the coordinates of the transformed point $\Psi(g(\vec{\alpha}))p$ by $\tilde{x}_k$

$$\tilde{x}_k \equiv \Psi_k(\vec{\alpha})(\vec{x}) = x_k\big(\Psi\big(g(\vec{\alpha})\big)p\big). \tag{9}$$

We consider $\tilde{x}_k$ to be a function of both the group parameters $\vec{\alpha}$ and the coordinates $\vec{x}$ of the original point p.

Method of moving frames, cont'd

Step 2

Choice of a section cutting through the orbits of the action Ψ.

We choose in a smooth way a single point on each of the (generic) orbits of the action of the group G. Typically this is done as follows: we find a subset of r coordinates, say $(x_{\pi(i)})_{i=1}^{r}$, on which the group G acts transitively, at least locally in an open neighborhood of chosen values $(x^0_{\pi(i)})_{i=1}^{r}$. Here π denotes a suitable injection $\pi\colon \{1,\dots,r\} \to \{1,\dots,N\}$ and r is the rank of the matrix C in (**??**). Points whose coordinates satisfy

$$x_{\pi(i)} = x^0_{\pi(i)}, \quad 1 \le i \le r \tag{10}$$

form our section Σ, intersecting each generic orbit once.

Method of moving frames, cont'd

Step 3

Construction of invariants.

For a given point $p \in \mathfrak{g}^*$ we find group elements transforming p into $\tilde{p} \in \Sigma$ by the action Ψ. We express as many of their parameters as possible (i.e., r of them) in terms of the original coordinates $\vec{x}$ and substitute them back into (9). This gives us $\tilde{x}_k$ as functions of $\vec{x}$ only. Out of them, $\tilde{x}_{\pi(i)}$, $i = 1, \ldots, r$ have the prescribed fixed values $x^0_{\pi(i)}$. The remaining $N - r$ functions $\tilde{x}_k$ are by construction invariant under the coadjoint action of G, i.e., define the invariants of the coadjoint representation.

Technically, it may not be necessary to evaluate all the functions $\tilde{x}_k$ so that a suitable choice of the basis in $\mathfrak{g}$ can substantially simplify the whole procedure.

Method of moving frames, cont'd

This happens when only a smaller subset of say r_0 group parameters α_k enters into the computation of $N - r + r_0$ functions $\tilde{x}_k$, $k = 1, \ldots, N - r + r_0$ (possibly after a rearrangement of x_k's). In this case the remaining parameters can be ignored throughout the computation since they do not enter into the expressions for $\tilde{x}_k$, $1 \leq k \leq N - r + r_0$ which define the invariants.

The method of moving frames exploits the fact that the flows can be computed easily due to the linear dependence on the coordinates in the coefficients of the vector fields (1). Thus the problem is reduced to a choice of the section and the elimination of the group parameters, i.e., to a system of algebraic equations. Unfortunately, these equations may be difficult or impossible to solve (and strongly depend on the choice of the section).

Example

Let us consider the Lie algebra $\mathfrak{g}$ with the nonvanishing Lie brackets

$$[e_2, e_3] = e_1, \quad [e_2, e_4] = e_3, \quad [e_3, e_4] = -e_2. \tag{11}$$

The vector fields (1) are

$$\begin{aligned} \widehat{E}_1 &= 0, & \widehat{E}_2 &= e_1\partial_{e_3} + e_3\partial_{e_4}, \\ \widehat{E}_3 &= -e_1\partial_{e_2} - e_2\partial_{e_4}, & \widehat{E}_4 &= -e_3\partial_{e_2} + e_2\partial_{e_3}. \end{aligned} \tag{12}$$

Consequently, the matrix C takes the form

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e_1 & e_3 \\ 0 & -e_1 & 0 & -e_2 \\ 0 & -e_3 & e_2 & 0 \end{pmatrix}. \tag{13}$$

Example, cont'd

The generic rank of C is 2 and the number (**??**) of functionally independent Casimir invariants is $n_I = 4 - 2 = 2$. Since the first column of C consists of zeros, e_1 is a solution. We take $\widehat{E}_2$ as the first vector field to which we apply the method of characteristics. We have

$$\frac{\mathrm{d}e_3}{e_1} = \frac{\mathrm{d}e_4}{e_3},$$

the invariants of $\widehat{E}_2$ are e_1, e_2 and $\xi = e_3^2 - 2e_1e_4$. Any invariant must take the form $J = J(e_1, e_2, \xi)$. Applying $\widehat{E}_3$ to J we find

$$\widehat{E}_3 J = e_1 \left(2e_2 \frac{\partial J}{\partial \xi} - \frac{\partial J}{\partial e_2} \right) \tag{14}$$

and we obtain two independent solution of $\widehat{E}_3 J = 0$ in the form $\eta = e_2^2 + e_3^2 - 2e_1e_4$ and e_1. Both e_1 and η are annihilated by $\widehat{E}_4$. Altogether, we have found that our algebra has 2 Casimir invariants

$$I_1 = e_1, \quad I_2 = e_2^2 + e_3^2 - 2e_1e_4. \tag{15}$$

Example, using the method of moving frames

The flows of the vector fields $\widehat{E}_1, \ldots, \widehat{E}_4$ are

$$\Psi^{\alpha_1}_{\widehat{E}_1}(e_1, e_2, e_3, e_4) = (e_1, e_2, e_3, e_4),$$

$$\Psi^{\alpha_2}_{\widehat{E}_2}(e_1, e_2, e_3, e_4) = \left(e_1, e_2, \alpha_2 e_1 + e_3, \frac{\alpha_2^2}{2} e_1 + \alpha_2 e_3 + e_4\right),$$

$$\Psi^{\alpha_3}_{\widehat{E}_3}(e_1, e_2, e_3, e_4) = \left(e_1, -\alpha_3 e_1 + e_2, e_3, \frac{\alpha_3^2}{2} e_1 - \alpha_3 e_2 + e_4\right),$$

$$\Psi^{\alpha_4}_{\widehat{E}_4}(e_1, e_2, e_3, e_4) = (e_1, e_2 \cos\alpha_4 - e_3 \sin\alpha_4, e_2 \sin\alpha_4 + e_3 \cos\alpha_4, e_4).$$

We compose the flows as in (8) and obtain

$$\begin{aligned}
\Psi(g(\alpha_1, \alpha_2, \alpha_3, \alpha_4)) &= \Psi^{\alpha_4}_{\widehat{E}_4} \circ \Psi^{\alpha_3}_{\widehat{E}_3} \circ \Psi^{\alpha_2}_{\widehat{E}_2} \circ \Psi^{\alpha_1}_{\widehat{E}_1}, \\
\Psi(g(\alpha_1, \alpha_2, \alpha_3, \alpha_4))(e_1, e_2, e_3, e_4) &= \Big(e_1, \cos\alpha_4(-\alpha_3 e_1 + e_2) - \sin\alpha_4(\alpha_2 e_1 + e_3), \\
&\quad \sin\alpha_4(-\alpha_3 e_1 + e_2) + \cos\alpha_4(\alpha_2 e_1 + e_3), \frac{\alpha_2^2 + \alpha_3^2}{2} e_1 - \alpha_3 e_2 + \alpha_2 e_3 + e_4\Big).
\end{aligned}$$

Example, using the method of moving frames, cont'd

We choose a section Σ given by the equations

$$e_2 = 0, \quad e_3 = 1. \tag{16}$$

The intersection of our section Σ with the orbit $\Psi(g(\alpha_1, \alpha_2, \alpha_3, \alpha_4))(e_1, e_2, e_3, e_4)$ starting from the point (e_1, e_2, e_3, e_4) has the following values of α_2, α_3

$$\alpha_2 = \frac{\cos\alpha_4 - e_3}{e_1}, \quad \alpha_3 = \frac{e_2 - \sin\alpha_4}{e_1} \tag{17}$$

(generically, i.e., when $e_1 \neq 0$).

Example, using the method of moving frames, cont'd

The coordinates of the intersection

$$\left(e_1, 0, 1, \frac{2e_1 e_4 - e_2^2 - e_3^2 + 1}{2e_1}\right). \tag{18}$$

are independent of the remaining two parameters α_1, α_4. That means that we have found using the method of moving frames that two functionally independent functions

$$e_1, \qquad (2e_1 e_4 - e_2^2 - e_3^2 + 1)/(2e_1)$$

are generalized Casimir invariants. Equivalently, taking functional combinations we find that

$$I_1 = e_1, \quad I_2 = e_2^2 + e_3^2 - 2e_1 e_4$$

of (15) are Casimir invariants.

Application of generalized Casimir invariants to the Lie algebra identification

The invariants of the coadjoint representation belong among important characteristics of any given Lie algebra. Their knowledge may help us to distinguish Lie algebras whose nonequivalence may be difficult to establish by other means, as the following example shows.

Let us consider two real 6–dimensional solvable Lie algebras $\mathfrak{s}_1, \mathfrak{s}_2$ with the nonvanishing Lie brackets

$\mathfrak{s}_1$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	0	0	0	0	$-e_1$
e_2		0	0	0	0	e_2
e_3			0	e_2	e_1	0
e_4				0	e_3	$-e_2+e_4$
e_5					0	$-e_5$

Application of generalized Casimir invariants to the Lie algebra identification, cont'd

$\mathfrak{s}_2$:

	$\tilde{e}_1$	$\tilde{e}_2$	$\tilde{e}_3$	$\tilde{e}_4$	$\tilde{e}_5$	$\tilde{e}_6$
$\tilde{e}_1$	0	0	0	0	0	$-\tilde{e}_2$
$\tilde{e}_2$		0	0	0	0	$\tilde{e}_1$
$\tilde{e}_3$			0	$\tilde{e}_2$	$\tilde{e}_1$	0
$\tilde{e}_4$				0	$\tilde{e}_3$	$\tilde{e}_5$
$\tilde{e}_5$					0	$-\tilde{e}_2 - \tilde{e}_4$

These two algebras $\mathfrak{s}_1, \mathfrak{s}_2$ are real forms of a single complex Lie algebra $\mathfrak{s}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{s}_1 \simeq \mathbb{C} \otimes \mathfrak{s}_2$, related by a complex isomorphism

$$e_1 = -\tilde{e}_1 - \mathrm{i}\tilde{e}_2, e_2 = \frac{\mathrm{i}\tilde{e}_1 + \tilde{e}_2}{2}, e_3 = -\tilde{e}_3, e_4 = \frac{\mathrm{i}}{4}\tilde{e}_1 - \frac{\tilde{e}_4 + \mathrm{i}\tilde{e}_5}{2},$$

$$e_5 = -\frac{1}{2}\tilde{e}_1 + \mathrm{i}\tilde{e}_4 + \tilde{e}_5, e_6 = -\frac{1}{2}\tilde{e}_3 + \mathrm{i}\tilde{e}_6.$$

Application of generalized Casimir invariants to the Lie algebra identification, cont'd

The question is whether they are equivalent also as real algebras, i.e. a real homomorphism between them exists, or they define two distinct real forms of $\mathfrak{s}_{\mathbb{C}}$.

A short calculation along the lines indicated above shows that their independent generalized Casimir invariants can be written as

$$\mathfrak{s}_1: \qquad e_1 e_2, \quad e_1^2 \exp\left(\frac{e_3^2 - 2e_1 e_4 + 2e_2 e_5}{e_1 e_2}\right),$$

$$\mathfrak{s}_2: \qquad \tilde{e}_1^2 + \tilde{e}_2^2, \quad (\tilde{e}_1^2 + \tilde{e}_2^2) \arctan\frac{\tilde{e}_2}{\tilde{e}_1} - \tilde{e}_1\tilde{e}_2 - 2\tilde{e}_1\tilde{e}_4 + 2\tilde{e}_2\tilde{e}_5 + \tilde{e}_3^2.$$

Since no real transformation can convert trigonometric functions into exponentials and vice versa we immediately see that the algebras $\mathfrak{s}_1, \mathfrak{s}_2$ cannot be isomorphic, i.e. they must define two different real forms of $\mathfrak{s}_{\mathbb{C}}$.

Summary

- We have presented two methods of their determination, namely the direct method using the method of characteristics repeatedly, and the method of moving frames, and
- we have illustrated both methods on an example, and
- we have shown an example of the use of Casimir invariants in the problem of identification of the given Lie algebra.

Classification and Identification of Lie Algebras

Lecture 5: Identification of Lie Algebras through the Use of Invariants

August 3, 2015

Identification of isomorphic algebras

We shall discuss the possible ways of identification of a given Lie algebra in a known list of algebras. The methods explained here can be used to demonstrate whether two given algebras are isomorphic or not.

An obvious way of establishing the equivalence of two algebras is an explicit construction of a change of basis which transforms the structure constants between them. Such a task in general involves a solution of a nonlinear set of equations.

Identification of isomorphic algebras, cont'd

Namely, we have two algebras of dimension n spanned by (e_i) and $(\tilde{e}_i)$, respectively, given by their Lie brackets

$$[e_i, e_j] = f_{ij}{}^k e_k, \quad [\tilde{e}_i, \tilde{e}_j] = g_{ij}{}^k \tilde{e}_k$$

and look for a regular transformation A such that

$$\bar{e}_i = e_k A_i^k, \quad [\bar{e}_i, \bar{e}_j] = g_{ij}{}^k \bar{e}_k \tag{1}$$

which reduces to the set of $n^2(n-1)/2$ quadratic equations for the components of A_i^k

$$A_i^k A_j^l f_{kl}{}^m = g_{ij}{}^k A_k^m, \quad \det A \neq 0. \tag{2}$$

The algebras are by definition isomorphic if and only if such a transformation exists. Unfortunately, the existence of a solution of (2) is in general rather difficult to establish.

Identification of isomorphic algebras, cont'd

Consequently one uses invariant, i.e., basis independent, characteristics of Lie algebras in order to distinguish classes of algebras which are mutually nonisomorphic. Examples of these are dimensions of unique ideals, properties of the Killing form etc. In the following we review some of them. For practical purposes we split our discussion into two parts: elementary invariants, most of which can be computed by hand once we write down the Lie brackets, and more sophisticated invariants whose efficient computation necessarily involves use of computer algebra systems in all but the lowest dimensions.

The first, very rough, invariant characteristic which one should establish is the type of the Lie algebra in question. Namely whether it is indecomposable or decomposable; semisimple, solvable or has a nontrivial Levi decomposition.

Elementary invariants

The solvability and semisimplicity are easily established from Cartan's criteria. Namely, the algebra is semisimple if and only if its Killing form is nondegenerate; the algebra is solvable if and only if its Killing form restricted to the derived algebra vanishes. The decomposability can be established by the algorithm described in next lecture. The Levi decomposition into the radical and a semisimple subalgebra can be performed as described in Lecture 7. Both these algorithms are often realized on a computer since they involve the factorization of polynomials and the solution of a relatively large system of linear equations, respectively. The factors obtained by either of these algorithms should themselves be identified. For Levi decomposable algebras one also has to analyze how the radical and semisimple subalgebra are combined together since more than one nonequivalent possibility may exist.

Dimensions of characteristic series

Another class of invariants which we have already encountered is the dimension of ideals in the characteristic series of the Lie algebra. We recall that we use the symbols DS, CS and US for sets of integers denoting the dimensions of subalgebras in the derived, lower central and upper central series, respectively.

These invariants are very easy to compute, essentially by inspection of the Lie brackets. The amount of information they contain depends strongly on the type of Lie algebra. They are useful for nilpotent and solvable algebras but do not provide any information for semisimple ones.

Dimensions of characteristic series

One can of course analyze the structure of characteristic series in more detail, e.g., identify the structure of each lower-dimensional ideal contained in them by calculating its respective characteristic series. This may provide more information in certain cases. However, the equality of dimensions often implies that the algebraic structure of the ideals is the same. The difference between algebras then lies in different ways these ideals are arranged inside the entire algebra and cannot be identified in this way.

When the algebra in question is solvable but not nilpotent we shall identify its nilradical by the algorithm of Lecture 8. The (faithful) representation of the solvable factor algebra $\mathfrak{g}/C(\mathfrak{g})$ on the nilradical can provide further invariant characteristics, e.g., whether the representation is completely reducible or not, what are the dimensions of its invariant subspaces etc. This appears to be the most straightforward way of identifying the given solvable algebra.

Rank

The invariant of particular importance for simple and semisimple algebras is the rank of the Lie algebra, i.e., the dimension of its Cartan subalgebra. Its determination together with the dimension of the algebra brings us very close to full identification of a simple complex Lie algebra in the Cartan's classification. The only remaining ambiguity to be resolved is the differentiation between B_l and C_l algebras, i.e., between $\mathfrak{so}(2l+1,\mathbb{C}))$ and $\mathfrak{sp}(2l,\mathbb{C})$. Exceptionally, for $l=6$ also E_6 shares the same values of rank and dimension with B_6 and C_6. The rank may provide nontrivial information also for other nonnilpotent algebras.

N.B.: For a nilpotent algebra the Cartan subalgebra coincides with the entire algebra and the term rank is often used for a different property, namely for the dimension of the maximal tori of derivations, i.e., maximal Abelian subalgebras of the algebra of derivations consisting of semisimple elements.

Dynkin diagram

For semisimple Lie algebras another well-known invariant is the Dynkin diagram and other properties related to the structure of the simple roots. For example, in order to distinguish B_l and C_l algebras given in an arbitrary basis one may find the Cartan subalgebra, construct the corresponding root system, identify the simple roots and consider their relative lengths. In the case of B_l one of the simple roots is shorter than the rest, C_l has one longer simple root. E_6 has all the simple roots of the same lengths.

Casimir invariants

Several invariant properties can be extracted also from the invariants of coadjoint representation, i.e., generalized Casimir invariants. An obvious invariant property is the number of functionally independent generalized Casimir invariants.

Another possibility available for solvable and Levi decomposable Lie algebras is the maximal number of proper, i.e., polynomial, Casimir invariants. For semisimple and nilpotent algebras it coincides with the total number of independent generalized Casimir invariants.

Under certain assumptions about the chosen basis of Casimir invariants we can go even further. Provided we choose the lowest degree polynomials as the generators, their degrees become further invariant characteristics of the given Lie algebra $\mathfrak{g}$. Indeed a change of basis in $\mathfrak{g}$ induces a linear transformation of the functionals $e_1, \ldots, e_n$ which does not change the degrees of polynomials expressed in terms of them.

Identification of real forms

Most of the invariants introduced up to now, in particular all the ones determined by the dimension of some distinguished subalgebra, may allow us to distinguish complex Lie algebras but are not suitable for identification of nonisomorphic real forms of the same complex algebra. They necessarily give the same answer for all of them. The most elementary invariant allowing such a discrimination is the signature of the Killing form, i.e., the number of its positive, negative and zero eigenvalues once it is brought to diagonal form. Because over the field $\mathbb{C}$ the Killing form is bilinear, not sesquilinear, its signature may change under a complex linear transformation but remains invariant once we restrict ourselves to real transformations. Therefore, it may allow the identification of a particular real form of a semisimple or solvable Lie algebra. For simple Lie algebras it is known that it allows complete identification of the real form.

Identification of real forms, cont'd

For nonsemisimple algebras one may find it useful to establish the signature of the Killing form of some algebra constructed out of the given algebra, e.g., of the Killing form of the algebra of derivations. E.g., this procedure allows to distinguish all nonisomorphic real forms of nilpotent algebras up to dimension 6 (in higher dimensions characteristically nilpotent algebras with nilpotent algebras of derivations appear, thus it fails in some cases).

Also, the type of transcendent functions present in the invariants can be helpful; e.g., as we have already seen last time different real forms may be distinguishable by presence of exponential versus trigonometric functions in the generalized Casimir invariants.

More computationally demanding invariants

Let us review some of the more involved constructions of invariants of Lie algebras. The list cannot be exhaustive since new ways of characterizing Lie algebras suited for their particular classes are constantly being developed.

Going in the other direction than the characteristic series is the construction of Lie algebras which are typically bigger than the one we started with. The most natural possibility is to construct the algebra of derivations of the given Lie algebra and investigate its properties. Although it is of no use for semisimple Lie algebras whose all derivations are inner, for other classes of algebras we may obtain interesting information in this way. For example, a low-dimensional (up to 6) complex nilpotent Lie algebra can be fully identified once we determine its DS, CS and US and the dimension of its algebra of derivations.

Distinguishing algebras in classes involving parameters

Other invariants address the problem of the identification of possible isomorphisms inside a class of algebras involving a parameter. Such invariants can be constructed, e.g., using the notion of (α, β, γ)-derivations introduced by J. Hrivnák and P. Novotný.

For given $\alpha, \beta, \gamma \in \mathbb{F}$ we call a linear operator $A \colon \mathfrak{g} \to \mathfrak{g}$ an (α, β, γ)-derivation when

$$\alpha A[x, y] = \beta[Ax, y] + \gamma[x, Ay] \tag{3}$$

for every $x, y \in \mathfrak{g}$. The vector spaces of A are denoted by $\mathcal{D}(\alpha, \beta, \gamma)$. In fact only the following values of (α, β, γ)

$$(\alpha, 0, 0), \quad (\alpha, 1, -1), \quad (\alpha, 1, 0), \quad (\alpha, 1, 1), \quad \alpha \in \mathbb{F}$$

should be used; any vector space $\mathcal{D}(\alpha, \beta, \gamma)$ with different values of parameters α, β, γ is equal to one in the above introduced range.

(α, β, γ)-derivations

The dimension of each of these vector spaces provides us with basis independent information about the given Lie algebra $\mathfrak{g}$. Their importance lies in the fact that they depend on the continuous parameter α; we have the "invariant function"

$$\psi_{\mathfrak{g}}(\alpha) = \dim \mathcal{D}(\alpha, 1, 1). \tag{4}$$

If we are given a family of Lie algebras depending on one or more continuous parameters and want to establish possible isomorphisms between members of the family, we can compute the invariant function $\psi_{\mathfrak{g}}(\alpha)$. The members of the family whose invariant functions differ are necessarily nonisomorphic. When the invariant functions of two algebras are the same, the algebras may be isomorphic. Other criteria or an explicit search for a basis transformation must be employed in order to complete the analysis in this case.

(α, β, γ)-derivations and identification of 3-dimensional algebras

To illustrate the use of (α, β, γ)-derivations let us consider the problem of identifying 3-dimensional solvable Lie algebras up to isomorphisms. The fact that 3-dimensional complex Lie algebras are completely characterized by the function $\psi_{\mathfrak{g}}(\alpha)$ was first observed J. Hrivnák and P. Novotný.

Let us first consider the 3-dimensional Lie algebras parameterized by two parameters a, b with nonvanishing Lie brackets in the following form

$$\mathfrak{g}_{a,b}: \quad [e_1, e_3] = ae_1, \quad [e_2, e_3] = be_2 \tag{5}$$

where $(a, b) \neq (0, 0)$. We will show how $(\alpha, 1, 1)$-derivations allow us to split the algebras $\mathfrak{g}_{a,b}$ into mutually nonisomorphic subclasses.

(α, β, γ)-derivations and identification of 3-dimensional algebras, cont'd

Let us write an $(\alpha, 1, 1)$-derivation A as

$$A(e_j) = \sum_{k=1}^{3} A_j^k e_k. \tag{6}$$

Equation (3) with $\beta = \gamma = 1$ implies that

$$\begin{aligned} 0 &= -bA_1^3 e_2 + aA_2^3 e_1, \\ \alpha a \sum_k A_1^k e_k &= a(A_1^1 + A_3^3)e_1 + bA_1^2 e_2, \\ \alpha b \sum_k A_2^k e_k &= aA_2^1 e_1 + b(A_2^2 + A_3^3)e_2. \end{aligned} \tag{7}$$

(α, β, γ)-derivations and identification of 3-dimensional algebras, cont'd

Thus the vector of constants
$\vec{A} = (A_1^1, A_1^2, A_1^3, A_2^1, A_2^2, A_2^3, A_3^1, A_3^2, A_3^3)$ defines an $(\alpha, 1, 1)$-derivation of algebra (5) if and only if it is annihilated by the following matrix

$$M = \begin{pmatrix} 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 \\ a(\alpha-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a \\ 0 & a\alpha - b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b\alpha - a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(\alpha-1) & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 & \alpha b & 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

(α, β, γ)-derivations and identification of 3-dimensional algebras, cont'd

In order to evaluate the function $\psi_{\mathfrak{g}_{a,b}}(\alpha)$ we have to compute the rank of the matrix M as a function of the parameters α, a, b,

$$\psi_{\mathfrak{g}_{a,b}}(\alpha) = 9 - rankM. \tag{9}$$

We find that

$$\begin{aligned}
\psi_{\mathfrak{g}_{a,b}}(\alpha) &= 3, \quad (\alpha a - b)(\alpha b - a) \neq 0, \quad ab \neq 0, \quad \alpha \neq 0, 1, \\
&= 4, \quad (\alpha a - b)(\alpha b - a) = 0, \quad ab \neq 0, \quad a \neq -b, \quad \alpha \neq 0, \\
&\quad \text{or} \quad ab = 0, \quad \alpha \neq 0, 1, \\
&= 5, \quad a = -b, \quad a \neq 0, \quad \alpha = -1, \\
\psi_{\mathfrak{g}_{a,b}}(0) &= 3, \quad ab \neq 0, \\
&= 6, \quad ab = 0, \\
\psi_{\mathfrak{g}_{a,b}}(1) &= 4, \quad a \neq b, \\
&= 6, \quad a = b.
\end{aligned} \tag{10}$$

(α, β, γ)-derivations and identification of 3-dimensional algebras, cont'd

We divide the algebras $\mathfrak{g}_{a,b}$ of the form (5) into classes C_p with different values of $\psi_{\mathfrak{g}_{a,b}}(\alpha)$

$$C_p = \{\mathfrak{g}_{a,pa}, \mathfrak{g}_{pa,a} \mid a \in \mathbb{F}, a \neq 0\}, \quad p \in \mathbb{F}. \tag{11}$$

and conclude that

$$C_p = C_{1/p} \quad (p \neq 0), \qquad C_p \cap C_q = 0 \quad \text{whenever } p \neq q,\ p \neq 1/q, \tag{12}$$

i.e., up to the identification $p \simeq 1/p$ algebras belonging to different classes (11) are not isomorphic. By an explicit construction of transformations one may demonstrate that each of the different classes (11) contains precisely one Lie algebra up to isomorphism. The algebras in the class C_0 are decomposable.

(α, β, γ)-derivations and identification of 3-dimensional algebras, cont'd

Next, one similarly considers the remaining complex 3-dimensional solvable and nilpotent non-Abelian algebras which can be cast in the form

$$\mathfrak{g}_a : \quad [e_1, e_3] = ae_1, \quad [e_2, e_3] = e_1 + ae_2 \tag{13}$$

by a change of basis. We find that

$$\begin{aligned} \psi_{\mathfrak{g}_a}(\alpha) &= 3, \qquad a \neq 0, \quad \alpha \neq 1, \\ &= 6, \qquad a = 0, \\ \psi_{\mathfrak{g}_a}(1) &= 4, \qquad a \neq 0, \end{aligned} \tag{14}$$

i.e., we have two distinct classes $\widetilde{C} = \{\mathfrak{g}_0\}$ and $\widehat{C} = \{\mathfrak{g}_a \mid a \neq 0\}$. An explicit computation shows that all algebras in class $\widehat{C}$ are mutually isomorphic, i.e., the class $\widehat{C}$ can be represented by just one algebra, e.g., $\mathfrak{g}_1$.

(α, β, γ)-derivations and identification of 3-dimensional algebras, cont'd

Altogether, all nonisomorphic 3-dimensional solvable and nilpotent algebras over the field of complex numbers can be identified by their values of the invariant function $\psi_{\mathfrak{g}}(\alpha)$ as follows:

$$
\begin{array}{rll}
\mathfrak{n}_{3,1}: [e_2, e_3] = e_1 & \psi_{\mathfrak{n}_{3,1}}(\alpha) = 6, & \\
\mathfrak{s}_{3,1}(a): [e_1, e_3] = e_1 & \psi_{\mathfrak{s}_{3,1}(a)}(\alpha) = 3, & a \neq 0, -1,\ \alpha \neq 1, a, 1/a, \\
{[e_2, e_3] = a e_2} & = 4, & a \neq 0, \pm 1,\ \alpha = 1, a, 1/a, \\
 & = 6, & a = 1, \quad \alpha = 1, \\
 & = 3, & a = -1, \quad \alpha \neq \pm 1, \\
 & = 4, & a = -1, \quad \alpha = 1, \\
 & = 5, & a = -1, \quad \alpha = -1, \\
\mathfrak{s}_{3,2}: [e_1, e_3] = e_1 & \psi_{\mathfrak{s}_{3,2}}(\alpha) = 3, & \alpha \neq 1, \\
{[e_2, e_3] = e_1 + e_2} & = 4, & \alpha = 1.
\end{array}
$$

Summary

- We have introduced the Lie algebra identification problem,
- we have reviewed some of the invariants that can be computed easily and provide us the first rough splitting,
- we have addressed the problem of identification of real forms,
- and presented one example of a computationally more complex invariant that allows us to identify isomorphic algebras inside parametric classes together with its application to 3-dimensional Lie algebras.

Classification and Identification of Lie Algebras

Lecture 6: Direct decomposition

August 3, 2015

Decomposable Lie algebras

We are given a Lie algebra $\mathfrak{g}$ of dimension n with basis $(x_1, \ldots, x_n)$ and Lie brackets

$$[x_i, x_k] = \sum_{l=1}^{n} c_{ik}{}^{l} x_l, \quad 1 \leq i,\ k \leq n. \tag{1}$$

The first step in identifying a Lie algebra $\mathfrak{g}$ is to decide whether $\mathfrak{g}$ is decomposable, i.e., whether, by an appropriate change of basis, we can decompose this algebra into a direct sum of two or more nonzero indecomposable Lie algebras, i.e. ideals,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \ldots \mathfrak{g}_m. \tag{2}$$

If $\mathfrak{g}$ is decomposable then we wish to find explicitly a basis transformation that will realize the decomposition. Below we shall give a simple criterion of decomposability and an algorithm for achieving a decomposition.

The basic idea

Our algorithm is based on efficient computation of idempotents in the centralizer of the adjoint representation of the given algebra $\mathfrak{g}$.

The commuting algebra (or the centralizer of S in R) $C_R(S)$ of a set S of square $f \times f$ matrices over the field $\mathbb{F}$ is the set of all elements of the full ring $R = \mathbb{F}^{f\times f}$ of square matrices of degree f, commuting elementwise with all elements of S:

$$C_R(S) = \{x \in R \mid [x, S] = 0\}. \tag{3}$$

An idempotent E is a nonzero element of the ring R satisfying

$$E^2 = E. \tag{4}$$

The unit element $E = \mathbf{1}$ is called a trivial idempotent. This is the only idempotent of rank f and hence the only nonsingular idempotent. Any idempotent is conjugated to a diagonal matrix with its diagonal matrix elements equal to 1 or 0.
Two idempotents E_1 and E_2 are orthogonal if $E_1E_2 = E_2E_1 = 0$.

Absolute indecomposability

When the field $\mathbb{F}$ is the field of real numbers we distinguish between absolutely indecomposable Lie algebras and indecomposable, but not absolutely indecomposable ones. A real Lie algebra $\mathfrak{g}$ is defined to be absolutely indecomposable if and only if it is indecomposable and stays indecomposable after complexification. For complex Lie algebras the notions of indecomposability and absolute indecomposability are equivalent. A convenient criterion of absolute indecomposability is given by the following theorem.

Theorem 1

The finite-dimensional Lie algebra $\mathfrak{g}$ is absolutely indecomposable if and only if the traceless subalgebra A_0 of the centralizer algebra $A = C_R(\mathrm{ad}\,\mathfrak{g})$ of the adjoint representation of $\mathfrak{g}$ is closed under multiplication.

Jacobson radical

In order to actually find the decomposition we need the following notion:

The Jacobson radical $J(S)$ of an associative algebra $S \subset R$ is the maximal nilpotent ideal of S. Since the field $\mathbb{F}$ is of characteristic zero, we have

$$J(S) = \{x \in S \mid Tr(xy) = 0, \ \forall y \in S\} \tag{5}$$

and $J(S)$ consists entirely of nilpotent matrices. Here $Tr(Z)$ denotes the trace of the matrix Z.

We recall that division ring is a ring with unity in which every nonzero element has a multiplicative inverse.

Algorithm for establishing whether a Lie algebra is decomposable or not and how

Step 1

Remove the maximal central component of $\mathfrak{g}$*, if one exists. This is done using Theorem 2 below. From now on assume* $C(\mathfrak{g}) \subseteq D(\mathfrak{g})$.

Theorem 2

A central decomposition of $\mathfrak{g}$*, if it exists, can be obtained by choosing a complement* $\mathfrak{g}_1$ *of the intersection* $C(\mathfrak{g}) \cap D(\mathfrak{g})$ *in* $C(\mathfrak{g})$*, thus decomposing*

$$C(\mathfrak{g}) = \mathfrak{g}_1 \oplus C(\mathfrak{g}) \cap D(\mathfrak{g}). \tag{6}$$

The algebra $\mathfrak{g}_2$ *is then chosen so that it satisfies*

$$D(\mathfrak{g}) \subseteq \mathfrak{g}_2, \quad \mathfrak{g}_2 \cap C(\mathfrak{g}) \subseteq D(\mathfrak{g}). \tag{7}$$

Algorithm, cont'd

Step 2

Determine the $n \times n$ matrices of the adjoint representation of $\mathfrak{g}$ and find the centralizer $A = C_R(\mathrm{ad}\,\mathfrak{g})$ of the adjoint representation in $R = \mathbb{F}^{n \times n}$. Choose a basis for A in the form $\{a_1 = \mathbf{1}_n, a_2, \ldots, a_s\}$ with $\mathrm{Tr}\, a_i = 0$, $2 \leq i \leq s$. Denote by A_0 the traceless subset of A: $A_0 = \{a_2, \ldots, a_s\}$.

Algorithm, cont'd

Step 3

Determine whether $\mathfrak{g}$ is absolutely indecomposable by calculating the traces of the products $a_i a_k$, $2 \leq i, k \leq s$ (cf. Theorem 1). The algebra $\mathfrak{g}$ is absolutely indecomposable if and only if

$$Tr a_i a_k = 0, \quad 2 \leq i, k \leq s. \tag{8}$$

If $\mathbb{F} = \mathbb{C}$ and (8) holds, then $\mathfrak{g}$ is indecomposable. If (8) does not hold, or if $\mathbb{F} = \mathbb{R}$, proceed further.

Algorithm, cont'd

Step 4

Determine the Jacobson radical $J(A)$ using the definition (5) *(for $S = A$). Choose a basis $x_1, \dots, x_\nu$ for $J(A)$ and its complement $b_1 = \mathbf{1}_n, \dots, b_\mu \in A = C_R(ad\mathfrak{g})$ such that*

$$b_1 = \mathbf{1}_n, \quad Trb_i = 0, \quad 2 \leq i \leq \mu. \tag{9}$$

If $\mathbb{F} = \mathbb{R}$ and $\mu = 2$, then verify whether the relation

$$b_2^2 = k\mathbf{1}_n \bmod J(A), \quad k < 0, \tag{10}$$

holds. If (10) *does hold, then $\mathfrak{g}$ is indecomposable but not absolutely indecomposable. In all other cases the algebra $\mathfrak{g}$ is decomposable and we proceed to decompose it.*

Algorithm, cont'd

Step 5

Run through the basis elements $b_2, \ldots, b_\mu$ until one is found with a reducible minimal polynomial. Call this element b_r. We now have a nonnilpotent traceless matrix b_r in $C_R(A)$. Using the invariant factors of b_r, or the rational roots theorem, and if necessary a more powerful factorization procedure, factor the minimal polynomial m_b into two mutually prime monic nonconstant polynomials

$$m_{b_i} = f_1 f_2, \quad f_j = \gcd(m_{b_i}, \bar{f}_j^{\nu}), \quad j = 1, 2. \tag{11}$$

We define the polynomials P_1, P_2 via $P_1 f_1 + P_2 f_2 = 1$. The matrix

$$M = P_1(b_r) f_1(b_r) \tag{12}$$

is a nontrivial idempotent in $A/J(A)$, i.e., $M^2 = M \bmod J(A)$.

Algorithm, cont'd

Step 6

Perform a change of basis that diagonalizes M and realizes the decomposition of $\mathfrak{g}$. This is done using a matrix G obtained by placing the row space of M in its first r rows and the row space of $M - \mathbf{1}$ in the last $n - r$ rows. Thus columns $1, \ldots, r$ and $r + 1, \ldots, n$ of G^{-1} are the bases of the eigenspaces of M corresponding to the eigenvalues 1 and 0, respectively. The new basis of $\mathfrak{g}$ is then given as

$$e'_i = G_i^j e_j. \tag{13}$$

Algorithm, cont'd

Step 7

We have decomposed $\mathfrak{g}$ *into the direct sum of two algebras,* $\mathfrak{g}_1$ *and* $\mathfrak{g}_2$*. Repeat the algorithm, starting at Step 2, for each component and continue until we arrive at a decomposition into indecomposable components.*

Example: central decomposition

Let us consider the algebra

$$\mathfrak{g} = span\{e_1, e_2, e_3, e_4, e_5\}$$

with the Lie brackets

$$[e_2, e_3] = e_1 + e_4, \qquad [e_2, e_5] = e_3, \qquad [e_3, e_5] = -e_2. \tag{14}$$

The center is $C(\mathfrak{g}) = span\{e_1, e_4\}$, the derived algebra is $D(\mathfrak{g}) = span\{e_1 + e_4, e_2, e_3\}$, and

$$C(\mathfrak{g}) \cap D(\mathfrak{g}) = span\{e_1 + e_4\}. \tag{15}$$

Thus, any $\mathfrak{g}_1 = span\{e_1 + \kappa e_4\}$, $\kappa \neq 1$ together with the algebra $\mathfrak{g}_2$ in the form

$$\mathfrak{g}_2 = span\{e_1 + e_4, e_2, e_3, e_5 + \alpha e_1\}, \quad \text{where } \alpha \in \mathbb{F} \text{ arbitrary} \tag{16}$$

performs the direct decomposition of $\mathfrak{g}$ into the direct sum

$$\mathfrak{g} = span\{e_1 + \kappa e_4\} \oplus span\{e_1 + e_4, e_2, e_3, e_5 + \alpha e_1\}. \tag{17}$$

Example: $\mathfrak{so}(4)$ vs. $\mathfrak{so}(1,3)$

Next, let us consider the algebras $\mathfrak{so}(4)$ and $\mathfrak{so}(1,3)$. It is well known that $\mathfrak{so}(4)$ is decomposable and $\mathfrak{so}(1,3)$ is indecomposable but not absolutely indecomposable. Let us derive these conclusions using the algorithm described above and also find an explicit decomposition.

The algebras $\mathfrak{so}(4)$ and $\mathfrak{so}(1,3)$ have the Lie brackets

$$\begin{aligned} &[x_1,x_2]=x_3, && [x_1,x_3]=-x_2, && [x_2,x_3]=x_1, && [x_1,x_5]=x_6,\\ &[x_1,x_6]=-x_5, && [x_2,x_4]=-x_6, && [x_2,x_6]=x_4, && [x_3,x_4]=x_5,\\ &[x_3,x_5]=-x_4, && [x_4,x_5]=\epsilon x_3, && [x_4,x_6]=\epsilon x_2, && [x_5,x_6]=\epsilon x_1 \end{aligned} \tag{18}$$

where $\epsilon=1$ for $\mathfrak{so}(4)$ and $\epsilon=-1$ for $\mathfrak{so}(1,3)$.

Example: $\mathfrak{so}(4)$ vs. $\mathfrak{so}(1,3)$, cont'd

Step 1

The center of $\mathfrak{so}(4)$ and $\mathfrak{so}(1,3)$ vanishes, i.e., there is no central component.

Step 2

The centralizer $C_R(ad\mathfrak{g})$ of the adjoint representation is

$$C_R(ad\mathfrak{g}) = \left\{ \begin{pmatrix} u\mathbf{1}_3 & \epsilon v\mathbf{1}_3 \\ v\mathbf{1}_3 & u\mathbf{1}_3 \end{pmatrix} \,\middle|\, u, v \in \mathbb{F} \right\}. \tag{19}$$

We set

$$a_1 = \mathbf{1}_6, \quad a_2 = \begin{pmatrix} 0 & \epsilon\mathbf{1}_3 \\ \mathbf{1}_3 & 0 \end{pmatrix}. \tag{20}$$

We have $A_0 = span\{a_2\}$.

Example: $\mathfrak{so}(4)$ vs. $\mathfrak{so}(1,3)$, cont'd

Step 3

We have $Tra_2^2 = 6\epsilon$. Therefore neither $\mathfrak{so}(4)$ nor $\mathfrak{so}(1,3)$ is absolutely indecomposable.

Step 4

The Jacobson radical $J\big(C_R(\mathrm{ad}\,\mathfrak{g})\big)$ vanishes and we can set $b_1 = a_1$, $b_2 = a_2$. We have

$$b_2^2 = \epsilon b_1. \tag{21}$$

Consequently, $\mathfrak{so}(4)$ is decomposable both as a real and as a complex Lie algebra, whereas the real algebra $\mathfrak{so}(1,3)$ is indecomposable but not absolutely indecomposable.

We proceed to decompose $\mathfrak{so}(4)$ and the complexification $\mathfrak{so}_{\mathbb{C}}(1,3)$ of $\mathfrak{so}(1,3)$.

Example: $\mathfrak{so}(4)$ vs. $\mathfrak{so}(1,3)$, cont'd

Step 5

The matrix b_2 has the reducible minimal polynomial

$$m_{b_2}(t) = t^2 - \epsilon = \begin{cases} (t-1)(t+1) & \textit{when } \epsilon = 1, \\ (t-\mathrm{i})(t+\mathrm{i}) & \textit{when } \epsilon = -1. \end{cases} \tag{22}$$

We consider the two cases separately:

When $\epsilon = 1$ we set $f_1(t) = t - 1$ and $f_2(t) = t + 1$. We find a particular solution of the equations determining the polynomials in the form $P_1(t) = \frac{1}{2}t$ and $P_2(t) = 1 - \frac{1}{2}t$. Thus we have

$$M_{\epsilon=1} = P_1(b_2)f_1(b_2) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

Example: $\mathfrak{so}(4)$ vs. $\mathfrak{so}(1,3)$, cont'd

When $\epsilon = -1$ we set $f_1(t) = t - \mathrm{i}$ and $f_2(t) = t + \mathrm{i}$. We find that $P_1(t) = \frac{1}{2}\mathrm{i}$ and $P_2(t) = -\frac{1}{2}\mathrm{i}$. Thus we have

$$M_{\epsilon=-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -\mathrm{i} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 1 & 0 & 0 & -\mathrm{i} \\ \mathrm{i} & 0 & 0 & 1 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{i} & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

Step 6

We find changes of bases diagonalizing the matrices (23), (24).

$$\begin{aligned} \epsilon = 1: \quad & x_1' = \tfrac{1}{2}(x_6 - x_3), \quad x_2' = \tfrac{1}{2}(x_5 - x_2), \quad x_3' = \tfrac{1}{2}(x_4 - x_1), \\ & x_4' = \tfrac{1}{2}(x_2 + x_5), \quad x_5' = \tfrac{1}{2}(x_1 + x_4), \quad x_6' = \tfrac{1}{2}(x_3 + x_6), \\ \epsilon = -1: \quad & x_1' = \tfrac{1}{2}(x_3 + \mathrm{i}x_6), \quad x_2' = \tfrac{1}{2}(x_2 + \mathrm{i}x_5), \quad x_3' = \tfrac{1}{2}(x_1 + \mathrm{i}x_4), \\ & x_4' = \tfrac{1}{2}(\mathrm{i}x_6 - x_3), \quad x_5' = \tfrac{1}{2}(\mathrm{i}x_5 - x_2), \quad x_6' = \tfrac{1}{2}(\mathrm{i}x_4 - x_1). \end{aligned} \tag{25}$$

Example: $\mathfrak{so}(4)$ vs. $\mathfrak{so}(1,3)$, cont'd

In these new bases the algebras $\mathfrak{so}(4)$ and $\mathfrak{so}_{\mathbb{C}}(1,3)$, respectively, split explicitly into direct sums of simple subalgebras spanned by x_1', x_2', x_3' and x_4', x_5', x_6'. We have

$$\begin{aligned} &[x_1', x_2'] = x_3', && [x_1', x_3'] = -x_2', && [x_2', x_3'] = x_1', \\ &[x_4', x_5'] = -x_6', && [x_4', x_6'] = x_5', && [x_5', x_6'] = -x_4' \end{aligned} \tag{26}$$

when $\epsilon = 1$, and

$$\begin{aligned} &[x_1', x_2'] = -x_3', && [x_1', x_3'] = x_2', && [x_2', x_3'] = -x_1', \\ &[x_4', x_5'] = x_6', && [x_4', x_6'] = -x_5', && [x_5', x_6'] = x_4' \end{aligned} \tag{27}$$

for $\epsilon = -1$.

Example: $\mathfrak{so}(4)$ vs. $\mathfrak{so}(1,3)$, cont'd

These new bases are not unique, any bases of the row spaces of the matrices M and $M - \mathbf{1}$ can be used. Thus, the Lie brackets can be obtained in a different form when the procedure is repeated.

Step 7

The subalgebras arising in the decompositions (26) *and* (27) *are simple, i.e., further indecomposable.*

Thus, we have decomposed the algebra $\mathfrak{so}(4)$ into a direct sum of two subalgebras, both of which turn out to be isomorphic to $\mathfrak{so}(3)$ upon further inspection

$$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3). \tag{28}$$

Similarly, the algebra $\mathfrak{so}(1,3)$ is indecomposable but not absolutely indecomposable. Its complexification $\mathfrak{so}_{\mathbb{C}}(1,3)$ decomposes into

$$\mathfrak{so}_{\mathbb{C}}(1,3) = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}). \tag{29}$$

Summary

- We have introduced the notion of (absolutely) decomposable Lie algebra,
- we presented a simple criterion for absolute indecomposability,
- we have explained the algorithm for determination whether the algebra is decomposablebut not absolutely indecomposable,
- in the case the algebra is decomposable we have shown how to decompose it explicitly,
- we have demonstrated the procedure on two examples.

Classification and Identification of Lie Algebras

Lecture 7: Levi decomposition

August 3, 2015

Levi theorem

In any Lie algebra $\mathfrak{g}$ there exists a unique maximal solvable ideal $R(\mathfrak{g})$ called the radical as was already mentioned. The radical satisfies $R(\mathfrak{g}) = 0$ if and only if $\mathfrak{g}$ is semisimple. On the other hand the radical $R(\mathfrak{g})$ coincides with $\mathfrak{g}$ if and only if $\mathfrak{g}$ is solvable.

Theorem (Levi theorem)

Any finite-dimensional Lie algebra $\mathfrak{g}$ can be decomposed into a semidirect sum

$$\mathfrak{g} = \mathfrak{p} \niplus R(\mathfrak{g}) \tag{1}$$

where the complement $\mathfrak{p}$ of the radical $R(\mathfrak{g})$ in $\mathfrak{g}$ is a semisimple Lie algebra, isomorphic to the factor algebra $\mathfrak{g}/R(\mathfrak{g})$. The semisimple Lie algebra $\mathfrak{p}$ is called the Levi factor or Levi subalgebra of $\mathfrak{g}$.

Examples of Levi decompositions, Mal'cev theorem

Let us review some important examples of Levi decompositions:

trivial: where the sum is in fact direct
$\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathfrak{u}(1)$, $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$

nontrivial:
- Poincaré algebra $\mathfrak{so}(1,3) \niplus \mathfrak{a}(4)$
- Symmetry algebra of the heat equation $\mathfrak{sl}(2) \niplus \mathfrak{h}(1)$ (see later)

A sequel to the Levi theorem is the result of Mal'cev

Theorem (Mal'cev theorem)

Any two Levi factors $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of the Lie algebra $\mathfrak{g}$ are isomorphically mapped one into the other by some inner automorphism Φ of the form

$$\Phi = \exp(adz) \tag{2}$$

where $z \in NR(\mathfrak{g})$.

Levi theorem expressed in a basis

Levi theorem in other words says that given a basis of $\mathfrak{g}$, say $(x_1, \ldots, x_n)$, it is always possible to find a new basis

$$\{s_1, s_2, \ldots, s_\sigma, r_1, r_2, \ldots, r_\rho\}, \quad \sigma + \rho = n, \tag{3}$$

such that $\mathfrak{r} = span\{r_1, \ldots, r_\rho\}$, $\mathfrak{p} = span\{s_1, \ldots, s_\sigma\}$, and the commutation relations are such that

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}, \quad [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r}, \quad [\mathfrak{p}, \mathfrak{r}] \subseteq \mathfrak{r}. \tag{4}$$

The question that we address here is: how does one find a convenient change of basis that realizes the Levi decomposition? Notice that a Levi decomposition can be performed for both decomposable and indecomposable Lie algebras. From the point of view of identifying a Lie algebra $\mathfrak{g}$, it is usually preferable to first perform a direct decomposition into indecomposable components and then construct a Levi decomposition for each component.

Since the factor algebra $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{p}$ is semisimple and hence perfect, i.e., its derived algebra $D(\mathfrak{p})$ satisfies $D(\mathfrak{p}) = \mathfrak{p}$, we have

$$D(\mathfrak{g}) + \mathfrak{r} = \mathfrak{g}. \tag{5}$$

From (5) we obtain the isomorphism

$$D(\mathfrak{g})/[D(\mathfrak{g}) \cap \mathfrak{r}] \cong \mathfrak{g}/\mathfrak{r} \tag{6}$$

and hence $\dim(D(\mathfrak{g})/[D(\mathfrak{g}) \cap \mathfrak{r}]) = n - \rho = \sigma$.

The problem of obtaining a Levi decomposition is one of linear algebra.

Algorithm

First we identify the radical and simplify our computation by construction of a perfect subalgebra via repeated commutation.

Next, the essential part of the algorithm is applied. Its basic idea is the construction of a proper subalgebra of the Lie algebra $\mathfrak{g}$ which contains $\mathfrak{p}$. Repeating the procedure finitely many times one obtains a subalgebra of $\mathfrak{g}$ which has vanishing radical, i.e., coincides with the sought Levi factor $\mathfrak{p}$.

Step 1

Find the radical $\mathfrak{r} = R(\mathfrak{g})$. This is a simple task of linear algebra, since we can use the property

$$R(\mathfrak{g}) = \{x \in \mathfrak{g} \mid K(x, y) = 0, \ \forall y \in D(\mathfrak{g})\}, \tag{7}$$

where $K(x, y)$ is the Killing form

$$K(x, y) = Tr\big(ad(x)ad(y)\big).$$

If $\mathfrak{g} = R(\mathfrak{g})$, then $\mathfrak{g}$ is solvable and $\mathfrak{p} = 0$ in (1). If $R(\mathfrak{g}) = 0$, then $\mathfrak{g} = \mathfrak{p}$ is semisimple. In both cases the Levi decomposition is trivial.

Step 2

If $0 \neq \mathfrak{p} \neq \mathfrak{g}$, we calculate the derived series of $\mathfrak{g}$:

$$\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \ldots, \mathfrak{g}^{(k+1)} = \left[\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}\right], \ldots$$

till we arrive, after a finite number of steps, at a perfect Lie algebra

$$\mathfrak{g}^{(k+1)} = \mathfrak{g}^{(k)}, \quad \mathfrak{g}^{(k)} \neq \mathfrak{g}^{(k-1)}. \tag{8}$$

If we know the Levi decomposition of $\mathfrak{g}^{(k)}$, i.e.,

$$\mathfrak{g}^{(k)} = \mathfrak{p} \ni R\left(\mathfrak{g}^{(k)}\right) \tag{9}$$

then we obtain the Levi decomposition of $\mathfrak{g}$ by extending the basis $(r_1, \ldots, r_\tau)$ of $R\left(\mathfrak{g}^{(k)}\right)$ to a basis of $R(\mathfrak{g})$: $(r_1, \ldots, r_\tau, r_{\tau+1}, \ldots, r_\rho)$.

Step 3

We observe that $D(\mathfrak{r}) \equiv \mathfrak{r}^2$ is a characteristic ideal of $\mathfrak{g}$ and consequently the subalgebra $\tilde{\mathfrak{g}} = \mathfrak{p} \dot{+} D(\mathfrak{r})$ of $\mathfrak{g}$ is a Levi decomposable algebra with the Levi factor $\mathfrak{p}$ and radical $\mathfrak{r}^2$. We shall proceed to construct its basis.
Let us choose a basis for $\mathfrak{g}$ in the form

$$(e_1, \ldots, e_\mu, r_1, \ldots, r_\nu, x_1, \ldots, x_\sigma), \qquad \mu + \nu = \rho \tag{10}$$

where $(e_1, \ldots, e_\mu)$ is a basis for $D\big(R(\mathfrak{g})\big)$, $(r_1, \ldots, r_\nu)$ for a complement of $D\big(R(\mathfrak{g})\big)$ in $R(\mathfrak{g})$, and $(x_1, \ldots, x_\sigma)$ for a complement of $R(\mathfrak{g})$ in $\mathfrak{g}$. Thus we have the following particular Lie brackets

$$[x_i, x_j] = c_{ij}{}^k x_k + d_{ij}{}^p r_p + f_{ij}{}^l e_l, \tag{11}$$
$$[x_i, r_p] = g_{ip}{}^q r_q + h_{ip}{}^m e_m. \tag{12}$$

Summation over repeated indices $k = 1, \ldots, \sigma$, $l, m = 1, \ldots, \mu$ and $p, q = 1, \ldots, \nu$ applies throughout.

Step 3, cont'd

A basis of $\tilde{\mathfrak{g}} = \mathfrak{p} \inplus D(\mathfrak{r})$ can be without loss of generality chosen in the form

$$\{e_1, \ldots, e_\mu, \hat{x}_1, \ldots, \hat{x}_\sigma\} \tag{13}$$

where $\hat{x}_k \in span\{x_k, r_1, \ldots, r_\nu\}$, i.e.,

$$\hat{x}_j = x_j + b_j^p r_p. \tag{14}$$

The span of the set (13) is by construction complementary to $span\{r_1, \ldots, r_\nu\}$. Thus it forms a basis of $\mathfrak{p} \inplus D(\mathfrak{r})$ if and only if it closes under the Lie bracket. That is always true for commutators of the type $[\hat{x}_i, e_l]$ since $D(\mathfrak{r})$ is an ideal in $\mathfrak{g}$. It remains to satisfy

$$[\hat{x}_i, \hat{x}_j] = c_{ij}{}^k \hat{x}_k + \widehat{f_{ij}}^l e_l \tag{15}$$

for some constants $\widehat{f_{ij}}^l$. Notice that the structure constants $c_{jk}{}^k$ are the same in (11) and (15) because they are the structure constants of the semisimple factor algebra $\mathfrak{g}/\mathfrak{r}$ in the same basis $x_j \bmod \mathfrak{r} = \hat{x}_j \bmod \mathfrak{r}$, $j = 1, \ldots, \sigma$.

Step 3, cont'd

Substituting (14) into (15) and dropping any term proportional to e_l we obtain the following set of equations, to be satisfied for all $1 \le i < j \le \sigma$

$$d_{ij}{}^{q} r_q + b_j^p g_{ip}{}^{q} r_q - b_i^p g_{jp}{}^{q} r_q = c_{ij}{}^{k} b_k^q r_q, \tag{16}$$

i.e., a set of $\frac{1}{2}\sigma(\sigma-1)\nu$ linear inhomogeneous equations

$$g_{jp}{}^{q} b_i^p - g_{ip}{}^{q} b_j^p + c_{ij}{}^{k} b_k^q = d_{ij}{}^{q} \tag{17}$$

for $\sigma\nu$ unknowns b_i^p. Due to the Levi theorem the set of equations (17) always has a solution. Thus, once we find any particular solution of (17) we have a basis of $\tilde{\mathfrak{g}} = \mathfrak{p} \niplus D(\mathfrak{r})$. We repeat the procedure until we arrive at $k \in \mathbb{N}$ such that

$$\mathfrak{r}^{(k)} = 0.$$

Example: the symmetry algebra of the heat equation

Let us consider the finite dimensional part of the algebra of infinitesimal point symmetries of the heat equation. It is spanned by the following vector fields in $\mathbb{R}^3$ with coordinates t, x, u

$$\begin{aligned} Y_1 &= 4t^2\partial_t + 4xt\partial_x - (2t + x^2)u\partial_u, & Y_2 &= 4t\partial_t + 2x\partial_x, \\ Y_3 &= \partial_t, & Y_4 &= -2t\partial_x + xu\partial_u, \\ Y_5 &= u\partial_u, & Y_6 &= \partial_x. \end{aligned} \tag{18}$$

Evaluation of the commutators gives the following Lie brackets of an abstract Lie algebra (with y_i replacing the vector fields Y_i)

$$\begin{aligned} [y_1, y_2] &= -4y_1, & [y_1, y_3] &= -2y_2 + 2y_5, & [y_1, y_6] &= 2y_4, \\ [y_2, y_3] &= -4y_3, & [y_2, y_4] &= 2y_4, & [y_2, y_6] &= -2y_6, \\ [y_3, y_4] &= -2y_6, & [y_4, y_6] &= -y_5. \end{aligned} \tag{19}$$

Example: the symmetry algebra of the heat equation, cont'd

In Steps 1 and 2 we find that the Lie algebra (19) is perfect and its radical is spanned by y_4, y_5, y_6. The radical (equal to the nilradical) is the Heisenberg algebra $\mathfrak{h}(1)$. The derived algebra of the radical is spanned by y_5.

The basis of the form (10) can be chosen as

$$e_1 = y_5, \quad r_1 = y_4, \quad r_2 = y_6, \quad x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3 \tag{20}$$

with the Lie brackets

$$\begin{aligned} &[r_1, r_2] = -e_1, \quad [r_1, x_2] = -2r_1, \quad [r_1, x_3] = 2r_2, \qquad [r_2, x_1] = -2r_1, \\ &[r_2, x_2] = 2r_2, \quad [x_1, x_2] = -4x_1, \quad [x_1, x_3] = 2e_1 - 2x_2, \quad [x_2, x_3] = -4x_3. \end{aligned} \tag{21}$$

Example: the symmetry algebra of the heat equation, cont'd

By an inspection of (21) we see that e_1, x_1, x_2, x_3 span a subalgebra of $\mathfrak{g}$. Thus, we have $\mathfrak{p} \supsetplus D(R(\mathfrak{g})) = span\{e_1, x_1, x_2, x_3\}$.
Next, we construct the Levi decomposition of $\mathfrak{p} \supsetplus D(R(\mathfrak{g})) = span\{e_1, x_1, x_2, x_3\}$ with the nonvanishing Lie brackets

$$[x_1, x_2] = -4x_1, \quad [x_1, x_3] = 2e_1 - 2x_2, \quad [x_2, x_3] = -4x_3. \tag{22}$$

The solution of the set of linear equations (17) is

$$b_1^1 = 0, \quad b_2^1 = -1, \quad b_3^1 = 0$$

and leads to the desired basis for the Levi factor in the form

$$x_1 = y_1, \quad x_2 - e_1 = y_2 - y_5, \quad x_3 = y_3. \tag{23}$$

Example: the symmetry algebra of the heat equation, cont'd

Notice that in this case the Levi factor of the subalgebra $span\{e_1, x_1, x_2, x_3\}$ coincides with its derived algebra and is therefore unique. However, as a Levi factor of the whole algebra $\mathfrak{g}$ it is not unique since a choice of the subalgebra $\mathfrak{p} \niplus D\big(R(\mathfrak{g})\big)$ different from $span\{e_1, x_1, x_2, x_3\}$ was possible. As stated in Theorem 2, the Levi factor $\mathfrak{p}$ is determined only up to inner automorphisms of $\mathfrak{g}$.
We conclude that the Levi decomposition of the "heat algebra" $\mathfrak{g}$, Eq. (18) is

$$\mathfrak{g} \simeq \mathfrak{sl}(2, \mathbb{R}) \niplus \mathfrak{h}(1) \tag{24}$$

where the Levi factor $\mathfrak{sl}(2, \mathbb{R})$ has the basis

$$\begin{gathered} Y_1 = 4t^2\partial_t + 4xt\partial_x - (2t + x^2)u\partial_u, \\ Y_2 - Y_5 = 4t\partial_t + 2x\partial_x - u\partial_u, \quad Y_3 = \partial_t \end{gathered} \tag{25}$$

and the radical is the Heisenberg algebra spanned by Y_4, Y_5, Y_6.

8-dimensional example

Let us consider an 8-dimensional Lie algebra $\mathfrak{g}$ with the Lie brackets

$$\begin{aligned}
&[y_1, y_2] = 2y_2, && [y_1, y_3] = -2y_3 + 2y_6, && [y_1, y_4] = 2y_4,\\
&[y_1, y_5] = 2y_5, && [y_1, y_7] = 2y_7, && [y_2, y_3] = y_1 + y_5 + y_8,\\
&[y_2, y_6] = y_5, && [y_2, y_8] = 2y_4, && [y_3, y_5] = -y_4 + y_6,\\
&[y_3, y_8] = y_6, && [y_4, y_8] = 2y_4, && [y_5, y_6] = y_4,\\
&[y_5, y_8] = y_5, && [y_6, y_8] = y_6, && [y_7, y_8] = 2y_7.
\end{aligned} \tag{26}$$

In Step 1 we find that its radical is spanned by y_4, y_5, y_6, y_7, y_8 and the nilradical by y_4, y_5, y_6, y_7.

8-dimensional example, cont'd

We proceed to Step 2. The derived series is

$$\begin{aligned} \mathfrak{g}^{(1)} &= D(\mathfrak{g}) = span\{y_1 + y_8, y_2, \ldots, y_7\}, \\ \mathfrak{g}^{(2)} &= \mathfrak{g}^{(3)} = span\{y_1 + y_8, y_2, y_3, y_4, y_5, y_6\}. \end{aligned} \tag{27}$$

Thus, the Levi factor of $\mathfrak{g}$ is found once the Levi factor of $\mathfrak{g}^{(2)}$ is constructed using the algorithm. The nilpotent non-Abelian radical of $\mathfrak{g}^{(2)}$ is spanned by y_4, y_5, y_6 with a single nonvanishing Lie bracket

$$[y_5, y_6] = y_4.$$

8-dimensional example, cont'd

We choose the basis as in (10)

$$e_1 = y_4, \quad r_1 = y_5, \quad r_2 = y_6, \quad x_1 = y_1 + y_8, \quad x_2 = y_2, \quad x_3 = y_3. \tag{28}$$

The Lie brackets of $\tilde{\mathfrak{g}} = \mathfrak{g}^{(2)}$ in this basis become

$$\begin{aligned}
&[r_1, r_2] = e_1, && [r_1, x_1] = -r_1, && [r_1, x_3] = e_1 - r_2, \\
&[r_2, x_1] = r_2, && [r_2, x_2] = -r_1, && [x_1, x_2] = -2e_1 + 2x_2, \\
&[x_1, x_3] = r_2 - 2x_3, && [x_2, x_3] = r_1 + x_1.
\end{aligned} \tag{29}$$

In order to find $\mathfrak{p} \supset D\big(R(\tilde{\mathfrak{g}})\big)$ we have to perform the change of basis (14). The conditions (17) reduce to the equations

$$b_2^2 = 0, \quad b_3^1 = 0, \quad b_2^1 + b_1^2 = 0, \quad b_3^2 - b_1^1 = 1.$$

8-dimensional example, cont'd

Thus, a particular solution of (17) is

$$b_1^1 = 1, \qquad b_1^2 = 0, \qquad b_2^p = b_3^p = 0, \quad p = 1, 2,$$

which corresponds to the change of basis

$$\hat{x}_1 = x_1 + r_1 = y_1 + y_5 + y_8, \quad \hat{x}_2 = x_2 = y_2, \quad \hat{x}_3 = x_3 = y_3. \quad (30)$$

The vectors $e_1, \hat{x}_1, \hat{x}_2, \hat{x}_3$ form a basis of $\mathfrak{p} \niplus D\big(R(\tilde{\mathfrak{g}})\big)$. In this basis we have the Lie brackets

$$[\hat{x}_1, \hat{x}_2] = -2e_1 + 2\hat{x}_2, \quad [\hat{x}_1, \hat{x}_3] = e_1 - 2\hat{x}_3, \quad [\hat{x}_2, \hat{x}_3] = \hat{x}_1. \quad (31)$$

The radical of $\mathfrak{p} \niplus D\big(R(\tilde{\mathfrak{g}})\big)$ is spanned by e_1 which coincides with the center of $\mathfrak{p} \niplus D\big(R(\tilde{\mathfrak{g}})\big)$. Thus, the Lie algebra (31) is decomposable into a direct sum of a simple algebra $\mathfrak{sl}(2, \mathbb{F})$ and a central component spanned by e_1, and the algorithm on the direct decomposition can be used.

8-dimensional example, cont'd

Alternatively, we use the change of basis (14)

$$\bar{x}_1 = \hat{x}_1 + \hat{b}_1^1 e_1, \quad \bar{x}_2 = \hat{x}_2 + \hat{b}_2^1 e_1, \quad \bar{x}_3 = \hat{x}_3 + \hat{b}_3^1 e_1$$

once again, arriving at the conditions (17) expressed as

$$\hat{b}_1^1 = 0, \quad \hat{b}_2^1 = -1, \quad 2\hat{b}_3^1 = -1.$$

Thus, we have constructed a basis $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ of the Levi factor $\mathfrak{p}$ of $\mathfrak{g}$ in the form

$$\begin{aligned} \bar{x}_1 &= \hat{x}_1 && = y_1 + y_5 + y_8, \\ \bar{x}_2 &= \hat{x}_2 - e_1 && = y_2 - y_4, \\ \bar{x}_3 &= \hat{x}_3 - \tfrac{1}{2} e_1 && = y_3 - \tfrac{1}{2} y_4. \end{aligned}$$

8-dimensional example, cont'd

To sum up, the Levi factor of the algebra (26) is

$$\mathfrak{p} = span\{y_1 + y_5 + y_8, y_2 - y_4, y_3 - \tfrac{1}{2}y_4\} \tag{32}$$

and is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$.

Let us recall that the Levi factor is generically far from unique. In our example, another choice for it is spanned by

$$\tilde{x}_1 = y_1 + y_8, \quad \tilde{x}_2 = y_2 - y_4, \quad \tilde{x}_3 = y_3 - y_6 \tag{33}$$

and, in fact, the choice (33) is more convenient because the Lie brackets of the algebra (26) are more compact when written in terms of $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$. Different choices of Levi factors arise through different choices of the particular solutions of the systems of linear equations involved.

Summary

- We have recalled the Levi theorem and expressed it explicitly in a basis,
- we have explained an algorithm which allows us to determine an explicit decomposition of the given Lie algebra, using only the solution of linear equations,
- we have presented two examples demonstrating the use of the algorithm.

Classification and Identification of Lie Algebras

Lecture 9-10: Classification of solvable Lie algebras with the given nilradical

August 3, 2015

Classification of solvable Lie algebras

There are two ways of proceeding in the classification of solvable Lie algebras: by dimension, or by structure.

The dimensional approach for real and complex Lie algebras was successful up to dimension 6 (S. Lie, L. Bianchi, G.M. Mubarakzyanov, P. Turkowski). Some partial classifications are known for solvable Lie algebras in dimension 7 and nilpotent algebras up to dimension 8 (M.P. Gong, Gr. Tsagas, A.R. Parry).

It seems to be neither feasible, nor fruitful to proceed by dimension in the classification of Lie algebras $\mathfrak{g}$ beyond $\dim \mathfrak{g} = 6$. It is however possible to proceed by structure, i.e. to classify all solvable Lie algebras with the nilradical of a given type.

Recall: Basic concepts and notation

Three series of ideals – characteristic series of $\mathfrak{g}$:

- derived series $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \ldots \supseteq \mathfrak{g}^{(k)} \supseteq \ldots$ defined

$$\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}],\ \mathfrak{g}^{(0)} = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^{(k)} = 0$, then $\mathfrak{g}$ is solvable.

- lower central series $\mathfrak{g} = \mathfrak{g}^1 \supseteq \ldots \supseteq \mathfrak{g}^k \supseteq \ldots$ defined

$$\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}],\ \mathfrak{g}^1 = \mathfrak{g}.$$

  If $\exists k \in \mathbb{N}$ such that $\mathfrak{g}^k = 0$, then $\mathfrak{g}$ nilpotent. The largest value of K s.t. $\mathfrak{g}^K \neq 0$ is the degree of nilpotency.

- upper central series $\mathfrak{z}_1 \subseteq \ldots \subseteq \mathfrak{z}_k \subseteq \ldots \subseteq \mathfrak{g}$ where $\mathfrak{z}_1$ is the center of $\mathfrak{g}$, $\mathfrak{z}_1 = C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0,\ \forall y \in \mathfrak{g}\}$ and $\mathfrak{z}_k$ are the higher centers defined recursively through

$$\mathfrak{z}_{k+1}/\mathfrak{z}_k = C(\mathfrak{g}/\mathfrak{z}_k).$$

Recall: Basic concepts and notation, continued

Any Lie algebra $\mathfrak{g}$ has a uniquely defined nilradical $\mathrm{NR}(\mathfrak{g})$, i.e. the maximal nilpotent ideal.

A derivation D of a given Lie algebra $\mathfrak{g}$ is a linear map

$$D: \; \mathfrak{g} \to \mathfrak{g}$$

such that for any pair x, y of elements of $\mathfrak{g}$

$$D([x,y]) = [D(x), y] + [x, D(y)]. \tag{1}$$

If an element $z \in \mathfrak{g}$ exists, such that

$$D = \mathrm{ad}_z, \;\; \text{i.e. } D(x) = [z,x], \; \forall x \in G,$$

the derivation is inner, any other one is outer.

An automorphism Φ of $\mathfrak{g}$ is a regular linear map

$$\Phi : \ \mathfrak{g} \to \mathfrak{g}$$

such that for any pair x, y of elements of $\mathfrak{g}$

$$\Phi([x, y]) = [\Phi(x), \Phi(y)]. \tag{2}$$

Ideals in the characteristic series as well as their centralizers are invariant with respect to all derivations and automorphisms, i.e. belong among characteristic ideals.

Construction of all solvable Lie algebras with the given nilradical

We assume that the nilradical $\mathfrak{n}$, $\dim \mathfrak{n} = n$ is known. That is, in some basis $(e_1, \ldots, e_n)$ we know the Lie brackets

$$[e_i, e_j] = N_{ij}{}^k e_k. \tag{3}$$

We wish to extend the nilpotent algebra $\mathfrak{n}$ to all possible indecomposable solvable Lie algebras $\mathfrak{s}$ having $\mathfrak{n}$ as their nilradical. Thus, we add further elements $f_1, \ldots, f_f$ to the basis $(e_1, \ldots, e_n)$ which together will form a basis of $\mathfrak{s}$. It follows from $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n}$ that

$$\begin{aligned} [f_a, e_i] &= (A_a)_i^j e_j, \ 1 \leq a \leq f, \ 1 \leq j \leq n, \\ [f_a, f_b] &= \gamma_{ab}{}^i e_i, \ 1 \leq a, b \leq f. \end{aligned} \tag{4}$$

Since $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{s}$, no nontrivial linear combination of A_a can be a nilpotent matrix, i.e. they are linearly nil–independent.

Consider the adjoint representation of $\mathfrak{s}$ restricted to the nilradical $\mathfrak{n}$. Then $\mathrm{ad}(f_a)|_{\mathfrak{n}}$ is a derivation of $\mathfrak{n}$. In other words, finding all sets of matrices A_a in (4) is equivalent to finding all sets of outer nil–independent derivations of $\mathfrak{n}$

$$D_1 = \mathrm{ad}(f_a)|_{\mathfrak{n}}, \ldots, D_f = \mathrm{ad}(f_f)|_{\mathfrak{n}}, \tag{5}$$

such that $[D_a, D_b]$ are inner derivations.

${\gamma_{ab}}^i$ are then determined up to elements in the center $C(\mathfrak{n})$ of $\mathfrak{n}$ by $[D_a, D_b] = {\gamma_{ab}}^i \mathrm{ad}(e_i)|_{\mathfrak{n}}$, i.e. the knowledge of all sets of such derivations almost amounts to the knowledge of all solvable Lie algebras with the given nilradical $\mathfrak{n}$.

Isomorphic Lie algebras with the given nilradical

If we

1. add any inner derivation to D_a, i.e. we consider outer derivations modulo inner derivations,
$$D'_a = D_a + \sum_{j=1}^{n} r_a^j \operatorname{ad}(e_j)|_{\mathfrak{n}}, \qquad r_a^j \in \mathbb{F}. \tag{6}$$
2. perform a change of basis in $\mathfrak{n}$ such that the Lie brackets (3) are not changed,
$$D'_a = \Phi \circ D_a \circ \Phi^{-1}, \qquad \Phi \in \operatorname{Aut}(\mathfrak{n}) \subseteq GL(n, \mathbb{F}). \tag{7}$$
i.e. we consider only conjugacy classes of sets of outer derivations (modulo inner derivations)
3. change the basis in the space $\operatorname{span}\{D_1, \ldots, D_f\}$,

the resulting Lie algebra is isomorphic to the original one.

Suitable basis of $\mathfrak{n}$ to begin with

Starting with any complement $\mathfrak{m}_1$ of $\mathfrak{n}^2$ in $\mathfrak{n}$ one can construct a sequence of subspaces $\mathfrak{m}_j$ such that

$$\mathfrak{n} = \mathfrak{m}_K \dot{+} \mathfrak{m}_{K-1} \dot{+} \ldots \dot{+} \mathfrak{m}_1 \tag{8}$$

where

$$\mathfrak{n}^j = \mathfrak{m}_j \dot{+} \mathfrak{n}^{j+1},\ \mathfrak{m}_j \subset [\mathfrak{m}_{j-1}, \mathfrak{m}_1]. \tag{9}$$

By construction of these subspaces, any derivation (automorphism) is determined once its action on $\mathfrak{m}_1$ is known. We shall assume that we work in a basis of $\mathfrak{n}$ which respects the decomposition (8).

Because $\mathfrak{n}^j = \mathfrak{m}_K \dot{+} \ldots \dot{+} \mathfrak{m}_j$, any derivation now takes a block triangular form

$$D = \begin{pmatrix} D_{\mathfrak{m}_K\mathfrak{m}_K} & \cdots & D_{\mathfrak{m}_K\mathfrak{m}_2} & D_{\mathfrak{m}_K\mathfrak{m}_1} \\ & \ddots & \vdots & \vdots \\ & & D_{\mathfrak{m}_2\mathfrak{m}_2} & D_{\mathfrak{m}_2\mathfrak{m}_1} \\ & & & D_{\mathfrak{m}_1\mathfrak{m}_1} \end{pmatrix}. \qquad (10)$$

where the elements of $D_{\mathfrak{m}_j\mathfrak{m}_k}, k \leq j = 2, \ldots, K$ are linear functions of elements in the last column blocks $D_{\mathfrak{m}_1\mathfrak{m}_1}, \ldots, D_{\mathfrak{m}_{j-k+1}\mathfrak{m}_1}$.

Now one can easily establish that:

- Any inner derivation has vanishing diagonal blocks.
- A derivation D is nilpotent if and only if $D_{\mathfrak{m}_1\mathfrak{m}_1}$ is nilpotent.
- Derivations $D_1, \ldots, D_f$ are linearly nilindependent if and only if $(D_1)_{\mathfrak{m}_1\mathfrak{m}_1}, \ldots, (D_f)_{\mathfrak{m}_1\mathfrak{m}_1}$ are linearly independent.
- If all pairwise commutators of the derivations $D_1, \ldots, D_f$ are inner derivations then necessarily $(D_1)_{\mathfrak{m}_1\mathfrak{m}_1}, \ldots, (D_f)_{\mathfrak{m}_1\mathfrak{m}_1}$ must pairwise commute.

Estimate on maximal value of f

Thus, $f = \dim \mathfrak{s} - \dim \mathfrak{n}$ is bounded by the maximal number of commuting $m_1 \times m_1$ matrices. i.e. satisfies

$$f \leq \dim \mathfrak{n} - \dim \mathfrak{n}^2. \tag{11}$$

The bound (11) is saturated for many classes of nilpotent Lie algebras whose solvable extensions were previously investigated – e.g. Abelian, naturally graded filiform $\mathfrak{n}_{n,1}$, $\mathcal{Q}_n$, a decomposable central extension of $\mathfrak{n}_{n,1}$, and of nilpotent triangular matrices. On the other hand, the bound (11) is not saturated in the case of Heisenberg nilradicals $\mathfrak{h}$ where the maximal number of non–nilpotent elements is in fact equal to $\frac{\dim \mathfrak{h}+1}{2} < \dim \mathfrak{h} - 1$.

Example: 3–dimensional solvable Lie algebras

In this case the restriction (11) shows that the dimension of the nilradical $\dim NR(\mathfrak{s})$ is 2 or 3. When $\dim NR(\mathfrak{s})$ is 3, the algebra is equal to its nilradical, i.e., nilpotent. When $\dim NR(\mathfrak{s}) = 2$ we have an Abelian nilradical and the solvable algebra $\mathfrak{s}$ is determined once the action of one nonnilpotent element f_1 on the nilradical $\mathfrak{n} = NR(\mathfrak{s}) = span\{e_1, e_2\}$ is specified. Any change of basis in the nilradical is allowed because any regular linear map is an automorphism of $\mathfrak{n}$ and consequently the task is reduced to the classification of 2×2 nonnilpotent matrices with respect to conjugation and overall rescaling. We find the following canonical forms for the matrix D_1.

Example: 3–dimensional solvable Lie algebras, cont'd

- Over the field of complex numbers the matrix D_1 has one of the following forms

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  where the parameter a satisfies $0 < |a| \leq 1$, if $|a| = 1$ then $\arg(a) \leq \pi$.

- Over the field of real numbers the matrix D_1 has one of the following forms

$$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  where the parameters a, α satisfy $-1 \leq a \leq 1$, $a \neq 0$, $\alpha \geq 0$.

Example: 3–dimensional solvable Lie algebras, cont'd

The condition $a \neq 0$ arises from the restriction to indecomposable algebras. The matrix $\begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}$ is present only over the field $\mathbb{R}$ because over the field $\mathbb{C}$ it is upon rescaling conjugated to $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with the choice $a = (\alpha + \mathrm{i})/(\alpha - \mathrm{i})$.

Example: 3–dimensional solvable Lie algebras, cont'd

The corresponding solvable algebras are

- $\mathfrak{s}_{3,1}$

	e_1	e_2	e_3
e_1	0	0	$-e_1$
e_2		0	ae_2

,

- $\mathfrak{s}_{3,2}$

	e_1	e_2	e_3
e_1	0	0	$-\alpha e_1 + e_2$
e_2		0	$-e_1 - \alpha e_2$

,

  which is isomorphic to $\mathfrak{s}_{3,1}$ over the field $\mathbb{C}$, and

- $\mathfrak{s}_{3,3}$

	e_1	e_2	e_3
e_1	0	0	$-e_1$
e_2		0	$-e_1 - e_2$

.

A similar investigation can be performed in any dimension when the nilradical is Abelian and has codimension one in $\mathfrak{s}$.

Types of nilradicals investigated so far

- Nilradicals with low degree of nilpotency (J.C. Ndogmo, J. Rubin, P. Winternitz)

  The algebras already investigated in this class are the Abelian and Heisenberg algebras (in arbitrary finite dimensions). These algebras possess large algebras of derivations that have well-understood properties. E.g., for an Abelian nilradical, any linear transformation is a derivation and any regular linear map is an automorphism. Consequently, the construction of solvable extensions is reduced to the consideration of Abelian subalgebras in $\mathfrak{gl}(n)$ and their equivalence. Similarly, for Heisenberg algebras $\mathfrak{h}(n)$, the task is reduced to the study of Abelian subalgebras of $\mathfrak{sp}(2n)$.

Types of nilradicals investigated so far, continued

- Nilradicals of Borel subalgebras of simple Lie algebras (L. Šnobl, S. Tremblay, P. Winternitz)

  Nilpotent algebras in this class have a very particular structure given by the corresponding root diagram. Consequently, all derivations of such algebras can be found in explicit form using cohomological arguments. This was done by G.F. Leger and E.M. Luks. A prime example of a nilradical in this class is the algebra of strictly upper triangular matrices.

Types of nilradicals investigated so far, continued

- Nilradicals with a high degree of nilpotency (J.M. Ancochea, R. Campoamor–Stursberg, L. Garcia Vergnolle, D. Karásek, L. Šnobl, P. Winternitz and others)

  The structure of Lie brackets of such algebras usually significantly restricts the algebra of derivations. Therefore the algebras of derivations can often be written down explicitly in arbitrary dimension and similarly for the automorphisms. Many explicit lists of solvable algebras with nilradicals in this class are known.

Example: Solvable extensions of the model filiform nilradical

We consider a class of nilpotent algebras $\mathfrak{n}_{n,1}$, the so-called model filiform algebra, where $\dim \mathfrak{n}_{n,1} = n = 3, 4, \ldots$ and the Lie brackets are given by

$$[e_1, e_n] = 0, \quad [e_k, e_n] = e_{k-1}, \; 2 \leq k \leq n-1. \tag{12}$$

We shall consider this algebra over the field $\mathbb{F} = \mathbb{R}, \mathbb{C}$.
The dimensions of the subalgebras in the characteristic series are

$$DS = [n, n-2, 0], \; CS = [n, n-2, n-3, \ldots, 1, 0], \; US = [1, 2, \ldots, n-2, n]. \tag{13}$$

The maximal Abelian ideal $\mathfrak{a}$ of $\mathfrak{n}_{n,1}$ is unique; it is the centralizer of the highest center $\mathfrak{z}_{n-2} = \mathrm{span}\{e_1, \ldots, e_{n-2}\}$, i.e.
$\mathfrak{a} = \mathrm{span}\{e_1, \ldots, e_{n-1}\}$.

We mention that for $n = 3$ we have $\mathfrak{n}_{3,1} \simeq \mathfrak{h}(1)$. The algebra $\mathfrak{n}_{4,1}$ is the only 4–dimensional indecomposable nilpotent Lie algebra.
For $n \geq 4$ $\mathfrak{n}_{n,1}$ is no longer isomorphic to $\mathfrak{h}(N)$.

Example: Solvable extensions of the model filiform nilradical, continued

Using the estimate (11) we immediately find any solvable Lie algebra with the nilradical $\mathfrak{n}_{n,1}$ has dimension $\dim \mathfrak{s} = n + 1$, or $\dim \mathfrak{s} = n + 2$. Classifying the outer derivations of $\mathfrak{n}_{n,1}$ into equivalence classes we find

Theorem

Three types of solvable Lie algebras of dimension $\dim \mathfrak{s} = n + 1$ exist for any $n \geq 4$. They are represented by the following three cases:

1. *The matrix $A = A_1$ of the derivation D_1 in Eq. (4) diagonal*

$$[f, e_k] = ((n - k - 1)\alpha + \beta)\, e_k,\ \ k \leq n - 1, [f, e_n] = \alpha e_n. \tag{14}$$

Example: Solvable extensions of the model filiform nilradical, continued

Theorem (continued)

The mutually nonisomorphic algebras of this type are

$$
\begin{aligned}
\mathfrak{s}_{n+1,1}(\beta): &\quad \alpha = 1, \beta \in \mathbb{F}\backslash\{0, n-2\}, \\
DS &= [n+1, n, n-2, 0],\ \ CS = [n+1, n, n, \ldots], US = [0], \\
\mathfrak{s}_{n+1,2}: &\quad \alpha = 1, \beta = 0, \\
DS &= [n+1, n-1, n-3, 0],\ \ CS = [n+1, n-1, n-1, \ldots], \\
US &= [0], \\
\mathfrak{s}_{n+1,3}: &\quad \alpha = 1, \beta = 2-n, \\
DS &= [n+1, n, n-2, 0],\ \ CS = [n+1, n, n, \ldots], \\
US &= [1, 1, \ldots], \\
\mathfrak{s}_{n+1,4}: &\quad \alpha = 0, \beta = 1, \\
DS &= [n+1, n-1, 0],\ \ CS = [n+1, n-1, n-1, \ldots], \\
US &= [0].
\end{aligned}
$$

Example: Solvable extensions of the model filiform nilradical, continued

Theorem (continued)

2. *The matrix $A = A_1$ of the derivation D_1 in Eq. (4) nondiagonal, its diagonal determined by $\alpha = \beta = 1$. We have*

$$\begin{aligned} \mathfrak{s}_{n+1,5}: &\quad [f, e_k] = (n-k)e_k, \ k \leq n-1, [f, e_n] = e_n + e_{n-1}, \\ DS &= [n+1, n, n-2, 0], \ CS = [n+1, n, n, \ldots], \ US = [0]. \end{aligned}$$

Example: Solvable extensions of the model filiform nilradical, continued

Theorem (continued)

3. *The matrix $A = A_1$ of the derivation D_1 in Eq. (4) nondiagonal, its diagonal determined by $\alpha = 0, \beta = 1$.*

$$\mathfrak{s}_{n+1,6}(a_3, \ldots, a_{n-1}) : \qquad [f, e_k] = e_k + \sum_{l=1}^{k-2} a_{k-l+1} e_l, \;\; k \leq n-1,$$
$$[f, e_n] = 0,$$

$a_j \in \mathbb{F}$, at least one a_j satisfies $a_j \neq 0$. Over $\mathbb{C}$: the first nonzero a_j satisfies $a_j = 1$. Over $\mathbb{R}$: the first nonzero a_j for even j satisfies $a_j = 1$. If all $a_j = 0$ for j even, then the first nonzero a_j (j odd) satisfies $a_j = \pm 1$. We have

$$DS = [n+1, n-1, 0], \;\; CS = [n+1, n-1, n-1, \ldots], US = [0].$$

Example: Solvable extensions of the model filiform nilradical, continued

Theorem

Precisely one class of solvable Lie algebras $\mathfrak{s}_{n+2}$ of $\dim \mathfrak{s} = n+2$ with nilradical $\mathfrak{n}_{n,1}$ exists. It is represented by a basis $(e_1, \ldots, e_n, f_1, f_2)$ and the Lie brackets involving f_1 and f_2 are

$$\begin{aligned} [f_1, e_k] &= (n-1-k)e_k,\ 1 \le k \le n-1,\ [f_1, e_n] = e_n, \\ [f_2, e_k] &= e_k,\ 1 \le k \le n-1,\ [f_2, e_n] = 0,\ [f_1, f_2] = 0. \end{aligned}$$

For this algebra we have

$$DS = [n+2, n, n-2, 0],\ CS = [n+2, n, n, \ldots], US = [0].$$

Solvable extensions of Borel nilradicals

Let us now concentrate on nilpotent Lie algebras $\mathfrak{n}$ that are isomorphic to the nilradicals of the Borel subalgebras of a complex simple Lie algebra. Such nilpotent Lie algebra $\mathfrak{n}$ can be interpreted as the one consisting of all positive root spaces. We shall present general structural properties of all solvable extensions of $\mathfrak{n}$.

The motivation for such an investigation comes from the particular case of triangular nilradicals which are Borel nilradicals of simple Lie algebras $A_l = \mathfrak{sl}(l+1, \mathbb{F})$.

Triangular nilradicals – summary

The results for triangular nilradicals can be summarized as follows:

- Every solvable Lie algebra $\mathfrak{s}(n_{NR}, q)$ with the triangular nilradical $\mathfrak{t}(l+1)$ has the dimension

$$d = q + n_{NR}, \qquad 1 \leq q \leq l$$

  where $n_{NR} = \frac{l(l+1)}{2}$ is the dimension of the nilradical $\mathfrak{t}(l+1)$ and l is the rank of the simple Lie algebra A_l.
- A "canonical basis" $\{X^\alpha, N_{ik}\}$ of $\mathfrak{s}(n_{NR}, q)$ exists in which the commutation relations are

$$\begin{aligned} [N_{ik}, N_{ab}] &= \delta_{ka} N_{ib} - \delta_{bi} N_{ak}, \\ [X^\alpha, N_{ik}] &= \sum_{p<q} A^\alpha_{ik\,,\,pq} N_{pq}, \\ [X^\alpha, X^\beta] &= \sigma^{\alpha\beta} N_{1(l+1)}. \end{aligned}$$

- The matrices A^α are linearly nilindependent and upper triangular. For $q \geq 2$ they pairwise commute. The only off–diagonal matrix elements in A^α that may not vanish are

$$A^\alpha_{12\,,\,2(l+1)}, \quad A^\alpha_{j(j+1)\,,\,1(l+1)}, \quad A^\alpha_{l(l+1)\,,\,1l} \tag{15}$$

The diagonal elements $A^\alpha_{i(i+1)\,,\,i(i+1)}$, $1 \leq i \leq l$ are free and determine the rest of the diagonal elements

$$A^\alpha_{ik\,,\,ik} = \sum_{j=i}^{k-1} A^\alpha_{j(j+1)\,,\,j(j+1)}, \qquad i+1 < k$$

- All constants $\sigma^{\alpha\beta}$ vanish unless we have $A^{\gamma}_{1(l+1)\,,\,1(l+1)} = 0$ for $\gamma = 1, \ldots, q$. The remaining off-diagonal elements $A^{\alpha}_{ik\,,\,ab}$ in equation (15) also vanish, unless the diagonal elements satisfy $A^{\beta}_{ik\,,\,ik} = A^{\beta}_{ab\,,\,ab}$ for all $\beta = 1, \ldots, q$.
- The maximal value $q = l$ corresponds to exactly one solvable Lie algebra for which all matrices A^{α} are diagonal and all elements X^{α} commute. This algebra is isomorphic to the Borel subalgebra of A_l.
- For the minimal value $q = 1$ at most $l - 1$ off-diagonal elements of A^1 are nonvanishing. They can be normalized to 1 when $\mathbb{F} = \mathbb{C}$ and to ± 1 when $\mathbb{F} = \mathbb{R}$.

We shall show that essentially the same results hold for solvable Lie algebras with any Borel nilradical. This simultaneous treatment is made possible by the fact that all outer derivations of these nilradicals are known, due to the work of G.F. Leger and E.M. Luks (Leger G F, Luks E M 1974 Cohomology of nilradicals of Borel subalgebras. *Trans. Amer. Math. Soc.* **195** 305–316.)

Borel nilradicals

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{g}_0$ its Cartan subalgebra, $l = \mathit{rank}\,\mathfrak{g} = \dim \mathfrak{g}_0$. Let us denote by Δ the set of all roots, by Δ^+ the set of all positive roots and by $\Delta^S = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots. Let $\mathfrak{g}_\lambda$ denote the root subspace of the root λ. Let S_β denote the Weyl reflection with respect to the root β,

$$S_\beta(\alpha) = \alpha - 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}\beta, \qquad \alpha \in \Delta.$$

Every (semi)simple complex Lie algebra $\mathfrak{g}$ contains a unique (up to isomorphisms) maximal solvable subalgebra, its Borel subalgebra $\mathfrak{b}(\mathfrak{g})$. It contains the Cartan subalgebra and all positive root vectors

$$\mathfrak{b}(\mathfrak{g}) = \mathfrak{g}_0 \dotplus \left(\dotplus \{\mathfrak{g}_\lambda | \lambda \in \Delta^+\} \right).$$

The properties of root systems imply that the Borel subalgebra is indeed a solvable subalgebra of $\mathfrak{g}$ with the nilradical

$$\mathrm{NR}(\mathfrak{b}(\mathfrak{g})) = \dot{+}\{\mathfrak{g}_\lambda \,|\, \lambda \in \Delta^+\}.$$

For the sake of brevity we shall call the nilpotent Lie algebra $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ the Borel nilradical (although it is not the nilradical of the simple Lie algebra $\mathfrak{g}$).

Let

$$\mathfrak{g}_m = \dot{+}\{\mathfrak{g}_\lambda \,|\, \lambda = \sum_{i=1}^{l} m_i \alpha_i, \ \sum_{i=1}^{l} m_i \geq m\}.$$

The vectors e_α, $\alpha \in \Delta^S$ generate the entire $\mathrm{NR}(\mathfrak{b}(\mathfrak{g})) = \dot{+}\{\mathfrak{g}_\lambda | \, \lambda \in \Delta^+\}$ through commutators

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] = \mathfrak{g}_{\lambda+\mu} \qquad \text{whenever} \quad \lambda, \mu, \lambda + \mu \in \Delta^+$$

and this implies that the ideals in the lower central series of the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ of the Borel subalgebra are

$$(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))^m = \mathfrak{g}_m.$$

The center $\mathfrak{z}$ of $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ is one–dimensional and is spanned by e_λ where λ is the highest root of $\mathfrak{g}$, i.e. the only root such that no root $\lambda + \alpha, \quad \alpha \in \Delta^+$ exists. The center $\mathfrak{z}$ coincides with the last nonvanishing ideal in the lower central series.

Outer derivations of Borel nilradicals

All derivations of the nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ were found by G. F. Leger and E. M. Luks and the result is as follows.

Theorem

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l, $\mathfrak{g}_0$ its Cartan subalgebra, $\Delta^S = \{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots and $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$. Then the algebra of derivations of the nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ of the Borel subalgebra of a complex simple Lie algebra $\mathfrak{g}$ satisfies

- $\mathfrak{Der}(\mathfrak{n}) = \mathfrak{Out}(\mathfrak{n}) \dot{+} \mathfrak{Inn}(\mathfrak{n})$,
- $\dim \mathfrak{Out}(\mathfrak{n}) = 2l$,
- $\mathfrak{Out}(\mathfrak{n}) = \mathrm{span}\{D_i, \tilde{D}_i \,|\, i = 1, \ldots, l\}$ *where the derivations $D_i, \tilde{D}_i$ are defined below.*

Outer derivations of Borel nilradicals

Theorem (continued)

The derivations D_i act diagonally in the basis of $\mathfrak{n}$ consisting of positive root vectors e_α, $\alpha \in \Delta^+$

$$D_i(e_\alpha) = m_i e_\alpha, \qquad \alpha = \sum_{j=1}^{l} m_j \alpha_j \in \Delta^+.$$

$\tilde{D}_i$ are nilpotent outer derivations acting on simple root vectors

$$\begin{aligned} \tilde{D}_i(e_\beta) &= e_\gamma, \qquad \text{where} \quad \gamma = S_{\alpha_i}(\lambda), \qquad && \text{if } \beta = \alpha_i, \\ &= 0, && \text{if } \beta = \alpha_j,\ j \neq i. \end{aligned} \tag{16}$$

The action of $\tilde{D}_i$ on e_α, $\alpha \in \Delta^+ \backslash \Delta^S$ follows from the definition of a derivation (1).

For the sake of brevity, we shall write $S_i(\lambda)$ instead of $S_{\alpha_i}(\lambda)$ and introduce nonnegative integer constants s_i

$$S_i(\lambda) = \lambda - s_i \alpha_i .$$

We notice that for $\mathfrak{g} = A_l$ only two constants s_i, namely s_1 and s_l, are nonvanishing and equal to one; for all other simple algebras only one s_i is nonvanishing and turns out to be equal to 1 or 2.

It can be easily deduced that for any simple complex Lie algebra $\mathfrak{g}$ the derivations $\tilde{D}_i$ of the algebra $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ give zero whenever they act on e_β, $\beta \in \Delta^+ \backslash \Delta^S$.

Let us assume from now on that $l > 2$. Then we always have $S_i(\lambda) \notin \Delta^S$ for all $i = 1, \ldots, l$ and consequently

$$\tilde{D}_i \circ \tilde{D}_j(e_{\alpha_k}) = 0 \tag{17}$$

for every $\alpha_k \in \Delta^S$. The Leibniz property (1) allows us to conclude that equation (17) must hold for any $\alpha \in \Delta^+$, i.e. we have

$$\tilde{D}_i \circ \tilde{D}_j = 0, \qquad i, j = 1, \ldots, l.$$

The derivations D_i obviously commute among each other and act diagonally on $\tilde{D}_j$,

$$[D_i, \tilde{D}_j] \in \mathrm{span}\{\tilde{D}_j\}. \tag{18}$$

To conclude, under the assumption that l is greater than 2, the $2l$ outer derivations $D_i, \tilde{D}_i$ span a Lie subalgebra $\mathfrak{Out}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ of the algebra of all derivations $\mathfrak{Der}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$. This algebra can be further decomposed into a semidirect sum of an l–dimensional Abelian ideal spanned by the nilpotent derivations $\tilde{D}_i$ and an l–dimensional Abelian subalgebra spanned by D_i.

Solvable extensions of the Borel nilradicals $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$

Let us now study the structure of any solvable Lie algebra with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$, $l = \mathrm{rank}\,\mathfrak{g} > 2$.

From the fact that there are only l linearly nilindependent derivations D_i in $\mathfrak{Out}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ we conclude that the maximal number of nonnilpotent basis elements in any solvable Lie algebra $\mathfrak{s}$ with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ is l. One algebra with this number of nonnilpotent basis elements is already known, namely the Borel subalgebra $\mathfrak{b}(\mathfrak{g})$ of the simple Lie algebra $\mathfrak{g}$. Is it the only one?

Let us assume that we have a solvable Lie algebra $\mathfrak{s}$ with the nilradical $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ and $l = \mathrm{rank}\,\mathfrak{g}$ nonnilpotent basis elements f_i. They define l outer linearly nilindependent derivations $\hat{D}^i$ such that $\hat{D}^i = ad(f_i)|_{\mathfrak{n}}$. Using the transformation (6) we may choose the basis vectors f_i so that

$$\hat{D}^i = D_i + \sum_{j=1}^{l} \omega^i_j \tilde{D}_j$$

where $D_i, \tilde{D}_j$ are the derivations defined before.

Because $\hat{D}^i$ lie in the subalgebra $\mathfrak{Out}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ of $\mathfrak{Der}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ and at the same time $[\hat{D}^i, \hat{D}^j] \in \mathfrak{Inn}(\mathrm{NR}(\mathfrak{b}(\mathfrak{g})))$ must hold, we find that

$$[\hat{D}^k, \hat{D}^j] = 0. \tag{19}$$

This requirement together with equation (18) in turn implies

$$\omega_i^j [D_k, \tilde{D}_i] + \omega_i^k [\tilde{D}_i, D_j] = 0. \tag{20}$$

for every $i, j, k = 1, \ldots, l$ such that $k \neq j$ (no summation over i).

For any given i we can find $\tilde{i}$ such that $[\tilde{D}_i, D_{\tilde{i}}] \neq 0$. Consequently, the value of $\omega_i^{\tilde{i}}$ together with the root system specifying the Lie brackets $[D_k, \tilde{D}_i]$ completely determines all ω_i^j for $j \neq \tilde{i}$. Altogether, we still have one undetermined parameter $\omega_i^{\tilde{i}}$ for each $i = 1, \ldots, l$. Next, we show that one can eliminate these parameters through a suitable choice of automorphism in equation (7).

The essence of the argument is that for each $i = 1, \ldots, l$ we can find $\hat{D}^{\tilde{i}}$ which transforms nontrivially under the transformation

$$\begin{array}{rcl} D_j & \to & D_j + t_i[\tilde{D}_i, D_j], \\ \tilde{D}_j & \to & \tilde{D}_j, \\ \hat{D}^j = D_j + \sum_{k=1}^{l} \omega_k^j \tilde{D}_k & \to & D_j + t_i[\tilde{D}_i, D_j] + \sum_{k=1}^{l} \omega_k^j \tilde{D}_k. \end{array} \tag{21}$$

due to $[\tilde{D}_i, D_{\tilde{i}}] \neq 0$. We use it to set $\omega_i^{\tilde{i}} = 0$ after the transformation. Equation (20) then implies that after the transformation all $\omega_i^j = 0$.

Therefore we have found that our derivations $\hat{D}^j$ can be brought to the form

$$\hat{D}^j = D_j$$

through a conjugation by a suitable automorphism $\tilde{\Phi}$ of $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$.

Next, we show that we can always accomplish

$$[f_i, f_j] = 0. \tag{22}$$

We have

$$[f_i, f_j] = \gamma_{ij} e_\lambda, \qquad \gamma_{ij} = -\gamma_{ji}$$

which is the preimage of the relation $[ad(f_i)|_{\mathfrak{n}}, ad(f_j)|_{\mathfrak{n}}] = 0$. It can be shown that by a suitable transformation of the form

$$f_i \to f_i + \tau_i e_\lambda$$

one can always make f_i, f_j satisfy Eq. (22).

To sum up, we have found that for any complex simple Lie algebra $\mathfrak{g}$ such that $\operatorname{rank}\mathfrak{g} > 2$ the maximal solvable Lie algebra with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ is unique and isomorphic to the Borel subalgebra $\mathfrak{b}(\mathfrak{g})$ of $\mathfrak{g}$.

We notice that the same is true also when $\operatorname{rank}\mathfrak{g} = 1$ or $\operatorname{rank}\mathfrak{g} = 2$, i.e. $\mathfrak{g} = \mathfrak{sl}(2), \mathfrak{sl}(3), \mathfrak{so}(5)$ or G_2.

Thus, we have proven the following theorem:

Solvable extensions of Borel nilradicals of maximal dimension

Theorem

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{b}(\mathfrak{g})$ its Borel subalgebra and $\mathfrak{n} = \mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ the nilradical of $\mathfrak{b}(\mathfrak{g})$. The solvable Lie algebra with the nilradical $\mathrm{NR}(\mathfrak{b}(\mathfrak{g}))$ of the maximal dimension $\dim \mathfrak{n} +$ rank $\mathfrak{g}$ is unique and isomorphic to the Borel subalgebra $\mathfrak{b}(\mathfrak{g})$ of $\mathfrak{g}$.

Solvable extensions of Borel nilradicals of non–maximal dimension

A similar analysis can be performed also for non–maximal solvable extensions. In this case we have derivations

$$\hat{D}^a = \sum_{j=1}^{l} \left(\sigma_j^a D_j + \omega_j^a \tilde{D}_j \right), \qquad a = 1, \dots, q \tag{23}$$

representing the elements f_a in the adjoint representation of $\mathfrak{s}$ on $\mathfrak{n}$, $\hat{D}^a = ad(f_a)|_{\mathfrak{n}}$. The $q \times l$ matrix $\sigma = (\sigma_j^a)$ must have maximal rank, i.e. q, in view of the nilindependence of $\hat{D}^a$. However we can no longer set σ_j^a equal to the Kronecker delta δ_j^a as was the case for $q = l$. This leads to cumbersome complications. Therefore, we shall only present the resulting theorems whose proofs can be found in our paper.

Theorem

Any solvable extension $\mathfrak{s}$ of the nilradical $NR(\underline{(\mathfrak{g})})$ by q nonnilpotent elements f_a, $a = 1, \ldots, q \leq$ rank $\mathfrak{g}$ is defined by q commuting derivations $\hat{D}^a$ and a constant $q \times q$ antisymmetric matrix $\gamma = (\gamma_{ab})$. The derivations $\hat{D}^a$ determine the Lie brackets

$$[f_a, e_\alpha] = \hat{D}^a(e_\alpha), \qquad a = 1, \ldots, q, \ \alpha \in \Delta^+$$

and take the form

$$\hat{D}^a = ad(f_a)|_{\mathfrak{n}} = \sum_{j=1}^{l} \left(\sigma_j^a D_j + \omega_j^a \tilde{D}_j\right), \qquad a = 1, \ldots, q,$$

where $\sigma = (\sigma_j^a)$, $a = 1, \ldots, q$, $j = 1, \ldots, l$ has the rank q.

Theorem (continued)

For any given value of k all parameters ω_k^a are equal to zero when the condition

$$\sum_{j=1}^{l} \sigma_j^a \lambda_j - \sigma_k^a (1 + s_k) \neq 0 \tag{24}$$

is satisfied for at least one $a \in \{1, \ldots, q\}$. The condition (24) is always satisfied for at least q values of the index k, i.e. there are at most $l - q$ values of k such that some of the parameters ω_k^a are nonvanishing.

Theorem (continued)

The matrix $\gamma = (\gamma_{ab})$ *defines the Lie brackets*

$$[f_a, f_b] = \gamma_{ab} e_\lambda, \qquad a, b = 1, \ldots, q.$$

When

$$\sum_{j=1}^{l} \lambda^j \sigma_j^a \neq 0$$

holds for at least one $a \in \{1, \ldots, q\}$, *the constants* γ_{ab} *are all equal to* 0, *i.e.*

$$[f_a, f_b] = 0.$$

We remark that the conditions in Theorem 9 are sufficient, i.e. any set of constants σ_j^a, ω_j^a and γ_{ab} satisfying the properties listed in the theorem gives rise to a solvable extension of the nilradical $NR((\underline{\mathfrak{g}}))$. On the other hand, the description presented in Theorem 9 is not unique, i.e. different choices of σ_j^a, ω_j^a and γ_{ab} may lead to isomorphic algebras. As already noted, we may replace the derivations $\hat{D}^a$ by any linearly independent combination of them thus changing all the parameters σ_j^a, ω_j^a and γ_{ab}. Also we may employ the scaling automorphisms to change the values of ω_j^a and γ_{ab}.

We remark that by virtue of indecomposability of the Borel nilradicals, all solvable Lie algebras described in Theorem 9 are indecomposable.

We notice that the statements of Theorem 9 significantly resemble the results for triangular nilradicals which they generalize.

Dimension $n_{NR} + 1$ solvable extensions of the Borel nilradicals

Theorem

Any solvable extension of the nilradical $NR((\underline{\mathfrak{g}}))$ by one nonnilpotent element is up to isomorphism defined by a single derivation

$$\hat{D} = ad(f_1)|_{\mathfrak{n}} = \sum_{j=1}^{l} \left(\sigma_j D_j + \omega_j \tilde{D}_j \right)$$

chosen so that the first nonvanishing parameter σ_j is equal to one. ω_k vanishes whenever $\sum_{j=1}^{l} \sigma_j \lambda_j - \sigma_k(1 + s_k) \neq 0$. At most $l - 1$ parameters ω_k are nonvanishing. They are all equal to 1 over the field of complex numbers. Over the field of real numbers they are equal to ± 1 and all parameters ω_k with $s_k = 0$ have the same sign.

Summary

- We have presented the approach to the classification of solvable Lie algebras based on construction of solvable extensions of nilpotent algebras.
- We have reviewed which classes of solvable Lie algebras were already described in this way and demonstrated several examples.
- We have introduced the structure of Borel nilradicals and presented their solvable extensions.