Classical and Quantum Superintegrable Systems with N-th Order Integrals of Motion

Pavel Winternitz

E-mail address: wintern@crm.umontreal.ca
Centre de recherches mathématiques et
Département de Mathématiques et de Statistique, Université de Montréal,
CP 6128, Succ. Centre-Ville, Montréal, Quebec H3C 3J7, Canada
Marie Curie Fellow at Università Roma Tre

20-27 June 2015

PMNP 2015 - Physics and Mathematics of Nonlinear Phenomena

Outline

Joint work with Sarah Post, U. Hawai'i

We restrict ourselves to a two-dimensional real Euclidean plane and to Hamiltonians of the form

$$H = p_1^2 + p_2^2 + V(x, y).$$

In classical mechanics p_1 and p_2 are the components of the linear momentum, to which we add for use below the angular momentum

$$L_3 = xp_2 - yp_1.$$

In quantum mechanics, p_1 and p_2 (as well as H and L_3) will be Hermitian operators with

$$\hat{p}_1 = -i\hbar\partial_x, \qquad \hat{p}_2 = -i\hbar\partial_y.$$

In classical mechanics an Nth order integral of the motion can be written as

$$X = \sum_{k=0}^{N} \sum_{j=0}^{N-k} f_{j,k}(x, y) p_1^j p_2^{N-k-j}, \qquad f_{j,k}(x, y) \in \mathbb{R},$$

or simply

$$X = \sum_{j,k} f_{j,k}(x, y) p_1^j p_2^{N-k-j},$$

with $f_{j,k} = 0$ for $j < 0$, $k < 0$ and $k + j > N$. The leading terms (of order N) are obtained by restricting the summation to $k = 0$. In quantum mechanics we also take the integral of the form (3) (or (3)) but the p_i are operators as in (3) and we must symmetrize in order for X to be Hermitian.

Integrability and Superintegrability

We recall that a Hamiltonian with n degrees of freedom in classical mechanics is integrable (Liouville integrable) if it allows n integrals of motion (including the Hamiltonian) that are well defined functions on phase space, are in involution (Poisson commute) and are functionally independent. The system is superintegrable if it allows more than n integrals that are functionally independent and commute with the Hamiltonian. The system is maximally superintegrable if it allows $2n-1$ functionally independent, well defined integrals, though at most n of them can be in involution.

In quantum mechanics the definitions are similar. The integrals are operators in the enveloping algebra of the Heisenberg algebra $H_n \sim \{x_1, \ldots, x_n, p_1, \ldots, p_n, \hbar\}$ that are either polynomials or convergent power series. A set of integrals is algebraically independent if no Jordan polynomial (formed using only anti-commutators) in the operators vanishes.

Background

- Best known superintegrable systems: Kepler-Coulomb system $V = \alpha/r$ and the harmonic oscillator $V = \alpha r^2$.
- Bertrand's theorem gives that the only rotationally invariant potentials in which all bounded trajectories are closed are precisely these two potentials.
- A theorem proven by Nekhoroshev 1972 states that, away from singular points, the bounded trajectories of a maximally superintegrable Hamiltonian system are periodic. It follows that there are no other rotationally invariant maximally superintegrable systems in E_n.
- A systematic study of other superintegrable systems started with the construction of all quadratically superintegrable systems in E_2 and E_3 Fris, Makarov, Smorodinsky, Uhlir & W. 1965, Fris, Smorodinsky, Uhlir & W. 1966, Makarov, Smorodinsky, Valiev & W. 1967
- Superintegrable systems with second-order integrals of motion are by now well understood both in spaces of constant curvature and in more general spaces, see e.g. Kalnins, Kress & Miller, Evans, RodrÃguez & W, etc . Second-order superintegrability is related to multi-separability in the Hamilton-Jacobi equations and the Schrödinger equation. The superintegrable potentials are the same in classical and quantum

Background Cont.

- Superintegrable systems involving one third-order and one lower order integral of motion in E_2 have been studied more recently. The connection with multiple separation of variables is lost. The quantum potentials are not necessarily the same as the classical ones and can involve elliptic functions or Painlevé transcendents.
- The integrals of motion form polynomial algebras and these can be used to calculate energy spectra and wave functions, see e.g. Marquette and Marquette & W.
- A relation with supersymmetry has been established, Quesne and Marquette together and separately.
- It was recently shown that infinite families of two-dimensional superintegrable systems exist with integrals of arbitrary order, Tremblay, Turbiner & W 2009 2010, Kalnins, Kress, & Miller 2010, Kalnins, Miller, & Pogosyan 2010, Marquette 2011, Lévesque, Post & W 2011, Post & Riglioni 2014 . . .
- Superintegrable systems not allowing separation of variables have been constructed Post & W 2011, Maciejewski, Przybylska & Yoshida 2010, and Maciejewski, Przybylska &Tsiganov 2010.

The present results are to be viewed in the context of a systematic study of integrable and superintegrable systems with integrals that are polynomials in the momenta, especially for those of degree higher than two. Here we concentrate on the properties of one integral of order N in two-dimensional Euclidean space.

The plan for the remainder of the talk is as follows:

1. Classical Integrals and their determining equations
2. Quantum Integrals and their determining equations
3. Some comments on quantization and difference choices of symmetrization

Classical Integrals

Let us consider the classical Hamiltonian and the Nth order integral, which Poisson commutes with the Hamiltonian

$$\{H, X\}_{PB} = 0,$$

which leads directly to a simple but powerful theorem.

Theorem 1.

A classical Nth order integral for the Hamiltonian (3) has the form

$$X = \sum_{\ell=0}^{[\frac{N}{2}]} \sum_{j=0}^{N-2\ell} f_{j,2\ell} p_1^j p_2^{N-j-2\ell},$$

where $f_{j,2\ell}$ are real functions that are identically 0 for $j, \ell < 0$ or $j > N - 2\ell$, with the following properties:

1. The functions $f_{j,2\ell}$ and the potential $V(x,y)$ satisfy the determining equations

$$0 = 2\left(\partial_x f_{j-1,2\ell} + \partial_y f_{j,2\ell}\right) - \left((j+1) f_{j+1,2\ell-2}\partial_x V + (N-2\ell+2-j) f_{j,2\ell-2}\partial_y V\right).$$

2. As indicated above, all terms in the polynomial X have the same parity.
3. The leading terms in the integral (of order N obtained for $\ell = 0$) are polynomials of order N in the enveloping algebra of the Euclidean Lie algebra $e(2)$ with basis $\{p_1, p_2, L_3\}$.

Corollary 2.

The classical integral (1) can be rewritten as

$$X = \sum_{0\le m+n\le N} A_{N-m-n,m,n} L_3^{N-m-n} p_1^m p_2^n + \sum_{\ell=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{j=0}^{N-2\ell} f_{j,2\ell} p_1^j p_2^{N-j-2\ell},$$

where $A_{N-m-n,m,n}$ are constants.

Let us add some comments.

1. For physical reasons (time reversal invariance) we have assumed that the functions $f_{j,k}(x,y) \in \mathbb{R}$ from the beginning. This is actually no restriction. If X were complex, its real and imaginary parts would Poisson commute with H separately.
2. The number of determining equations (1) is equal to

$$\sum_{\ell=0}^{[\frac{N+1}{2}]} (N - 2\ell + 2) = \begin{cases} \frac{1}{4}(N+3)^2 & \text{N odd} \\ \frac{1}{4}(N+2)(N+4) & \text{N even.} \end{cases}$$

For a given potential, the equations are linear first-order partial differential equations for the unknowns $f_{j,2\ell}(x,y)$. The number of unknowns is

$$\sum_{\ell=0}^{[\frac{N+1}{2}]} (N - 2\ell + 1) = \begin{cases} \frac{(N+1)(N+3)}{4} & \text{N odd} \\ \frac{1}{4}(N+2)^2 & \text{N even.} \end{cases}$$

As is clear from Corollary 1, the determining equations for $f_{j,0}$ can be solved without knowledge of the potential and the solutions depend on $(N+1)(N+2)/2$ constants. Thus, $N+1$ of the functions $f_{j,2\ell}$, namely $f_{j,0}$, are known in terms of $(N+1)(N+2)/2$ constants. The remaining system is overdetermined and subject to further compatibility conditions.

If the potential is not a priori known, then the system becomes nonlinear and $V(x,y)$ must be determined from the compatibility conditions. We present the first set of compatibility conditions as a corollary.

Corollary 3.

If the Hamiltonian admits X as an integral then the potential function $V(x,y)$ satisfies the following linear partial differential equation (PDE)

$$0=\sum_{j=0}^{N-1}\partial_x^{N-1-j}\partial_y^{j}(-1)^j\left[(j+1)f_{j+1,0}\partial_x V+(N-j)f_{j,0}\partial_y V\right].$$

3. For N odd, the lowest-order determining equations are

$$f_{1,N-1} V_x + f_{0,N-1} V_y = 0.$$

In particular, for the $N = 3$ case, the compatibility conditions of these equation with the determining equations for $f_{j,2}$ lead to nonlinear equations for the potential.

Next, let's consider quantum systems.

Quantum Integrals

Theorem 4.

A quantum Nth order integral for the Hamiltonian (3) has the form

$$X = \frac{1}{2}\sum_{\ell=0}^{\lfloor \frac{N}{2} \rfloor}\sum_{j=0}^{N-2\ell}\{f_{j,2\ell}, \hat{p}_1^j \hat{p}_2^{N-2\ell-j}\},$$

where $f_{j,2\ell}$ are real functions that are identically 0 for $j, \ell < 0$ or $j > N-2\ell$, with the following properties:

1. *The functions $f_{j,2\ell}$ and the potential $V(x,y)$ satisfy the determining equations*

$$0 = M_{j,2\ell},$$

$$\begin{aligned} M_{j,2\ell} \;\; &\equiv \;\; 2\left(\partial_x f_{j-1,2\ell} + \partial_y f_{j,2\ell}\right) \\ &\quad - \left((j+1) f_{j+1,2\ell-2}\partial_x V + (N-2\ell+2-j) f_{j,2\ell-2}\partial_y V + \hbar^2 Q_{j,2\ell}\right), \end{aligned}$$

where $Q_{j,2\ell}$ is a quantum correction term given by:

$$Q_{j,2\ell} \equiv \big(2\partial_x\phi_{j-1,2\ell} + 2\partial_y\phi_{j,2\ell} + \partial_x^2\phi_{j,2\ell-1} + \partial_y^2\phi_{j,2\ell-1}\big)$$
$$-\sum_{n=0}^{\ell-2}\sum_{m=0}^{2n+3}(-\hbar^2)^n\binom{j+m}{m}\binom{N-2\ell+2n+4-j-m}{2n+3-m}(\partial_x^m\partial_y^{2n+3-m}V)f_{j+m,2\ell-2n-4}$$
$$-\sum_{n=1}^{2\ell-1}\sum_{m=0}^{n}(-\hbar^2)^{\lfloor(n-1)/2\rfloor}\binom{j+m}{m}\binom{N-2\ell+n+1+j-m}{n-m}(\partial_x^m\partial_y^{n-m}V)\phi_{j+m,2\ell-n-1},$$

where the $\phi_{j,k}$ are defined for $k>0$, $\epsilon=0,1$ as

$$\phi_{j,2\ell-\epsilon} = \sum_{b=1}^{\ell}\sum_{a=0}^{2b-\epsilon}\frac{(-\hbar^2)^{b-1}}{2}\binom{j+a}{a}\binom{N-2\ell+2b-j-a}{2b-\epsilon-a}\partial_x^a\partial_y^{2b-\epsilon-a}f_{j+a,2\ell-2b}.$$

In particular $\phi_{j,0}=0$, hence $Q_{j,0}=0$ so the $\ell=0$ determining are the same as in the classical case.

2. As indicated in the form of X, the symmetrized integral will have terms which are differential operators of the same parity.
3. The leading terms in X (of order N obtained for $\ell=0$) are polynomials of order N in the enveloping algebra of the Euclidean Lie algebra $e(2)$ with basis $\{\hat{p}_1, \hat{p}_2, \hat{L}_3\}$.

The functions $\phi_{j,k}$ can be understood by expanding out the integral X as

$$X = \sum_{\ell,j} \left(f_{j,2\ell} - \hbar^2 \phi_{j,2\ell}\right) \hat{p}_1^j \hat{p}_2^{N-2\ell-j} - i\hbar \sum_{\ell,j} \phi_{j,2\ell-1} \hat{p}_1^j \hat{p}_2^{N-2\ell+1-j}.$$

An important lemma used in the proof of this theorem is the following:

Lemma 5.

Given a general, self-adjoint Nth order differential operator, X, there exist real functions $f_{j,k}$ such that

$$X = \frac{1}{2} \sum_k \sum_j \{f_{j,k}, \hat{p}_1^j \hat{p}_2^{N-k-j}\}$$

For the form of the integral given above, so that it contains only terms of the same parity, can be shown by noticing that the Hamiltonian is invariant under complex conjugation.

The highest-order determining equations can be solved directly and the functions $f_{j,0}$ are the same as in the classical case. However, as will be discussed later, the choice of symmetrization of the leading order terms will lead to $\hbar^2$-dependent correction terms in the lower-order functions. The quantum version of Corollary 2 still holds:

Corollary 6.

If the quantum Hamiltonian H admits X as an integral then the potential function $V(x, y)$ satisfies the same linear PDE as in the classical case, namely

$$0 = \sum_{j=0}^{N-1} \partial_x^{N-1-j} \partial_y^j (-1)^j \left[(j+1) f_{j+1,0} \partial_x V + (N-j) f_{j,0} \partial_y V \right].$$

This does not imply that the quantum and classical potentials are necessarily the same since further (nonlinear) compatibility conditions exist.

Proof

The $\ell = 1$ set of determining equations are $M_{j,2} = 0$ with

$$M_{j,2} = 2\left(\partial_x f_{j-1,2} + \partial_y f_{j,2}\right) - \left[(j+1) f_{j+1,0}\partial_x V + (N-j) f_{j,0}\partial_y V + \hbar^2 Q_{j,2}\right],$$

with quantum correction term

$$Q_{j,2} = 2\partial_x \phi_{j-1,2} + 2\partial_y \phi_{j,2} + \partial_x^2 \phi_{j,1} + \partial_y^2 \phi_{j,1}.$$

The functions $\phi_{j,k}$, coming from expanding out the highest order terms are

$$\phi_{j,1} = \frac{j+1}{2}\partial_x f_{j+1,0} + \frac{N-j}{2}\partial_y f_{j,0},$$

$$\phi_{j,2} = \sum_{a=0}^{2} \frac{1}{2} \binom{j+a}{a} \binom{N-j-a}{2-a} \partial_x^a \partial_y^{2-a} f_{j+a,0}.$$

The linear compatibility condition of $M_{j,2} = 0$ is obtained as

$$0 = \sum_{j=0}^{N-1} \partial_y^{N-1-j} \partial_x^j (-1)^{j-1} M_{j,2}$$

$$= \sum_{j=0}^{N-1} \partial_x^{N-1-j} \partial_y^j (-1)^j \left[(j+1)\partial_x V f_{j+1,0} + (N-j)\partial_y V f_{j,0} + \hbar^2 Q_{j,2}\right].$$

The coefficient of $\hbar^2$ in the previous equation vanish, as the relevant terms are zero, i.e.

$$\partial_x^{N-1-j}\partial_y^j Q_{j,2} = 0,$$

since the functions $f_{j,0}$ are polynomials of total degree at most N. Thus, the potential satisfies the same linear compatibility condition as in the classical case.

In fact, this result can be obtain directly by considering instead the following form of the integral. Consider a general, homogeneous polynomial of degree N in the operators $\hat{p}_1, \hat{p}_2$ and $\hat{L}_3$ that is self-adjoint, call it $P_N(\hat{p}_1, \hat{p}_2, \hat{L}_3)$. By Lemma 2 above along with the requirement that this operator commute with the free (zero potential) Hamiltonian, this operator can be expanded out as

$$P_N(\hat{p}_1, \hat{p}_2, \hat{L}_3) = \frac{1}{2}\{f_{j,0}, \hat{p}_1^j \hat{p}_2^{N-j}\} + \sum_{\ell=1}^{\lfloor N/2 \rfloor} \frac{1}{2}\{\psi_{j,2\ell}, \hat{p}_1^j \hat{p}_2^{N-2\ell-j}\}.$$

Thus, the operator X can instead be expressed as

$$X = P_N(\hat{p}_1, \hat{p}_2, \hat{L}_3) + \sum_{\ell=1}^{\lfloor N/2 \rfloor} \frac{1}{2}\{\tilde{f}_{j,2\ell}, \hat{p}_1^j \hat{p}_2^{N-2\ell-j}\}.$$

In practice $P_N(\hat{p}_1, \hat{p}_2, \hat{L}_3)$ is often chosen as

$$P_N(\hat{p}_1, \hat{p}_2, \hat{L}_3) = \frac{1}{2} \sum_{m,n} A_{N-m-n,m,n} \{\hat{p}_1^m \hat{p}_2^n, \hat{L}_3^{N-m-n}\},$$

although this choice is not necessary for what follows. Let us now consider the determining equations for the $\tilde{f}_{j,2\ell}$. Recall the $(j, 2\ell)$ determining equation is obtained from the coefficient of $\hat{p}_1^j \hat{p}_2^{N-2\ell+1-j}$ in $[X, H]$. Expanding $[X, H]$ gives

$$[X, H] = [P_N, H_0] + [P_N, V] + \left[\sum_{\ell=1}^{\lfloor N/2 \rfloor} \sum_{j=0}^{N-2\ell} \frac{1}{2} \{\tilde{f}_{j,2\ell}, \hat{p}_1^j \hat{p}_2^{N-2\ell-j}\}, H \right]. \qquad (1)$$

By definition, the first commutator is 0. The third commutator will give exactly the determining equations with $f_{j,2\ell}$ replaced by $\tilde{f}_{j,2\ell}$, except the auxillary functions $\phi_{j,k}$ and therefore the quantum corrections will be missing terms coming from the $f_{j,0}$. To finish the determining equations, we would need to compute the coefficient of $\hat{p}_1^j \hat{p}_2^{N-2\ell+1-j}$ in $[P_N, V]$. This form of the determining equations would clearly depend on the choice of symmetrization, discussed later, and even in the standard case are not particularly illuminating.

More interesting to consider is the dependence on $\hbar$ of the determining equations for $\tilde{f}_{j,2\ell}$.

- The highest order terms in $\hbar$ are absent from the $\tilde{\phi}_{j,k}$'s, as they would come from the $f_{j,0}$'s and their contributions to lower-order terms are already contained in the P_N.
- Thus, whereas in general the functions $\phi_{j,2\ell+\epsilon}$ have terms of order ℓ in $\hbar^2$, the functions $\tilde{\phi}_{j,2\ell-\epsilon}$ will have terms of only $\ell - 1$.
- Therefore, the determining equations for the form of X with leading terms $\{f_{j,0}, \hat{p}_1^j \hat{p}_2^{N-j}\}$ will be one degree higher in $\hbar^2$ that the determining equations for the form of X with leading terms $P_N(\hat{p}_1, \hat{p}_2, \hat{L}_3)$.

Let's look at some lower dimensional examples to illustrate these results:

$N = 2$

In the notation above, a second-order integral is of the form

$$X = \sum_{j=0}^{2} f_{j,0}\hat{p}_1^j \hat{p}_2^{2-j} - i\hbar \sum_{j=0}^{1} \phi_{j,1}\hat{p}_1^j \hat{p}_2^{1-j} + f_{0,2} - \hbar^2 \phi_{0,2},$$

with

$$\phi_{0,1} = \partial_y f_{0,0} + \tfrac{1}{2}\partial_x f_{1,0}, \qquad \phi_{1,1} = \partial_x f_{2,0} + \frac{1}{2}\partial_y f_{1,0}$$
$$\phi_{0,2} = \tfrac{1}{2}\textstyle\sum_{a=0}^{2} \partial_x^a \partial_y^{2-a} f_{a,0}.$$

The highest-order determining equations $M_{j,0}$ are satisfied by taking the functions

$$f_{j,0} = \sum_{n=0}^{2-j}\sum_{m=0}^{j} \binom{2-n-m}{j-m} A_{2-n-m,m,n} x^{2-j-n}(-y)^{j-m},$$

The final determining equations are $M_{j,2} = 0$ with

$$\begin{aligned} M_{j,2} &= 2\partial_x f_{j-1,2} + 2\partial_y f_{j,2} - (j+1) f_{j+1,0}\partial_x V - (2-j) f_{j,0}\partial_y V \\ &\quad -\hbar^2 \left(2\partial_x \phi_{j-1,2} + 2\partial_y \phi_{j,2} + \partial_x^2 \phi_{j,1} + \partial_y^2 \phi_{j,1}\right). \end{aligned}$$

Note that the quantum correction term (the second line) depends on third derivatives of the $f_{j,0}$. Thus, the determining equations for second-order integrals of the motion are equivalent in both the classical and quantum cases. They reduce to

$$\begin{aligned} M_{0,2} &= 2\partial_y f_{0,2} - f_{1,0}\partial_x V - 2 f_{0,0}\partial_y V = 0 \\ M_{1,2} &= 2\partial_x f_{0,2} - 2 f_{2,0}\partial_x V - f_{1,0}\partial_y V = 0. \end{aligned}$$

$N = 3$

We now turn to the case $N = 3$. This case was investigated by Gravel and W. where the determining equations were given.

As for all N, the $\phi_{j,0}$ are identically 0. There are thus essentially two families of the $\phi's$: those that depend only on $f_{j,0}$

$$\phi_{j,1} = \sum_{a=0}^{1} \frac{1}{2} \begin{pmatrix} j+a \\ a \end{pmatrix} \begin{pmatrix} 3-j-a \\ 1-a \end{pmatrix} \partial_x^a \partial_y^{1-a} f_{j+a,0}$$

$$\phi_{j,2} = \sum_{a=0}^{2} \frac{1}{2} \begin{pmatrix} j+a \\ a \end{pmatrix} \begin{pmatrix} 3-j-a \\ 2-a \end{pmatrix} \partial_x^a \partial_y^{2-a} f_{j+a,0}.$$

and one that also depends on $f_{j,2}$,

$$\phi_{0,3} = \sum_{b=1}^{2} \sum_{a=0}^{2b-1} \frac{(-\hbar^2)^{b-1}}{2} \partial_x^a \partial_y^{2b-1-a} f_{a,4-2b}.$$

Let us now consider the quantum correction terms.

As always we have $Q_{j,0} = 0$ and so the determining equations $M_{j,0} = 0$ are the same as in the classical case with solutions

$$f_{j,0} = \sum_{n=0}^{3-j} \sum_{m=0}^{j} \binom{3-n-m}{j-m} A_{3-n-m,m,n} x^{3-j-n} (-y)^{j-m}.$$

The next set of quantum corrections are

$$Q_{j,2} = \left(2\partial_x \phi_{j-1,2} + 2\partial_y \phi_{j,2} + \partial_x^2 \phi_{j,1} + \partial_y^2 \phi_{j,1}\right),$$

which vanish on solutions of $M_{j,0}$ (i.e. $f_{j,0}$ taking the appropriate form). Thus, $Q_{j,2} = 0$ on solutions and so the next set of determining equations are again independent of $\hbar$ and given by

$$\begin{aligned} \partial_y f_{0,2} &= \frac{1}{2} f_{1,0} \partial_x V + \frac{3}{2} f_{0,0} \partial_y V \\ \partial_x f_{0,2} + \partial_y f_{1,2} &= f_{2,0} \partial_x V + f_{1,0} \partial_y V \\ \partial_x f_{1,2} &= \frac{3}{2} f_{3,0} \partial_x V + \frac{1}{2} f_{2,0} \partial_y V, \end{aligned}$$

which are equivalent to those of Gravel and W.

Turning now to the final determining equation. The quantum correction term is

$$\begin{aligned} Q_{0,4} &= \left(\partial_x^2\phi_{0,3} + \partial_y^2\phi_{0,3}\right) \\ &\quad - \sum_{m=0}^{3}(\partial_x^m\partial_y^{3-m}V)f_{m,0} - \sum_{m=0}^{2}(\partial_x^m\partial_y^{2-m}V)\phi_{m,1} - \sum_{m=0}^{1}(\partial_x^m\partial_y^{1-m}V)\phi_{m,2}. \end{aligned}$$

Note that from this expression it appears that $Q_{0,4}$ has a term depending on $\hbar^2$ (from $\phi_{0,3}$) which would lead to an $\hbar^4$ term to the final determining equation $M_{0,4}$. However, if we inspect this term, we can see that it contains only fifth-order derivatives of the functions $f_{j,0}$ and so will vanish. This leads to the simplification

$$\begin{aligned} 0 &= -f_{1,2}\partial_x V - f_{0,2}\partial_y V \\ &\quad -\hbar^2\left(-\frac{1}{4}\sum_{m=0}^{3}(\partial_x^m\partial_y^{3-m}V)f_{m,0} - \frac{1}{2}(\partial_x\partial_y f_{2,0})\partial_x V - \frac{1}{2}\left(\partial_x\partial_y f_{1,0}\right)\partial_y V.\right) \end{aligned}$$

Summarizing the results for $N = 3$,

- there are three families of determining equations.
- The first ensures that the leading order terms are in the enveloping algebra.
- This set as well as the second set of equations are the same in the classical and quantum case.
- The final set of equations (in this case one equation) does have a quantum correction term, linear in $\hbar^2$ which is also linear and homogeneous in the derivatives of the potential V.

$N = 4$

The structure of the fourth- order integrals is similar to that for $N = 3$.These determining equations were obtained by Post & W 2011. The equations for the $f_{j,2}$ functions are

$$\begin{aligned}
\partial_y f_{0,2} &= \frac{1}{2} f_{1,0}\partial_x V + 2f_{0,0}\partial_y V + \hbar^2(6yA_{400} - \frac{3}{2}A_{310}) \\
\partial_x f_{0,2} + \partial_y f_{1,2} &= f_{2,0}\partial_x V + \frac{3}{2} f_{1,0}\partial_y V + \hbar^2(6xA_{400} + \frac{3}{2}A_{301}) \\
\partial_x f_{1,2} + \partial_y f_{2,2} &= \frac{3}{2} f_{3,0}\partial_x V + f_{2,0}\partial_y V + \hbar^2(6yA_{400} - \frac{3}{2}A_{310}) \\
\partial_x f_{2,2} &= 2f_{4,0}\partial_x V + \frac{1}{2} f_{3,0}\partial_y V + \hbar^2(6xA_{400} + \frac{3}{2}A_{301}).
\end{aligned}$$

Notice that, unlike the case $N = 3$, the quantum corrections $Q_{j,2}$ are not identically 0. However, the equivalent computations obtained by P & W do not have any quantum corrections. As described above, this is due to the fact that the integral was assumed to have the form

$$X = P_4(\hat{p}_1, \hat{p}_2, \hat{L}_3) + \frac{1}{2}\{\tilde{f}_{2,2}, \hat{p}_1^2\} + \frac{1}{2}\{\tilde{f}_{1,2}, \hat{p}_1 p_2\} + \frac{1}{2}\{\tilde{f}_{0,2}, \hat{p}_2^2\} + \tilde{f}_{0,4}.$$

For the final set of determining equations, there are two that need to be satisfied, instead of one as in the case $N = 3$. They are given by

$$0 = 2\partial_x f_{0,4} - \left(2f_{2,2}\partial_x V + f_{1,2}\partial_y V + \hbar^2 Q_{1,4}\right),$$
$$0 = 2\partial_y f_{0,4} - \left(f_{1,2}\partial_x V + 2f_{0,2}\partial_y V + \hbar^2 Q_{0,4}\right),$$

with quantum corrections that depend on $\hbar^2$. Summarizing the results for $N = 4$, they are essentially the same as $N = 3$.

- There are three families of determining equations.
- The first ensures that the leading order terms are in the enveloping algebra and is the same as in the classical case.
- The second set of equations is the same in the classical and quantum case if the leading term of the quantum integral is of the form $P_3(\hat{p}_1, \hat{p}_2, \hat{L}_3)$.
- The final set of equations (in this case two equations) does have a quantum correction term, linear in $\hbar^2$ which is also linear and homogeneous in the derivatives of the potential V.

$N = 5$

The determining equations for the case $N = 5$ can be summarized as follows.

- There are four families of determining equations.
- The first ensures that the leading order terms are in the enveloping algebra and is the same as in the classical case.
- The second set of equations is the same in the classical and quantum case if the leading term of the quantum integral is of the form $P_3(\hat{p}_1, \hat{p}_2, \hat{L}_3)$.
- There are two remaining sets of determining equations with quantum correction terms of degree $\hbar^2$ and $\hbar^4$, respectively, for the form of the integral with leading order term $P_3(\hat{p}_1, \hat{p}_2, \hat{L}_3)$. Otherwise, the determining equations have quantum correction terms of two degrees higher.
- The fourth family of equations is a single equation given by

$$f_{1,4}V_x + f_{0,4}V_y - Q_{0,4} = 0.$$

Quantizing Classical Operators

In this section, we would like to make a few comments and observations about the relationship between classical and quantum integrals. It is clear from the theorems that, given a quantum integral of motion that commutes with a Hamiltonian H, and assuming that both the potential V and the functions $f_{j,2\ell}$ are independent of $\hbar$, then the classical integral

$$\mathcal{X} = \sum_{\ell=0}^{[\frac{N}{2}]} \sum_{j=0}^{N-2\ell} f_{j,2\ell}, p_1^j p_2^{N-2\ell-j},$$

will be an integral of the motion for a classical Hamiltonian with the same potential.

However, the implication is clearly not reversible. This observation and the general question of quantizing a classical system has received much interest over the years. One of the first questions that appear in the transition from the classical to the quantum case is the choice of symmetrization, assuming cannonical quantization.

Clearly, there are many possible choices of symmetrization. The most general choice would be

$$f_{j,2\ell} p_1^j p_2^{N-2\ell-j} \to S(f_{j,2\ell}, c_{a,b})$$

with

$$S(f_{j,2\ell}, c_{a,b}) \equiv \sum_{a,b} c_{a,b} \left(\hat{p}_1^a \hat{p}_2^{b-a} f_{j,2\ell} \hat{p}_1^{j-a} \hat{p}_2^{N-2\ell-j-b+a} + \hat{p}_1^{j-a} \hat{p}_2^{N-2\ell-j-b+a} f_{j,2\ell} \hat{p}_1^a \hat{p}_2^{b-a} \right)$$

with $\sum_{a,b} c_{a,b} = \frac{1}{2}$. It is true, however, that the choice of symmetrization does not affect the general form of the integral and furthermore the different choices will lead to quantum correction terms in the $f_{j,2\ell}$ as polynomials in $\hbar^2$. In particular, given a choice of symmetrization as above, this choice is equivalent to the standard one with quantum corrections to the coefficient functions. The determining equations for any choice of symmetrization are equivalent to the standard one up to appropriate modifications in the quantum corrections $Q_{j,2\ell}$. The formulas given in Theorem 4 only hold for for the canonical choice.

The following theorem gives these results.

Theorem 7.

Let $f_{j,2\ell} \in C^N(\mathbb{R}^2)$ *be polynomial in* $\hbar^2$*, then the self-adjoint differential operator* $S(f_{j,2\ell}, c_{a,b})$ *can be expressed as*

$$S(f_{j,2\ell}, c_{a,b}) \quad = \quad \frac{1}{2}\{f_{j,2\ell}, \hat{p}_1^j \hat{p}_2^{N-2\ell-j}\} - \hbar^2 \sum_{k=0}^{[N-2\ell]} \left(\frac{1}{2}\{g_{j,2\ell+2k}, \hat{p}_1^j \hat{p}_2^{N-2\ell-2k-j}\} \right),$$

where the $g_{j,2\ell+2k}$ *are polynomial in* $\hbar^2$*.*

Note that the case of quantizing classical integrals would correspond to the functions $f_{j,2\ell}$ being independent of $\hbar^2$, in the theorem. In what follows we give an example of a classical system where the choice of symmetrization is slightly non-intuitive.

Example 1

Let us consider an example of a potential that allows separation of variables in polar coordinates, has a nonzero $\hat{p}_2(\hat{L}_3)^2$ term, and has a non-zero classical limit. This example is in fact second order superintegrable and the third order integral given below can be obtained from the lower order integrals. The potential is

$$V = \frac{a}{\sqrt{x^2+y^2}} + \frac{\alpha_1}{x^2} + \frac{\alpha_2 y}{x^2\sqrt{x^2+y^2}}$$

and it allows separation of variables in polar and parabolic coordinates. The classical third-order integral is given by

$$\mathcal{X} = p_2 L_3^2 + f_{1,2}p_1 + f_{0,2}p_2$$

with

$$f_{1,2} = -\frac{x(ay-\alpha_2)}{2\sqrt{x^2+y^2}}$$

$$f_{0,2} = \frac{ax^2+2\alpha_2 y}{2\sqrt{x^2+y^2}} + \frac{\alpha_2 y^3}{x^2\sqrt{x^2+y^2}} + \frac{\alpha_1(x^2+y^2)}{x^2}.$$

The correct symmetrization which keeps $f_{1,2}$ and $f_{0,2}$ fixed is given by

$$X = \frac{1}{8}(\hat{p}_2\hat{L}_3^2 + 2\hat{L}_3\hat{p}_2\hat{L}_3 + \hat{L}_3^2\hat{p}_2) + \frac{1}{2}\{f_{1,2}, \hat{p}_1\} + \frac{1}{2}\{f_{0,2}, \hat{p}_2\}.$$

This integral can be expressed in the standard form via

$$f_{0,0} = x^2, \qquad f_{1,0} = -2xy, \qquad f_{2,0} = y^2, \qquad f_{3,0} = 0$$

and a quantum correction to $f_{0,2}$ of $7/4\hbar^2$ so that the integral becomes

$$X = \sum_{j=0}^{3} \frac{1}{2}\{f_{j,0}, \hat{p}_1^j\hat{p}_2^{3-j}\} + \frac{1}{2}\{f_{1,2}, \hat{p}_1\} + \frac{1}{2}\{f_{0,2} + \frac{7}{4}\hbar^2, \hat{p}_2\}.$$

Note that the appropriate symmetrization (keeping the lower order-terms free of $\hbar$ dependent terms) is neither $\{\hat{L}_3^2, \hat{p}_2\}$ as in the form generally assumed in previous literature nor simply $\{f_{j,0}, \partial_x^j\partial_y^{3-j}\}$ as above.

Example 2

The following examples are of systems that represent non-trivial quantum corrections to the harmonic oscillator and are particular examples of Marquette and Quesne 2013. These systems are composed of 1D exactly solvable Hamiltonians for exceptional Hermite polynomials discovered and analyzed by Gómez-Ullate, Grandati, and Milson 2014. We give an explicit form for the higher-order integrals and show that they are special cases of quantum potential obtained by Gravel 2004. The first system is

$$H = -\hbar^2 \left(\partial_x^2 + \partial_y^2\right) + \omega^2(x^2 + y^2) + \frac{8\hbar^2\omega(2\omega x^2 - \hbar)}{(2\omega x^2 + \hbar)^2},$$

which admits separation of variables in Cartesian coordinates and hence a second-order integral of the motion

$$X_1 = \hat{p}_2^2 + \omega^2 y^2.$$

Additionally, there are two, third-order integral of the motion given by

$$
\begin{aligned}
X_2 &= \frac{1}{2}\{\hat{L}_3, \hat{p}_1\hat{p}_2\} + \{\omega^2 x^2, \hat{L}_3\} \\
&+\hbar\left(\{2\omega y + \frac{12\hbar\omega y(2\omega x^2 - \hbar)}{(2\omega x^2 + \hbar)^2}, \hat{p}_1\} - \{2\omega x + \frac{12\hbar\omega y(2\omega x^2 - \hbar)}{(2\omega x^2 + \hbar)^2}, \hat{p}_2\}\right),
\end{aligned}
$$

$$
\begin{aligned}
X_3 &= \hat{L}_3^3 + \frac{3\hbar}{2\omega}\left[\frac{1}{2}\{\hat{L}_3, X_1\}\right. \\
&\qquad + \hbar\left(2\hat{L_3} - \{\frac{8\hbar\omega^2 y^3(2\omega x^2 + \hbar)}{(2\omega x^2 + \hbar)^2}, \hat{p_1}\} + \{\frac{8\hbar\omega^2 xy^2(2\omega x^2 + 3\hbar)}{(2\omega x^2 + \hbar)^2},\right.
\end{aligned}
$$

This system can thus be considered as a quantum deformation of the harmonic oscillator. As demonstrated by the existence of the third-order integrals, this system falls in the classification of Gravel, namely this potential is equivalent to V_e in the classification.

Another example is based on the fourth Hermite polynomial. The associated superintegrable Hamiltonian is given by

$$H = -\hbar^2\left(\partial_x^2+\partial_y^2\right)+\omega^2(x^2+y^2)+\frac{768\hbar^4\omega^2x^2}{\left(4\,\omega^2x^4+12\,\omega\,x^2\hbar+3\,h^2\right)^2}+\frac{16\hbar^2\omega\ \left(2\,\omega\,x^2\right.}{4\,\omega^2x^4+12\,\omega\,x^2\hbar}$$

The Hamiltonian admits the same second-order integral as the previous one, as well as the following third-order integral

$$\begin{aligned} X_2 \quad = \quad & \{\hat{L}_3,\hat{p}_1^2\}+\{\omega^2x^2,\hat{L}_3\} \\ & +\hbar\left[6\omega\hat{L}_3-\left\{\frac{24\hbar\omega y(8\omega^2x^6+12\hbar\omega^2x^4+18\hbar^2\omega x^2-9\hbar^3)}{4\omega^2x^4+12\hbar\omega x^2+3\hbar^2)^2},\hat{p}_1\right\}\right. \\ & \left.\qquad +\left\{\frac{8\hbar\omega y(8\omega^2x^6-12\hbar\omega^2x^4-6\hbar^2\omega x^2-27\hbar^3)}{4\omega^2x^4+12\hbar\omega x^2+3\hbar^2)^2},\hat{p}_2\right\}\right]. \end{aligned}$$

Unlike the previous case, this potential is not immediately recognizable in Gravel's classification. A remarkable fact is that this potential is associated with the fourth Painlevé equation. Indeed, the x-dependent part of the potential $W(x) = V(x, y) - \omega^2 y^2 + 4\omega\hbar$ satisfies

$$-\hbar^2 W^{(4)} - 12\omega^2 (xW)' + 3(W^2)'' - 2\omega^2 x^2 W'' + 4\omega^4 x^2 = 0,$$

which is equivalent to the relevant equation given by Gravel. It can also be shown that the potential of the previous Hamiltonian is also a solution of this non-linear equation. Thus, these two systems whose wave functions are given by exceptional Hermite polynomials have potentials that can be expressed in terms of rational solutions to the fourth Painlevé equation. Of course, many such particular solutions to the Painlevé equations exist but their connection to exceptional orthogonal polynomials as well as the harmonic oscillator is quite remarkable and will be investigated in future work.

Concluding Remarks

The main results are summed up in Theorems 1 and 4. They present the determining equations for the coefficients of an Nth order integral of the motion X in the Euclidean plane E_2 in classical and quantum mechanics, respectively. Both the similarities and differences between the two cases are striking.

- The number of determining equations $M_{j,2\ell} = 0$ to solve and the number of coefficient functions $F_{j,2\ell}(x, y)$ to determine is the same in the classical and quantum cases.
- If the potential $V(x, y)$ is known, the determining equations are linear. If the potential is not known a priori then in both cases we have a coupled system of nonlinear PDE for the potential $V(x, y)$ and the coefficients $f_{j,2\ell}$ $(0 \leq \ell \leq \lfloor \frac{N}{2} \rfloor$ $0 \leq j \leq N - 2\ell)$.
- In both cases the functions $f_{j,2\ell}$ are real, if the potential is real and the quantum integral is assumed to be a Hermitian operator.
- In both cases the integral X contains only terms of the same parity as the leading terms (obtained for $\ell = 0$).
- The leading terms in the integral lie in the enveloping algebra of the Euclidean Lie algebra.

- For $N \geq 3$ the quantum determining equations with $\ell \geq 1$ have quantum corrections.
- For $\ell = 1$ the determining equations $M_{j,2} = 0$ for j in the interval $0 \leq j \leq N$ must satisfy a compatibility condition. This linear compatibility condition for the potential to allow and Nth order integral is the same in the classical and quantum case.
- The determining equations for the classical case and the quantum case for $\ell \geq 2$ will differ. Moreover, new compatibility conditions on the potential arise for each higher value of ℓ. They will be nonlinear equations for $V(x, y)$ and will be considerably more complicated in the quantum case than in the classical one (see Gravel &W 2002, Gravel 2004, Marquette & W 2008, Tremblay & W 2010, Marchesiello, Popper, Post & W 2012, Post & Šnobl 2014 for the case $N = 3$ and Post & W 2015 for $N \geq 3$).

- The determining for $f_{j,0}$ have been solved for general N (there are $N+2$ such equations). Even so, solving the determining equations for $N>2$ is a formidable task, even for $N=3$. A much more manageable task is to use the determining equations in the context of superintegrability. In a two-dimensional space a Hamiltonian system is superintegrable if it allows two integrals of motion X and Y, in addition to the Hamiltonian. They satisfy $[X, H] = [Y, H] = 0$. Assuming that X and Y are polynomials in the momenta and that the system considered is defined on E_2, both will have the form studied in this article. The integrals H, X and Y are assumed to be polynomially independent. The integrals X and Y do not commute, $[X, Y] \neq 0$, and hence generate a non-Abelian polynomial algebra of integrals of the motion.
- The case that has recently been the subject of much investigation is that when one of the integrals is of order one or two and hence the potential will have a specific form that allows separation of variables in the Hamilton-Jacobi and Schrödinger equations. The potential $V(x, y)$ will then be written in terms of two functions of one variable each, the variables being either Cartesian, polar, parabolic or elliptic coordinates. Once such a potential is inserted into the determining equations, they become much more manageable.

- For $N = 3$ and $N = 4$ the assumption of separation of variables at least for Cartesian and polar coordinates leads to new superintegrable systems. In classical mechanics these potentials are expressed in terms of elementary functions or solutions of algebraic equations. In quantum mechanics one also obtains “exotic potentials" that do not satisfy any linear PDE, i.e. the linear compatibility condition is solved trivially. These exotic potentials are expressed in terms of elliptic functions or Painlevé transcendents. See citations above as well as I. Marquette, Sajedi & Winternitz 2015.
- This has so far been done systematically for $N = 3$ and $N = 4$. The formalism presented here makes it possible to investigate superintegrable separable potentials for all N.

- Another application of the presented formalism is to make assumptions about the form of the potential and then look for possible integrals of the motion. Hypotheses about integrability or superintegrability of a given potential can then be verified (or refuted) by solving a system of linear PDEs.
- Alternative approaches to the construction of superintegrable systems in two or more dimensions exist. In quantum mechanics they typically start from a one-dimensional Hamiltonian $H_1 = p_1^2 + V_1(x)$. See e.g. work of I. Marquette 2012, 2013... and Gungor, Kuru, Negro & Nieto 2014.
- There are many avenues open for further research. In addition to the already discussed applications to Nth order superintegrability already in progress, we mention the extension of the theory to spaces of non-zero curvature and to higher dimensions.

Thank You For Your Attention.

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