Superintegrable systems with spin

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Let us first consider a *classical* system in an n-dimensional Riemannian space with Hamiltonian

$$H = \sum_{i,k=1}^{n} g_{ik} p_i p_k + V(\vec{x}) \, , \vec{x} \in \mathbb{R}^n \tag{1}$$

The system is called *integrable* (or Liouville integrable) if it allows $n-1$ integrals of motion (in addition to H)

$$\begin{aligned} X_a &= f_a(\vec{x}, \vec{p}) \, , \qquad a = 1, \ldots, n-1 \\ \frac{dX_a}{dt} &= \{H, X_a\} = 0 \, , \{X_a, X_b\} = 0 \end{aligned} \tag{2}$$

This system is *superintegrable* if it allows further integrals

$$\begin{aligned} Y_b = f_b(\vec{x}, \vec{p}) \, , \qquad b = 1, \ldots, k \qquad 1 \leq k \leq n-1 \\ \frac{dY_b}{dt} = \{H, Y_b\} = 0 \, . \end{aligned} \tag{3}$$

The integrals must satisfy

1. The integrals H, X_a, Y_b are well defined functions on phase space, i.e. polynomials or convergent power series on phase space (or an open submanifold of phase space).
2. The integrals H, X_a are in involution, i.e. Poisson commute as indicated in (3). The integrals Y_b Poisson commute with H but not necessarily with each other, nor with X_a.
3. The entire set of integrals is functionally independent, i.e., the Jacobian matrix satisfies

$$\mathrm{rank}\frac{\partial(H, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_k)}{\partial(x_1, \ldots, x_n, p_1, \ldots, p_n)} = n + k \tag{4}$$

In *quantum mechanics* we define integrability and superintegrability in the same way, however in this case, H, X_a and Y_b are operators.

The condition on the integrals of motion must also be modified e.g. as follows :

1. H, X_a and Y_b are well defined Hermitian operators in the enveloping algebra of the Heisenberg algebra $H_n \sim \{\vec{x}, \vec{p}, \hbar\}$ or some generalization thereof.
2. The integrals satisfy the Lie bracket relations

$$[H, X_a] = [H, Y_b] = 0\,, [X_i, X_k] = 0 \tag{5}$$

3. No polynomial in the operators H, X_a, Y_b formed entirely using Lie anticommutators should vanish identically.

The two best known superintegrable systems are the Kepler-Coulomb system with potential $V(r) = \frac{\alpha}{r}$ and the isotropic harmonic oscillator $V(r) = \alpha r^2$. In both cases the integrals X_a correspond to angular momentum, the additional integrals Y_a to the Laplace-Runge-Lenz vector for $V(r) = \frac{\alpha}{r}$ and to the quadrapole tensor $T_{ik} = p_i p_k + \alpha x_i x_k$, respectively. No further ones were discovered until a 1940 paper by Jauch and Hill on the rational anisotropic harmonic oscillator $V(\vec{x}) = \alpha \sum_{i=1}^{n} n_i x_i^2$, $n_i \in \mathbb{Z}$.

A systematic search for superintegrable systems was started in 1965 and a real proliferation of them was observed during the last couple of years.

Let us just list some of the reasons why superintegrable systems are interesting both in classical and quantum physics.

In *classical* mechanics, superintegrability restricts trajectories to an $n-k$ dimensional subspace of phase space.

- For $k = n-1$ (maximal superintegrability), this implies that all finite trajectories are closed and motion is periodic.
- Moreover, at least in principle, the trajectories can be calculated without any calculus.
- Bertrand's theorem states that the only spherically symmetric potentials $V(r)$ for which all bounded trajectories are closed are $\frac{\alpha}{r}$ and αr^2, hence no other superintegrable systems are spherically symmetric.

- The algebra of integrals of motion $\{H, X_a, Y_b\}$ is a non-Abelian and interesting one. Usually it is a finitely generated polynomial algebra, only exceptionally a finite dimensional Lie algebra. In the special case of quadratic superintegrability (all integrals of motion are at most quadratic polynomials in the moments), integrability is related to separation of variables in the Hamilton-Jacobi equation, or Schrödinger equation, respectively.

In *quantum* mechanics,

- superintegrability leads to an additional degeneracy of energy levels, sometimes called "accidental degeneracy". The term was coined by Fok and used by Moshinsky and collaborators, though the point of their studies was to show that this degeneracy is certainly no accident.

- A conjecture, born out by all known examples, is that all maximally superintegrable systems are exactly solvable. If the conjecture is true, then the energy levels can be calculated algebraically. The wave functions are polynomials (in appropriately chosen variables) multiplied by some gauge factor.
- The non-Abelian polynomial algebra of integrals of motion provides energy spectra and information on wave functions. Interesting relations exist between superintegrability and supersymmetry in quantum mechanics.

As a comment, let us mention that superintegrability has also been called non-Abelian integrability. From this point of view, infinite dimensional integrable systems (soliton systems) described e.g. by the Korteweg-de-Vries equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili equation, etc. are actually superintegrable.

Indeed, the generalized symmetries of these equations form infinite dimensional non-Abelian algebras (the Orlov-Shulman symmetries) with infinite dimensional Abelian subalgebras of commuting flows.

The purpose of this talk is twofold. First, we present a method for generating superintegrable systems with a spin-orbital interaction in three-dimensional Euclidean space $\mathbb{E}_3$ from superintegrable scalar systems in $\mathbb{E}_2$. The method starts with a Hamiltonian of the form

$$H^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + V(\rho), \quad \rho = \sqrt{x_1^2 + x_2^2} \tag{6}$$

and uses coalgebra symmetry to generate systems of the form

$$H^{(3)} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V_0(r) + V_1(r)(\vec{\sigma}, \vec{L}) \tag{7}$$

where

$$p_k = -i\hbar\frac{\partial}{\partial x_k}, \quad L_k = \epsilon_{kab} x_a p_b, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{8}$$

and σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

Secondly, we use this coalgebra method to derive a maximally superintegrable system with the Hamiltonian

$$H = \frac{-\hbar^2}{2}\nabla^2 + \frac{2\gamma}{r^2}\vec{S}\cdot\vec{L} - \frac{\alpha}{r} + \frac{\hbar^2\gamma(\gamma+1)}{2r^2}, \quad \vec{S} = \frac{\hbar}{2}\vec{\sigma} \tag{10}$$

where α and γ are arbitrary constants. This Hamiltonian is integrable because it is spherically symmetric and hence the angular momentum

$$\vec{\mathcal{J}} = \vec{L} + \vec{S} \tag{11}$$

is an integral of the motion.

We shall show below that the system is also superintegrable. The additional integrals of motion are the components of a vector that is in general a third order polynomial in the momenta (a third order Hermitian operator). The Hamiltonian (10) can be viewed as describing the Coulomb interaction of a particle with spin $\frac{1}{2}$ with another of spin 0

A two-dimensional superintegrable Coulomb system with a velocity dependent potential

Let us start from the quantum Coulomb system in $\mathbb{E}_2$ in polar coordinates

$$\hat{H}_0^{(2)} = -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2\right) - \frac{\alpha}{r} \tag{12}$$

Its symmetry algebra is generated by the angular momentum $\hat{L}^2$ and the Laplace-Runge-Lenz vector $\vec{\mathcal{R}}$ with:

$$\hat{L}_0^{(2)} = -i\hbar\partial_\phi; \tag{13}$$

$$\mathcal{R}_{1,0} = \frac{1}{2}\left(\hat{p}_2\hat{L}_0^{(2)} + \hat{L}_0^{(2)}\hat{p}_2\right) - \frac{\alpha x_1}{r}; \mathcal{R}_{2,0} = -\frac{1}{2}\left(\hat{p}_1\hat{L}_0^{(2)} + \hat{L}_0^{(2)}\hat{p}_1\right) - \frac{\alpha x_2}{r}.$$

Let us use a gauge transformation $U(1) = e^{i\gamma\phi}$ to transform the superintegrable system (12) (13) into a velocity dependent Coulomb system that is also superintegrable, namely

$$e^{-i\gamma\phi}\hat{H}_0^{(2)}e^{i\gamma\phi} \equiv \hat{H}^{(2)} \tag{14}$$

$$\hat{H}^{(2)} = -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2\right) + \frac{\hbar\gamma}{r^2}(-i\hbar\partial_\phi) + \frac{\hbar^2\gamma^2}{2r^2} - \frac{\alpha}{r} \tag{15}$$

$$\begin{aligned}
\hat{L}^{(2)} &= \hbar(-i\partial_\phi + \gamma) \\
\hat{\mathcal{R}}_1^{(2)} &= \frac{1}{2}\left(\hat{p}_2\hat{L}^{(2)} + \hat{L}^{(2)}\hat{p}_2\right) + \frac{i\hbar^2\gamma x_2}{2r^2} + \frac{\hbar\gamma x_1}{r^2}\hat{L}^2 - \frac{\alpha x_1}{r} \qquad (16) \\
\hat{\mathcal{R}}_2^{(2)} &= -\frac{1}{2}\left(\hat{p}_1\hat{L}^{(2)} + \hat{L}^{(2)}\hat{p}_1\right) - \frac{i\hbar^2\gamma x_1}{2r^2} + \frac{\hbar\gamma x_2}{r^2}\hat{L}^2 - \frac{\alpha x_2}{r}.
\end{aligned}$$

The momentum (or velocity) dependent Hamiltonian (14) can be rewritten as

$$\hat{H}^{(2)} = -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r\right) - \frac{\alpha}{r} + \frac{1}{2r^2}\left(\hat{L}^{(2)}\right)^2. \tag{17}$$

We separate variables in the usual manner putting

$$\hat{H}^{(2)}\psi = E\psi \quad , \quad \hat{L}^{(2)}\psi = \hbar(m+\gamma)\psi \quad , \quad \psi = R_{Em}(r)e^{im\phi} \tag{18}$$

and obtain the radial equation

$$\begin{aligned} \hat{H}_m^{(2)} R_{E,m}(r) &= E R_{E,m}(r) \qquad (19) \\ \hat{H}_m^{(2)} &= -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{1}{r}\partial_r\right) - \frac{\alpha}{r} + \frac{\hbar^2}{2r^2}(m+\gamma)^2 . \end{aligned}$$

Let us introduce the "radial" ladder operators

$$\hat{a}_m^\dagger = \frac{1}{\sqrt{2}}\left(-i\hbar\partial_r + \frac{i\alpha}{\hbar(m+\frac{1}{2}+\gamma)} - i\hbar\frac{m+\gamma+1}{r}\right) \tag{20}$$

$$\hat{a}_m = \frac{1}{\sqrt{2}}\left(-i\hbar\partial_r - \frac{i\alpha}{\hbar(m+\frac{1}{2}+\gamma)} + i\hbar\frac{m+\gamma}{r}\right) \tag{21}$$

satisfying

$$\hat{a}_m^\dagger \hat{a}_m = \hat{H}_m^{(2)} + \frac{\alpha^2}{2\hbar^2(m+\frac{1}{2}+\gamma)^2}. \tag{22}$$

The Hamiltonian (19) satisfies

$$\hat{a}_m \hat{H}_m^{(2)} = \hat{H}_{m+1}^{(2)} \hat{a}_m \tag{23}$$

$$\hat{H}_m^{(2)} \hat{a}_m^\dagger = \hat{a}_m^\dagger \hat{H}_{m+1}^{(2)} \tag{24}$$

and can hence be viewed as being shape invariant. We shall sometimes replace the eigenvalue $\hbar(m+\gamma)$ by the differential operator $\hat{L}^{(2)}$.

To do this without introducing negative powers of differential operators we redefine the ladder operators as

$$\hat{A}^\dagger(\hat{L}) \equiv i\hat{a}^\dagger(\hat{L})(\hat{L}^{(2)} + \frac{\hbar}{2})$$

$$\hat{A}^\dagger(\hat{L}) = \frac{1}{\sqrt{2}}\left(\hbar(\hat{L}^{(2)} + \frac{\hbar}{2})\partial_r - \alpha + \frac{(\hat{L}^{(2)} + \frac{\hbar}{2})(\hat{L}^{(2)} + \hbar)}{r}\right)$$

$$\hat{A}(\hat{L}) \equiv -i\hat{a}(\hat{L})(\hat{L}^{(2)} + \frac{\hbar}{2})$$

$$\hat{A}(\hat{L}) = \frac{1}{\sqrt{2}}\left(-\hbar(\hat{L}^{(2)} + \frac{\hbar}{2})\partial_r - \alpha + \frac{(\hat{L}^{(2)} + \frac{\hbar}{2})(\hat{L}^{(2)})}{r}\right)$$

These operators satisfy

$$\hat{A}(\hat{L}^{(2)})\hat{H}^{(2)}(\hat{L}^{(2)}) = \hat{H}^{(2)}(\hat{L}^{(2)} + \hbar)\hat{A}(\hat{L}^{(2)}) \tag{25}$$

$$\hat{H}^{(2)}(\hat{L}^{(2)})\hat{A}^\dagger(\hat{L}^{(2)}) = \hat{A}^\dagger(\hat{L}^{(2)})\hat{H}^{(2)}(\hat{L}^{(2)} + \hbar) \tag{26}$$

We also define the ladder operators for the angular momentum, namely

$$L^+ = e^{i\phi} \quad ; \quad \hat{L}^{(2)} L^+ = L^+ (\hat{L}^{(2)} + \hbar) \tag{27}$$
$$L^- = e^{-i\phi} \quad ; \quad \hat{L}^{(2)} L^- = L^- (\hat{L}^{(2)} - \hbar) \tag{28}$$

The $\mathbb{E}_2$ Laplace-Runge-Lenz vector (16) is now expressed as

$$X \equiv \vec{\mathcal{R}}_1^{(2)} = \frac{1}{\sqrt{2}} \left(L^+ \hat{A}(\hat{L}) + \hat{A}^\dagger \hat{L}^- \right) \tag{29}$$
$$Y \equiv \vec{\mathcal{R}}_2^{(2)} = \frac{1}{i\sqrt{2}} \left(L^+ \hat{A}(\hat{L}) - \hat{A}^\dagger \hat{L}^- \right) \tag{30}$$

Finally, the superintegrable system that we are going to apply the co-algebra symmetry to is given by the Hamiltonian (17) and the integrals (16).

They form a symmetry algebra satisfying

$$\left[\hat{H}^{(2)}, \hat{\mathcal{R}}_1^{(2)}\right] = \left[\hat{H}^{(2)}, \hat{L}^{(2)}\right] = \left[\hat{H}^{(2)}, \hat{\mathcal{R}}_2^{(2)}\right] = 0 \qquad (31)$$

$$\left[\hat{\mathcal{R}}_1^{(2)}, \hat{L}^{(2)}\right] = -i\hbar\hat{\mathcal{R}}_2^{(2)} \qquad (32)$$

$$\left[\hat{\mathcal{R}}_2^{(2)}, \hat{L}^{(2)}\right] = i\hbar\hat{\mathcal{R}}_1^{(2)} \qquad (33)$$

$$\left[\hat{\mathcal{R}}_1^{(2)}, \hat{\mathcal{R}}_2^{(2)}\right] = -2i\hbar\hat{L}^{(2)}\hat{H}(\hat{L}). \qquad (34)$$

For $H(\hat{L}) = E < 0$ this algebra is isomorphic to $o(3)$.

Let us first introduce an abstract $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra with basis $\hat{J}_3, \hat{J}_+, \hat{J}_-$ and commutation relations

$$[\hat{J}_3, \hat{J}_+] = 2i\hbar\hat{J}_+ \quad ; \quad [\hat{J}_3, \hat{J}_-] = -2i\hbar\hat{J}_- \quad ; \quad [\hat{J}_-, \hat{J}_+] = 4i\hbar\hat{J}_3 \tag{35}$$

we equip this algebra with a trivial coproduct Δ defined by

$$\Delta(1) = 1 \otimes 1 \quad ; \quad \Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i \quad , \quad i = +, -, 3 \tag{36}$$

The action of the coproduct defines an isomorphism for the algebra

$$[\Delta(\hat{J}_3), \Delta(\hat{J}_+)] = 2i\hbar\Delta(\hat{J}_+) \ ; \ [\Delta(\hat{J}_3), \Delta(\hat{J}_-)] = -2i\hbar\Delta(\hat{J}_-)$$
$$[\Delta(\hat{J}_-), \Delta(\hat{J}_+)] = 4i\hbar\Delta(\hat{J}_3)$$

The coproduct can be used to generate multivariable realizations of a Lie algebra from single variable ones. We start from a single variable representation of $\mathfrak{sl}(2)$, namely

$$D(\hat{J}_+) = -\hbar^2 \partial_x^2 \quad ; \quad D(\hat{J}_-) = x^2;$$

$$D(\hat{J}_3) = -i\hbar(x\partial_x + \frac{1}{2})$$

Let us define an iteration of the coproduct $\hat{J}_i^{(n)} = \Delta(\Delta(...\Delta(\hat{J}_i)))$. The n-variable realization of $\mathfrak{sl}(2)$ is given by

$$D(\hat{J}_+^{(n)}) = -\hbar^2 \sum_{i=1}^{n} \partial_{x_i}^2 \quad ; \quad D(\hat{J}_-^{(n)}) = \sum_{i=1}^{n} x_i^2; \tag{37}$$

$$D(\hat{J}_3^{(n)}) = -i\frac{n\hbar}{2} - i\hbar \sum_{i=1}^{n} (x_i \partial_{x_i})$$

The coalgebra symmetry ensures that the algebra elements $\hat{J}_i^{(n)}, (i = +, -, 3)$ commute with a set of operators obtained by a k fold ($2 \leq k \leq n$) "left" or "right" application of the coproduct Δ :

$$\left[\hat{J}_{+,-,3}, \hat{\mathcal{C}}\right] = 0, \quad \hat{\mathcal{C}} = \frac{1}{2}\left(\hat{J}_-\hat{J}_+ + \hat{J}_+\hat{J}_-\right) - \hat{J}_3^2 \tag{38}$$

$$\hat{\mathcal{C}}^{(i)} = \Delta(\mathcal{C})^i \otimes \underbrace{1 \otimes ...1}_{n-i} \tag{39}$$

$$\hat{\mathcal{C}}_{(i)} = \underbrace{1 \otimes ...1}_{n-i} \otimes \Delta(\mathcal{C})^i \tag{40}$$

The operators $\mathcal{C}^{(k)}$ and $\mathcal{C}_{(k)}$ can be viewed as Casimir operators of $o(k)$ subalgebras of the $o(n)$ algebra of angular momentum. Since $\mathcal{C}^{(n)} = \mathcal{C}_{(n)}$ we have $2n - 3$ Casimirs $\mathcal{C}^{(2)}, ..., \mathcal{C}^{(n)}, \mathcal{C}_{(2)}, .., \mathcal{C}_{(n-1)}$.

Let us now return to the realization of $\mathfrak{sl}(2,\mathbb{R})$ (35) . We rewrite the ladder operators A , $A^\dagger$ and the Hamiltonian H in terms of the generators J_k of $\mathfrak{sl}(2,\mathbb{R})$ and its Casimir $\hat{\mathcal{C}}$. We shall use the n variable realization $J_k^{(n)}$

$$(\hat{L}^{(n)})^2 \equiv \hat{\mathcal{C}}^{(n)} + \hbar^2 \tag{41}$$

$$\hat{A}^{(n)} \equiv \frac{-1}{\sqrt{2}}\left(i\frac{(\hat{L}^{(n)}+\hbar\gamma+\frac{\hbar}{2})}{\sqrt{J_-^{(n)}}}\hat{J}_3^{(n)}+\alpha-\frac{(\hat{L}^{(n)}+\hbar\gamma+\frac{\hbar}{2})(\hat{L}^{(n)}+\hbar\gamma+\hbar)}{\sqrt{J_-^{(n)}}}\right)$$

$$\hat{A}^{\dagger(n)} \equiv \frac{1}{\sqrt{2}}\left(i\frac{(\hat{L}^{(n)}+\hbar\gamma+\frac{\hbar}{2})}{\sqrt{J_-^{(n)}}}\hat{J}_3^{(n)}-\alpha+\frac{(\hat{L}^{(n)}+\hbar\gamma+\frac{\hbar}{2})(\hat{L}^{(n)}+\hbar\gamma)}{\sqrt{J_-^{(n)}}}\right)$$

$$\hat{H} \equiv \frac{1}{2}\left(\frac{1}{J_-^{(n)}}(\hat{J}_3^{(n)}+i\hbar)^2+\frac{(\hat{L}+\hbar\gamma)^2}{\hat{J}_-^{(n)}}\right)-\frac{\alpha}{\sqrt{\hat{J}_-^{(n)}}} \tag{42}$$

For $n=2$ this coincides with the formulas of Section 2. Thus $\hat{L}^{(2)}$ in (41) coincides with $\hat{L}_0^{(2)}$ in (13) , $\hat{A}^{(n)}$ and $\hat{A}^{\dagger(n)}$ coincide with (25 , 25) .The Hamiltonian (42) reduces to (17) for $n=2$. Equation (41) defines $(\hat{L}^{(n)})^2$ rather than $\hat{L}^{(n)}$ itself. For $n=2$ this is no problem since we have $\hat{L}_0^{(2)}=-i\hbar\partial_\phi$ which is the square root of $\mathcal{C}^{(2)}+\hbar^2=-\hbar^2\partial_\phi^2$. In Section 4 we shall need $\hat{L}^{(3)}=\sqrt{\mathcal{C}^{(3)}+\hbar^2}$ and obtain it as a linear operator satisfying $[J_k,\hat{L}^{(n)}]=0(k=+,-,3)$ For any analytical function $F(z)$ we have also the commutation relations

$$[J_k,F(\hat{L^{(n)}})]=0,\quad [J_3,F(J_\pm)]=\pm 2i\hbar J_\pm F'(J_\pm) \tag{43}$$

and hence

$$\begin{aligned}\hat{A}^{(n)}\hat{H}(\hat{J}_-^{(n)},\hat{J}_3^{(n)},\hat{L}^{(n)}) &= \hat{H}(\hat{J}_-^{(n)},\hat{J}_3^{(n)},\hat{L}^{(n)}+\hbar)\hat{A}^{(n)} & (44)\\ \hat{H}(\hat{J}_-^{(n)},\hat{J}_3^{(n)},\hat{L}^{(n)})\hat{A}^{\dagger(n)} &= \hat{A}^{\dagger(n)}\hat{H}(\hat{J}_-^{(n)},\hat{J}_3^{(n)},\hat{L}^{(n)}+\hbar) & (45)\end{aligned}$$

We wish to apply the coalgebra formalism to objects that are not necessarily purely radial, such as for instance the "angular" ladder operators $\hat{L}^+$, $\hat{L}^-$. To achieve this we extend the algebra $sl(2,\mathbb{R})$ to a semidirect product with the Heisenberg algebra. More specifically we take the n-variable realization of $\mathfrak{sl}(2)$ (37), obtained from the single variable one via the iterated co-product and extend it by the Heisenberg algebra $\mathbb{H}_n$ in $\mathbb{E}_n$ .We put

$$\left[\hat{J}_+^{(n)}, x_k\right] = -2i\hbar\hat{p}_k \quad ; \quad \left[\hat{J}_+^{(n)}, \hat{p}_k\right] = 0 \tag{46}$$

$$\left[\hat{J}_-^{(n)}, x_k\right] = 0 \quad ; \quad \left[\hat{J}_-^{(n)}, \hat{p}_k\right] = 2i\hbar x_k \tag{47}$$

$$\left[\hat{J}_3^{(n)}, x_k\right] = -i\hbar x_k \quad ; \quad \left[\hat{J}_3^{(n)}, \hat{p}_k\right] = i\hbar\hat{p}_k \tag{48}$$

$$[\hat{p}_k, x_k] = -i\hbar\delta_{k,l} \tag{49}$$

The angular ladder operators (27) (28) can be expressed in terms of the $\mathfrak{sl}(2) \ni \mathbb{H}_n$ operators as

$$\hat{L}_k^{-(n)} \equiv \frac{x_k}{\sqrt{\hat{J}_-^{(n)}}}\hat{L}^{(n)} - \frac{i}{\sqrt{\hat{J}_-^{(n)}}}\left(x_k \hat{J}_3^{(n)} - \hat{J}_-^{(n)}\hat{p}_k + i\hbar x_k\right) \quad (50)$$

$$\hat{L}_k^{+(n)} \equiv \hat{L}^{(n)}\frac{x_k}{\sqrt{\hat{J}_-^{(n)}}} + \frac{i}{\sqrt{\hat{J}_-^{(n)}}}\left(x_k \hat{J}_3^{(n)} - \hat{J}_-^{(n)}\hat{p}_k\right) \quad (51)$$

and they satisfy

$$(\hat{L}^{(n)})^2\hat{L}^- = \hat{L}^-(\hat{L}^{(n)} - \hbar)^2 \quad (52)$$

$$(\hat{L}^{(n)})^2\hat{L}^+ = \hat{L}^+(\hat{L}^{(n)} + \hbar)^2. \quad (53)$$

The three dimensional system with spin and its third order integrals

Because of the coalgebra symmetry we can state that the system $\hat{H}^{(3)}$ commutes with the Casimirs $\hat{\mathcal{C}}^{(2)}, \hat{\mathcal{C}}_{(2)}, \hat{\mathcal{C}}^{(3)}$ and furthermore with the three components of

$$X_j = \frac{1}{\sqrt{2}} \left(\hat{L}_j^{+(3)} \hat{A}^{(3)} + \hat{A}^{(3)\dagger} \hat{L}_j^{-(3)} \right), \quad 1 \leq j \leq 3 \tag{54}$$

$$Y_j = \frac{1}{i\sqrt{2}} \left(\hat{L}_j^{+(3)} \hat{A}^{(3)} - \hat{A}^{(3)\dagger} \hat{L}_j^{-(3)} \right), \quad 1 \leq j \leq 3 \tag{55}$$

Equations (54) , (55) are three dimensional analogs of (29), (30), however X_j, Y_j are third order differential operators whereas X and Y are second order ones. The operators $\hat{H}^{(3)}$ and X_j, Y_j depend linearly on the operator $\hat{L}^{(3)}$. Its square was defined in (41). We need a realization of the operator $\hat{L}^{(3)}$ itself in order to obtain the three dimensional Hamiltonian $\hat{H}^{(3)}$ and its integrals of motion X_j, Y_j.

For a two dimensional system this is not an issue. In the 3-dimensional case $(\hat{L}^{(3)})^2$ turns out to be:

$$(\hat{L}^{(3)})^2 = \hat{\mathcal{C}}^{(3)} + \hbar^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 + \frac{\hbar^2}{4} \tag{56}$$

As in the case of the Dirac equation, the square root of a sum of squared differential operators can be computed considering this operation on a space of anticommuting matrices, which in this case are the Pauli sigma matrices (9).
They allow us to introduce the following representation of $\hat{L}^{(3)}$

$$\hat{L}^{(3)} = \sqrt{\hat{\mathcal{C}}^{(3)} + \hbar^2} = \sigma_i \hat{L}_i + \frac{\hbar}{2} \tag{57}$$

It can be verified by a direct calculation that $\hat{L}_i^{\pm}$ satisfy

$$[\hat{L}^{(3)}, \hat{L}_i^{(\pm)}] = \pm\hbar \hat{L}_i^{\pm} \tag{58}$$

This leads directly to the representation (10) of $\hat{H}^{(3)}$. The relations $[X_j, H] = [Y_j, H] = 0$ can be verified directly or by using the algebraic relations (44, 45, 52, 53, 58) . The operators $\hat{L}_j^{+(3)}$ and $\hat{L}_j^{-(3)}$ are the three dimensional versions of the operators defined in (50), (51), and $\hat{A}^{(3)}$, $\hat{A}^{\dagger(3)}$of those defined in (41)-(42). Similarly as in the case of the Dirac equation, the calculation of the square root of a Casimir operator leads to the introduction of a spin term, in this case the spin-orbital interaction in (10). As mentioned in the introduction, a Hamiltonian of the form (10) was obtained in [OWY] as part of a systematic search for superintegrable systems with second order integrals of motion. This lead to (10) with $\gamma = \frac{1}{2}$. Using the coalgebra symmetry approach we have obtained a more general result . The price of this generality is that the additional vector integrals of motion are third order operators $\vec{X}, \vec{Y}$ that reduce to second order ones for $\gamma = \frac{1}{2}$ or 1.

Finally let us give an explicit representation for the constant of motion $\vec{X}$ which can be regarded as the generalization of the Laplace-Runge-Lenz vector for the hydrogen atom

$$\vec{X} = \frac{1}{2}((\vec{L}\cdot\sigma)\vec{\mathcal{A}} + \vec{\mathcal{A}}(\vec{L}\cdot\sigma)) + \hbar\vec{\mathcal{A}} \tag{59}$$

$$\hat{\vec{\mathcal{A}}} \equiv \frac{1}{2}\left(\hat{\vec{p}}\wedge\hat{\vec{L}} - \hat{\vec{L}}\wedge\hat{\vec{p}}\right) + 2\gamma\vec{p}\wedge\vec{S} + \frac{1}{2}\left(\vec{x}\hat{\mathcal{V}} + \hat{\mathcal{V}}\vec{x}\right) \tag{60}$$

$$\hat{\mathcal{V}} \equiv -\frac{\alpha}{r} + \frac{2\gamma}{r^2}\vec{L}\cdot\vec{S} + \frac{\hbar^2\gamma(2\gamma+1)}{2r^2} \tag{61}$$

The algebra generated by the set of constants of the motion for $\hat{H}_G^{(3)}$ defines a closed polynomial algebra under the operation of commutation.

It is easy to compute this algebra if we consider the following fundamental identities:

$$
\begin{aligned}
\mathcal{J}_j &= {}_j + \frac{\hbar}{2}\sigma_j \\
L_j^- L_k^+ &= -\frac{1}{2}\left(\mathcal{J}_j\mathcal{J}_k + \mathcal{J}_k\mathcal{J}_j\right) - i\epsilon_{jkl}\mathcal{J}_l(\vec{L}\cdot\vec{\sigma} + 2\hbar) + \\
&\quad +\delta_{jk}(\mathcal{J}^2 + \hbar(\vec{L}\cdot\vec{\sigma} + \frac{3}{2}\hbar)) \\
L_j^+ L_k^- &= -\frac{1}{2}\left(\mathcal{J}_j\mathcal{J}_k + \mathcal{J}_k\mathcal{J}_j\right) + i\epsilon_{jkl}\mathcal{J}_l(\vec{L}\cdot\vec{\sigma}) + \delta_{jk}(\mathcal{J}^2 + \hbar(\vec{L}\cdot\vec{\sigma} + \frac{1}{2}\hbar))
\end{aligned}
$$

$$\hat{A}^\dagger(\hat{L})\hat{A}(\hat{L}) = (\hat{L} + \frac{\hbar}{2} + \hbar\gamma)^2\hat{H}_G + \frac{\alpha^2}{2} \tag{62}$$

$$\hat{A}(\hat{L}-\hbar)\hat{A}^\dagger(\hat{L}-\hbar) = (\hat{L} - \frac{\hbar}{2} + \hbar\gamma)^2\hat{H}_G + \frac{\alpha^2}{2} \tag{63}$$

$$\hat{\vec{L}}^\pm \cdot \hat{\vec{\mathcal{J}}}_{\frac{1}{2}} = 0 \tag{64}$$

Taking into account (62) - (64) we obtain the following polynomial symmetry algebra

$$
\begin{aligned}
[\mathcal{J}_i, \mathcal{J}_j] &= i\hbar\epsilon_{ijk}\mathcal{J}_k \\
[X_i, \mathcal{J}_j] &= i\hbar\epsilon_{ijk}X_k \\
[Y_i, \mathcal{J}_j] &= i\hbar\epsilon_{ijk}Y_k \\
[X_i, X_j] &= -i\hbar\epsilon_{ijk}\mathcal{J}_k\mathcal{F}(H, L\cdot\sigma) \\
[Y_i, Y_j] &= -i\hbar\epsilon_{ijk}\mathcal{J}_k\mathcal{F}(H, L\cdot\sigma) \\
[X_i, Y_j] &= i\hbar(L\cdot\sigma + \hbar(\gamma + \frac{1}{2}))(\mathcal{J}_i\mathcal{J}_j + \mathcal{J}_j\mathcal{J}_i)H + \delta_{ij}\mathcal{G}(H, L\cdot\sigma, \mathcal{J}^{\mathbf{2}}) \\
[X_i, L\cdot\sigma] &= -i\hbar Y_i \\
[Y_i, L\cdot\sigma] &= i\hbar X_i
\end{aligned}
$$

where

$$
\mathcal{F}(H, L\cdot\sigma) = \alpha^2 + H\left(4(L\cdot\sigma)^2 + \hbar(L\cdot\sigma)(6\gamma+5) + 2\hbar^2(\gamma+1)^2\right)
$$

$$\mathcal{G} = \frac{-i\hbar}{2}(2\alpha^2(L\cdot\sigma+\hbar) + H(4(L\cdot\sigma)(\mathcal{J}^2+(L\cdot\sigma)^2) + 2\hbar(\mathcal{J}^2(1+2\gamma)+4(L\cdot\sigma)^2(1+\gamma)) + 4\hbar^2(L\cdot\sigma)(1+\gamma)(2+\gamma) + \hbar^3(3+6\gamma+4\gamma^2))$$

All commutators not shown above vanish. The basis elements of the algebra are $\{H, \mathcal{J}_i, X_i, Y_i, (\vec{\sigma}, \vec{L}), 1\}$ and the right hand sides are at most fourth order polynomials in the basis elements.

Exact bound states solutions of the Schrödinger-Pauli equation

Let us conclude the analysis of this Hamiltonian system by evaluating explicitly its eigenfunctions and its spectrum for bound states. We construct the wavefunction as a complete set of commutative operators

$$
\begin{aligned}
H\psi(r,\theta,\phi)_{q,n,j,k} &= E\psi(r,\theta,\phi)_{q,n,j,k} && (65)\\
\hat{J}^2\Omega(\theta,\phi)_{q,j,k} &= j(j+1)\Omega(\theta,\phi)_{q,j,k} && (66)\\
\hat{\mathcal{J}}_3\Omega(\theta,\phi)_{q,j,k} &= k\Omega(\theta,\phi)_{q,j,k} && (67)\\
\vec{L}\cdot\vec{S}\Omega(\theta,\phi)_{q,j,k} &= \frac{q}{2}\Omega(\theta,\phi)_{q,j,k};\, q=\begin{cases} l \\ -l-1 \end{cases} && (68)\\
\hat{L}^2\Omega(\theta,\phi)_{q,j,k} &= q(q+1)\Omega(\theta,\phi)_{q,j,k} && (69)
\end{aligned}
$$

$$
\begin{aligned}
\psi(r,\theta,\phi)_{q,n,j,k} &= \rho(r)_{q,n,j}\Omega(\theta,\phi)_{q,j,k} && (70)\\
\Omega(\theta,\phi)_{l,j,k} &= \frac{1}{\sqrt{2j}}\begin{pmatrix} \sqrt{j+k}\,Y_{j-\frac{1}{2},k-\frac{1}{2}}(\theta,\phi) \\ \sqrt{j-k}\,Y_{j-\frac{1}{2},k+\frac{1}{2}}(\theta,\phi) \end{pmatrix} && (71)\\
\Omega(\theta,\phi)_{-l-1,j,k} &= \frac{1}{\sqrt{2j+2}}\begin{pmatrix} \sqrt{j-k+1}\,Y_{j+\frac{1}{2},k-\frac{1}{2}}(\theta,\phi) \\ \sqrt{j+k+1}\,Y_{j+\frac{1}{2},k+\frac{1}{2}}(\theta,\phi) \end{pmatrix} && (72)
\end{aligned}
$$

and the functions $Y_{l,m}(\theta,\phi)$ are the usual spherical harmonic functions:

$$
\begin{aligned}
\hat{L}^2 Y_{l,m}(\theta,\phi) &= l(l+1)\,Y_{l,m}(\theta,\phi) && (73)\\
\hat{L}_3 Y_{l,m}(\theta,\phi) &= mY_{l,m}(\theta,\phi). && (74)
\end{aligned}
$$

In view of (66) - (72) we can reduce the 3-dimensional Hamiltonian operator $\hat{H}$ to the following radial one:

$$\hat{\mathcal{H}} = < \Omega(\theta,\phi)_{q,j,k} | \hat{H} | \Omega(\theta,\phi)_{q,j,k} > = \quad (75)$$

$$= \begin{cases} q = l \to -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{(l+\gamma)(l+\gamma+1)}{r^2}\right) - \frac{\alpha}{r} \\ q = -l-1 \to -\frac{\hbar^2}{2}\left(\partial_r^2 + \frac{2}{r}\partial_r + \frac{(l-\gamma)(l-\gamma+1)}{r^2}\right) - \frac{\alpha}{r}. \end{cases} \quad (76)$$

It is straightforward to get the explicit expression for the bound state eigenfunctions of $\hat{\mathcal{H}}$

$$\rho_{l,n,j} \propto r^{j+\gamma-\frac{1}{2}} e^{-\frac{\alpha}{\hbar^2(n+\gamma+j+\frac{1}{2})}r} L_n^{2j+2\gamma}\left(\frac{2\alpha r}{\hbar^2(n+\gamma+j+\frac{1}{2})}\right) \quad (77)$$

$$\rho_{-l-1,n,j} \propto r^{j-\gamma+\frac{1}{2}} e^{-\frac{\alpha}{\hbar^2(n-\gamma+j+\frac{3}{2})}r} L_n^{2j-2\gamma+2}\left(\frac{2\alpha r}{\hbar^2(n-\gamma+j+\frac{3}{2})}\right) \quad (78)$$

$L_n^k(x)$ are Laguerre polynomials.

Finally we have

$$\hat{\mathcal{H}}\rho_{l,n,j}(r) = -\frac{\alpha^2}{2\hbar^2(n+j+\gamma+\frac{1}{2})^2}\rho_{l,n,j}(r) \tag{79}$$

$$\hat{\mathcal{H}}\rho_{-l-1,n,j}(r) = -\frac{\alpha^2}{2\hbar^2(n+j-\gamma+\frac{1}{2})^2}\rho_{-l-1,n,j}(r) \tag{80}$$

The main physical result of this paper is the new superintegrable system with spin described by the Hamiltonian (10) and the polynomial algebra of integrals of motion (65)-(65). The corresponding Schrödinger Pauli equation is solved in Section 5 where we give the bound state energies and wave functions. The radial parts are Laguerre polynomials times factors ensuring appropriate behaviour for $r \to 0$ and $r \to \infty$. The angular parts are expressed in terms of spherical spinors. A special case of the Hamiltonian (10)with $\gamma = \frac{1}{2}$ was obtained in [13] as part of a systematic search for superintegrable system of the form (7) with integrals of motion of degree at most 2 in the momenta [17] [18] . Another special case of (10) with $\gamma = 1$ is implicit in [17] [18] . It is obtained from the spinless Coulomb Hamiltonian $H = -\frac{1}{2}\Delta + \frac{\alpha}{r}$ in $\mathbb{E}_3$ by a gauge transformation (given in [14]) that transforms the total angular momentum $\vec{L}$ into a new integral of motion which depends linearly on the total angular momentum $\vec{\mathcal{J}}$

The power of the coalgebra approach is that it leads to a general independent parameter γ in the spin orbital potential and that it leads directly to the third order integrals of motion $\vec{X}$, $\vec{Y}$. A systematic search for systems with third order integrals would be very difficult . The explicit form of the integrals $\vec{X}$, $\vec{Y}$ is actually quite simple , however it is not linear in the Pauli matrices σ_i. The quadratic terms can be eliminated, that leads to quite complicated expressions. For instance we obtain

$$\vec{X}=\vec{x}\left(\frac{-\alpha}{r}+\frac{\hbar^2\gamma(\gamma+1)}{r^2}+(\vec{L}\cdot\vec{\sigma})\mathbf{p}^2+\hbar(1+2\gamma)\mathbf{p}^2+\frac{2i\hbar^2\gamma}{r^2}(\vec{x}\cdot\vec{p})-\frac{\hbar\gamma}{r^2}(\vec{x}\cdot\vec{p})^2\right)$$

$$\left((\vec{L}\cdot\vec{\sigma})(i\hbar-(\vec{x}\cdot\vec{p}))-i\hbar^2\gamma-\hbar(1+\gamma)(\vec{x}\cdot\vec{p})+i\hbar^2(1+\gamma)+\left(\frac{\hbar^2\gamma^2}{r^2}-\frac{\alpha}{r}\right)(\vec{\sigma}\cdot\vec{L})\right)\vec{p}$$

$$+\frac{i\hbar}{2}(\vec{x}\wedge\vec{\sigma})\left(\mathbf{p}^2+\frac{\alpha}{r}+\frac{i\hbar\gamma}{r^2}(\vec{x}\cdot\vec{p})\right)+\left(\frac{\hbar^2(1+2\gamma)}{2}+\frac{i\hbar}{2}(\vec{x}\cdot\vec{p})\right)(\vec{p}\wedge\vec{\sigma})$$

In the past systematic searches for second order superintegrable systems were conducted for purely scalar potentials $V_0(\vec{r})$ in $\mathbb{E}_2$ and $\mathbb{E}_3$ [26], [27], [28] and also in more general conformally flat spaces [29]. Searches for third order supereintegrable systems in $\mathbb{E}_2$, allowing separation of variables were also succesful [30], [31], [32], but were considerably more difficult. For particle with spin a systematic approach to searching for higher order integrable and superintegrable systems is a prohibitive task. Hence the development of other techniques is imperative, in particular those involving coalgebra symmetry. Further applications and generalizations of the coalgebra techniques are in progress, for instance to obtain suprintegrable systems in n dimensions, and systems involving higher spins, or two particle with nonzero spin.

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