

Superintegrability and exact solvability. Concepts and perspectives

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Introduction

Let us first consider a *classical* system in an n -dimensional Riemannian space with Hamiltonian

$$H = \sum_{i,k=1}^n g_{ik} p_i p_k + V(\vec{x}), \quad \vec{x} \in \mathbb{R}^n \quad (1.1)$$

The system is called *integrable* (or Liouville integrable) if it allows $n - 1$ integrals of motion (in addition to H)

$$\begin{aligned} X_a &= f_a(\vec{x}, \vec{p}), \quad a = 1, \dots, n-1 \\ \frac{dX_a}{dt} &= \{H, X_a\} = 0, \quad \{X_a, X_b\} = 0 \end{aligned} \quad (1.2)$$

This system is *superintegrable* if it allows further integrals

$$\begin{aligned} Y_b &= f_b(\vec{x}, \vec{p}), \quad b = 1, \dots, k \quad 1 \leq k \leq n-1 \\ \frac{dY_b}{dt} &= \{H, Y_b\} = 0. \end{aligned} \quad (1.3)$$

The integrals must satisfy

1. The integrals H, X_a, Y_b are well defined functions on phase space, i.e. polynomials or convergent power series on phase space (or an open submanifold of phase space).
2. The integrals H, X_a are in involution, i.e. Poisson commute as indicated in (1.3). The integrals Y_b Poisson commute with H but not necessarily with each other, nor with X_a .
3. The entire set of integrals is functionally independent, i.e., the Jacobian matrix satisfies

$$\text{rank} \frac{\partial(H, X_1, \dots, X_{n-1}, Y_1, \dots, Y_k)}{\partial(x_1, \dots, x_n, p_1, \dots, p_n)} = n + k \quad (1.4)$$

In *quantum mechanics* we define integrability and superintegrability in the same way, however in this case, H , X_a and Y_b are operators.

The condition on the integrals of motion must also be modified e.g. as follows :

1. H , X_a and Y_b are well defined Hermitian operators in the enveloping algebra of the Heisenberg algebra $H_n \sim \{\vec{x}, \vec{p}, \hbar\}$ or some generalization thereof.
2. The integrals satisfy the Lie bracket relations

$$[H, X_a] = [H, Y_b] = 0, [X_i, X_k] = 0 \quad (1.5)$$

3. No polynomial in the operators H , X_a , Y_b formed entirely using Lie anticommutators should vanish identically.

The two best known superintegrable systems are the Kepler-Coulomb system with potential $V(r) = \frac{\alpha}{r}$ and the isotropic harmonic oscillator $V(r) = \alpha r^2$. In both cases the integrals X_a correspond to angular momentum, the additional integrals Y_a to the Laplace-Runge-Lenz vector for $V(r) = \frac{\alpha}{r}$ and to the quadrupole tensor

$T_{ik} = p_i p_k + \alpha x_i x_k$, respectively. No further ones were discovered until a 1940 paper by Jauch and Hill on the rational anisotropic harmonic oscillator $V(\vec{x}) = \alpha \sum_{i=1}^n n_i x_i^2$, $n_i \in \mathbb{Z}$.

A systematic search for superintegrable systems was started in 1965 and a real proliferation of them was observed during the last couple of years.

Let us just list some of the reasons why superintegrable systems are interesting both in classical and quantum physics.

In *classical* mechanics, superintegrability restricts trajectories to an $n - k$ dimensional subspace of phase space.

- For $k = n - 1$ (maximal superintegrability), this implies that all finite trajectories are closed and motion is periodic.
- Moreover, at least in principle, the trajectories can be calculated without any calculus.
- Bertrand's theorem states that the only spherically symmetric potentials $V(r)$ for which all bounded trajectories are closed are $\frac{\alpha}{r}$ and αr^2 , hence no other superintegrable systems are spherically symmetric.
- The algebra of integrals of motion $\{H, X_a, Y_b\}$ is a non-Abelian and interesting one. Usually it is a finitely generated polynomial algebra, only exceptionally a finite dimensional Lie algebra. In the special case of quadratic superintegrability (all integrals of motion are at most quadratic polynomials in the moments), integrability is related to separation of variables in the Hamilton-Jacobi equation, or Schrödinger equation, respectively.

In *quantum* mechanics,

- superintegrability leads to an additional degeneracy of energy levels, sometimes called "accidental degeneracy". The term was coined by Fok and used by Moshinsky and collaborators, though the point of their studies was to show that this degeneracy is certainly no accident.
- A conjecture, born out by all known examples, is that all maximally superintegrable systems are exactly solvable. If the conjecture is true, then the energy levels can be calculated algebraically. The wave functions are polynomials (in appropriately chosen variables) multiplied by some gauge factor.
- The non-Abelian polynomial algebra of integrals of motion provides energy spectra and information on wave functions. Interesting relations exist between superintegrability and supersymmetry in quantum mechanics.

As a comment, let us mention that superintegrability has also been called non-Abelian integrability. From this point of view, infinite dimensional integrable systems (soliton systems) described e.g. by the Korteweg-de-Vries equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili equation, etc. are actually superintegrable.

Indeed, the generalized symmetries of these equations form infinite dimensional non-Abelian algebras (the Orlov-Shulman symmetries) with infinite dimensional Abelian subalgebras of commuting flows.

Second Order Superintegrability

Let us consider the Hamiltonian (1.1) in the Euclidian space E_2 and search for second order integrals of motion. We have

$$H = \frac{1}{2} (p_1^2 + p_2^2) + V(x_1, x_2), \quad X = \sum_{j+k=0}^2 \left\{ f_{jk}(x_1, x_2), p_1^j p_2^k \right\} \quad (2.1)$$

In the quantum case we have

$$p_j = -i\hbar \frac{\partial}{\partial x_j}, \quad L_3 = x_1 p_2 - x_2 p_1 \quad (2.2)$$

The commutativity condition $[H, X] = 0$ implies that the even terms $j + k = 0, 2$ and odd terms $j + k = 1$ in X commute with H separately. Hence we can, with no loss of generality, set $f_{10} = f_{01} = 0$. Further we find that the leading (second order) term in X lies in the enveloping algebra of the Euclidian algebra $e(2)$. Thus we obtain

$$X = aL_3^2 + b_1(L_3 p_1 + p_1 L_3) + b_2(L_3 p_2 + p_2 L_3) + c_1(P_1^2 - P_2^2) + 2c_2 P_1 P_2 + \phi(x_1, x_2) \quad (2.3)$$

where a, b_j, c_i are constants.

The function $\phi(x_1, x_2)$ must satisfy the determining equations

$$\begin{aligned}\phi_{x_1} &= -2(ax_2^2 + 2b_1x_2 + c_1)V_{x_1} + 2(ax_1x_2 + b_1x_1 - b_2x_2 - c_2)V_{x_2} \\ \phi_{x_2} &= -2(ax_1x_2 + b_1x_1 - b_2x_2 - c_2)V_{x_1} + 2(-ax_1^2 + 2b_2x_1 + c_1)V_{x_2}\end{aligned}\quad (2.4)$$

The compatibility condition $\phi_{x_1x_2} = \phi_{x_2x_1}$ implies

$$\begin{aligned} &(-ax_1x_2 - b_1x_1 + b_2x_2 + c_2)(V_{x_1x_1} - V_{x_2x_2}) \\ &- (a(x_1^2 + x_2^2) + 2b_1x_1 + 2b_2x_2 + 2c_1)V_{x_1x_1} \\ &- (ax_2 + b_1)V_{x_1} + 3(ax_1 - b_2)V_{x_2} = 0\end{aligned}\quad (2.5)$$

Eq. (2.5) is exactly the same equation that we would have obtained if we had required that the potential should allow the separation of variables in the Schrödinger equation in one of the coordinate system in which the Helmholtz equation allows separation. Another important observation is that (2.4) and (2.5) do not involve the Planck constant. Indeed, if we consider the classical functions H and X in (2.1) and require that they Poisson commute, we arrive at exactly the same conclusions and to equations (2.4) and (2.5).

Thus for quadratic integrability (and superintegrability) the potentials and integrals of motion coincide in classical and quantum mechanics (up to a possible symmetrization).

The Hamiltonian (1.1) is form invariant under Euclidian transformations, so we can classify the integrals X into equivalence classes under rotations, translations and linear combinations with H . There are two invariants in the space of parameters a, b_i, c_i , namely

$$I_1 = a, \quad I_2 = (2ac_1 - b_1^2 + b_2^2)^2 + 4(ac_2 - b_1b_2)^2 \quad (2.6)$$

Solving (2.1) for different values of I_1 and I_2 we obtain :

$$\begin{array}{lll} I_1 = I_2 = 0 & V_C = f_1(x_1) + f_2(x_2) & \\ I_1 = 1, I_2 = 0 & V_R = f(r) + \frac{1}{r^2}g(\phi) & x_1 = r \cos \phi, x_2 = r \sin \phi \\ I_1 = 0, I_2 = 1 & V_P = \frac{f(\xi) + g(\eta)}{\xi^2 + \eta^2} & x_1 = \frac{\xi^2 - \eta^2}{2}, x_2 = \xi\eta \\ I_1 = 1, I_2 = I^2 \neq 0 & V_E = \frac{f(\sigma) + g(\eta)}{\cos^2 \sigma - \cosh^2 \rho} & \begin{array}{l} x_1 = I \cosh \rho \cos \sigma \\ x_2 = I \sinh \rho \sin \sigma \\ 0 < I < \infty \end{array} \end{array} \quad (2.7)$$

We see that V_C, V_R, V_P and V_E correspond to separation of variables in Cartesian, polar, parabolic and elliptic coordinates, respectively and that second order integrability (in E_2) implies separation of variables. For second order superintegrability, two integrals of the form (2.4) exist and the Hamiltonian separates in at least two coordinate systems.

Four three-parameter families of superintegrable systems exist namely

$$\begin{aligned}
 V_I &= \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2}, & V_{II} &= \alpha(x^2 + 4y^2) + \frac{\beta}{x^2} + \gamma y \\
 V_{III} &= \frac{\alpha}{r} + \frac{1}{r^2} \left(\frac{\beta}{\cos^2 \frac{\phi}{2}} + \frac{\gamma}{\sin^2 \frac{\phi}{2}} \right), & V_{IV} &= \frac{\alpha}{r} + \frac{1}{\sqrt{r}} \left(\beta \cos \frac{\phi}{2} + \gamma \sin \frac{\phi}{2} \right) \quad (2.8)
 \end{aligned}$$

The classical trajectories, quantum energy levels and wave functions for all of these systems are known. The potentials V_I and V_{II} are isospectral deformations of the isotropic and an anisotropic harmonic oscillator, respectively, whereas V_{III} and V_{IV} are isospectral deformations of the Kepler-Coulomb potential. In n -dimensional space E_n , a set of n commuting second order integrals corresponds to a separable coordinate system.

All of the above results on quadratic superintegrability have been generalized to arbitrary dimensions, to spaces of constant curvature and to other real and complex spaces.

Third order superintegrability

General setting

In 1935, J. Drach published an article in which he studied the classical Hamiltonian (2.1) in a two dimensional *complex* Euclidian space $E_2(\mathbb{C})$ and found 10 potentials for which a third order integral of motion exists. Much more recently, Rañada and Tsiganov showed that 7 of them are "reducible", i.e. they are actually second order superintegrable and that the third order integral of motion found in is the Poisson commutator of the two second order ones.

A systematic search for third order quantum and classical superintegrable systems in $E_2(\mathbb{R})$ was started in 2004. The Hamiltonian was taken as in (2.1) with p_1, p_2 as in (2.2) and the study was carried out in a quantum mechanics setting.

The integral was taken in the form

$$X = \sum_{j+k=0}^3 \left\{ f_{jk}(x_1, x_2), p_1^j p_2^k \right\} \quad (3.1)$$

where the curly brackets signify and anticommutator. The commutativity condition $[H, X] = 0$ implies the even and odd terms commute separately and that the highest order terms lie in the enveloping algebra of $e(2)$. The same two conclusions hold for integrals that are appropriately symmetrized polynomials of any order n in the momenta.

We can hence, with no loss of generality, write a third order integral as

$$X = \sum_{i+j+k=3} A_{ijk} \{L_3^i, p_1^j p_2^k\} + \{g_1(x, y), p_1\} + \{g_2(x, y), p_2\} \quad (3.2)$$

where $A_{ijk} = A_{ikj}$ are constants (for convenience, we switch to the notation $x_1 = x, x_2 = y$).

The 10 constants A_{ijk} and 3 unknown functions g_1, g_2 and V are solutions of an overdetermined system of determining equations, namely :

$$g_1 V_x + g_2 V_y - \frac{\hbar^2}{4} \left[f_1 V_{xxx} + f_2 V_{xxy} + f_3 V_{xyy} + f_4 V_{yyy} + 8A_{300}(xV_y - yV_x) + 2(A_{210} V_x + A_{201} V_y) \right] = 0 \quad (3.3)$$

$$(g_1)_x = 3f_1(y)V_x + f_2(x, y)V_y \quad (3.4)$$

$$(g_2)_y = f_3(x, y)V_x + 3f_4(x, y)V_y \quad (3.5)$$

$$(g_1)_y + (g_2)_x = 2[f_2(x, y)V_x + f_3(x, y)V_y] \quad (3.6)$$

The functions f_j are defined as

$$\begin{aligned} f_1(y) &= -A_{300}y^3 + A_{210}y^2 - A_{120}y + A_{030}, \\ f_2(x, y) &= 3A_{300}xy^2 - 2A_{210}xy + A_{201}y^2 + A_{120}x - A_{111}y + A_{021}, \\ f_3(x, y) &= -3A_{300}x^2y + A_{210}x^2 - 2A_{201}xy + A_{111}x - A_{102}y + A_{012}, \\ f_4(x) &= A_{300}x^3 + A_{201}x^2 + A_{102}x + A_{003}. \end{aligned} \quad (3.7)$$

The first conclusion from (3.3-3.6) is that third order integrability (as opposed to second order integrability) is very different in quantum and classical mechanics. Indeed, the Planck constant \hbar enters into the determining equation (3.3). For $\hbar \rightarrow 0$ we obtain the classical integrability condition (for H and X to Poisson commute). Thus, unless the expression in square brackets multiplying \hbar^2 in (3.3) vanishes, classical and quantum integrable (and superintegrable) systems will differ. The same is true for integrals of motion of any order $n \geq 3$.

The compatibility of equation (3.4),(3.5),(3.6) is the same in classical and quantum mechanics, namely a third-order linear equation for the potential

$$\begin{aligned}
 0 = & -f_3 V_{xxx} + (2f_2 - 3f_4) V_{xxy} + (-3f_1 + 2f_3) V_{xyy} - f_2 V_{yyy} \\
 & + 2(f_{2y} - f_{3x})(V_{xx} - V_{yy}) + 2(-3f_{1y} + f_{2x} + f_{3y} - 3f_{4x}) V_{xy} \\
 & + (-3f_{1yy} + 2f_{2xy} - f_{3xx}) V_x + (-f_{2yy} + 2f_{3xy} - 3f_{4xx}) V_y.
 \end{aligned} \tag{3.8}$$

Compatibility between (3.3) and the other 3 determining equations imposes 3 further partial differential equations on the potential. They are, however, nonlinear and of order 4. It is very difficult to solve the determining equations (3.3) , . . . ,(3.6) and the nonlinear compatibility conditions are of little help. Instead, we consider a simpler problem, namely that of finding superintegrable systems with one first or second order integral and one third order one.

One first order integral

Let us first assume that the second integral Y is a first order one. This implies that the potential allows a geometrical symmetry. For translational or rotational invariance we have respectively

$$\begin{aligned} Y &= P_2, & V(x, y) &= V(x) \\ Y &= L_3, & V(x, y) &= V(r), & r &= \sqrt{x^2 + y^2} \end{aligned} \quad (3.9)$$

In the case of rotational invariance, we have $V(r) = \frac{\alpha}{r}$ or $V(r) = \alpha r^2$, in agreement with Bertrand's theorem. Both allow two independent second order invariants and the third order one is their commutator.

Translational invariance in classical mechanics implies $V(x) = ax$, or $V(x) = \frac{a}{x^2}$, both quadratically superintegrable.

However, in the quantum case, we get one more third order superintegrable potential satisfying

$$\hbar^2 V'(x)^2 = 4(V(x) - A_1)(V(x) - A_2)(V(x) - A_3). \quad (3.10)$$

where A_1, A_2 and A_3 are constants, so that $V(x)$ is expressed in terms of elliptic functions (or degenerate cases thereof if some of the roots A_i coincide). For instance, for $A_1 \leq V \leq A_2 \leq A_3$, we obtain

$$V = \frac{(\hbar\omega)^2}{\cosh^2(\omega x)}, \quad (3.11)$$

a soliton solution of the Korteweg-de-Vries equation. The integrals for all solutions of (3.10) are

$$\begin{aligned} X &= \{L_3, p_1^2\} + \{(\alpha - 3V(x))y, p_1\} + \{-\alpha x + 2xV(x) + \int V(x)dx, p_2\}, \quad Y = P_2 \\ \alpha &= A_1 + A_2 + A_3 \end{aligned} \quad (3.12)$$

We see that even though the potential $V(x)$ is one-dimensional, the problem is really two-dimensional, since the integral X involves angular momentum L_3 .

Second order "Cartesian" integral

A more interesting case arises if the additional integral Y is second order in the momenta.

In this case, the potential $V(x, y)$ will allow separation of variables. So far, two separable coordinate systems have been investigated systematically, namely Cartesian and polar coordinates. In the case of Cartesian coordinates, we have

$$\begin{aligned} V(x, y) &= V_1(x) + V_2(y) \\ Y &= \frac{1}{2}(p_1^2 - p_2^2) + V_1(x) - V_2(y) \end{aligned} \quad (3.13)$$

The determining equations and compatibility condition (3.7) simplify greatly. Indeed, the compatibility equations are no longer partial differential equations, since they involve only function of 1 variable $V_1(x)$ and $V_2(y)$.

A complete analysis shows that in the classical case, one obtains two known second order superintegrable potentials, an anisotropic harmonic oscillator plus 4 new irreducible ones, namely

$$\begin{aligned}
 V_I &= \beta_1^2 \sqrt{|x|} + \beta_2^2 \sqrt{|y|} \\
 V_{II} &= \frac{1}{2} \omega^2 y^2 + V(x), & -9V^4 + 14\omega^2 x^2 V^3 + (6d - \frac{15}{2} \omega^4 x^4) V^2 \\
 & & + \frac{3\omega^6}{2} x^6 - 2\omega^2 dx^2) V + cx^2 - d^2 - d \frac{\omega^4}{2} x^4 \\
 & & - \frac{1}{16} \omega^8 x^8 = 0 \\
 V_{III} &= a^2 |y| + b^2 \sqrt{|x|} \\
 V_{IV} &= ay + V(x) & V^3 - 2bxV^2 + b^2 x^4 V - d = 0
 \end{aligned} \tag{3.14}$$

where $a, b, c, d, \beta_i, \omega$ are constants.

The trajectories in all of these potentials have been calculated and all bounded trajectories are periodic.

The quantum case is much richer.

One obtains 13 new irreducible superintegrable potentials. Among them, 6 are expressed in terms of elementary functions, 2 in terms of elliptic ones and 5 in terms of Painlevé transcendents.

Examples of the elementary superintegrable potentials are

$$\begin{aligned}
 V &= \hbar^2 \left(\frac{1}{8\alpha^4} (x^2 + y^2) + \frac{1}{(y - \alpha)^2} + \frac{1}{(x - \alpha)^2} + \frac{1}{(y + \alpha)^2} + \frac{1}{(x + \alpha)^2} \right) \\
 V &= \hbar^2 \left(\frac{1}{8\alpha^4} (9x^2 + y^2) + \frac{1}{(y + \alpha)^2} + \frac{1}{(y - \alpha)^2} \right)
 \end{aligned} \tag{3.15}$$

and an elliptic function one is

$$V = \hbar^2 (\mathcal{P}(x) + \mathcal{P}(y)) \tag{3.16}$$

where $\mathcal{P}(x)$ is the Weierstrass elliptic function.

The most interesting potentials obtained in this manner are those expressed in terms of Painlevé transcendents. They provide one of the relationships between the theory of quantum superintegrable systems and soliton theory. By this we mean the theory of infinite dimensional integrable systems, usually described by nonlinear partial differential equations that are compatibility conditions for certain linear equations obtained from Lax pairs. The "Painlevé conjecture" states that all reductions of soliton type equations to ordinary differential equations should have the Painlevé property (possibly after a change of variables). That means they should be single-valued about movable singularities. The 6 Painlevé transcendents were discovered in a study of second order ODEs with the Painlevé property and they are the only equations of the studied class that can not be expressed in terms of elliptic functions, or solutions of linear differential equations.

The Painlevé equations come up as solutions of many of the nonlinear equations of classical physics, such as the Korteweg-de-Vries, Boussinesq, Kadomtsev-Petviashvili or three-wave interaction equations, to name just a few examples. Here they appear as superintegrable potentials in the linear Schrödinger equation in quantum mechanics.

Example of Schrödinger equation with Painlevé potential

The Painlevé transcendents that occur as potentials when the Schrödinger equation separates in Cartesian coordinates are \mathcal{P}_I , \mathcal{P}_{II} and \mathcal{P}_{IV} . Here we shall just consider the case of \mathcal{P}_{IV} .

The Hamiltonian and two integrals of motion in this case are

$$\begin{aligned}
 H &= \frac{1}{2} [p_1^2 + p_2^2 + \omega^2(x^2 + y^2)] + V(x) \\
 A &= p_1^2 - p_2^2 + \omega^2(x^2 - y^2) + V(x) \\
 B &= \frac{1}{2} \{L_3, P_1^2\} + \frac{1}{2} \left\{ \frac{\omega^2}{2} x^2 y - 3yV_1', P_1 \right\} - \\
 &\quad \frac{1}{\omega^2} \left\{ \frac{\hbar^2}{4} V_1''' + \left(\frac{\omega^2}{2} x^2 - 3V_1 \right) V_1', P_1 \right\}
 \end{aligned} \tag{3.17}$$

with

$$\begin{aligned}
 V &= \epsilon \hbar (\omega \mathcal{P}'_{IV} + 2\omega^2 \mathcal{P}_{IV}^2 + 2\omega^2 x \mathcal{P}_{IV}) \\
 &\quad + \frac{1}{6} (-\hbar^2 K_1 + 3\epsilon \hbar \omega), \quad \epsilon = \pm 1 \\
 \mathcal{P}_{IV} &= \mathcal{P}_{IV}(\rho x, K_1, K_2), \quad \rho = -\frac{2\omega^2}{\hbar^2}
 \end{aligned} \tag{3.18}$$

The integrals of motion form a polynomial (cubic) algebra, satisfying

$$\begin{aligned}
 [A, B] &= C & [A, C] &= 16\omega^2 \hbar^2 B \\
 [B, C] &= -2\hbar^2 A^3 - 6\hbar^2 HA^2 + 8\hbar^2 H^3 + \frac{1}{3}(\hbar^6 K_1^2 - 20\hbar^4 \omega^2 \\
 &\quad - 24\hbar^2 K_2 \omega^2 - 4\epsilon \hbar^5 K_1 \omega)A - 8\omega^2 \hbar^4 H + \frac{1}{27}[-\hbar^8 K_1^3 \\
 &\quad - 12\hbar^6 K_1(1 + 6K_2)\omega^2 + 6\epsilon \hbar^7 K_1^2 \omega + 8\epsilon \hbar^5(1 + 18K_2)\omega^3] \quad (3.19)
 \end{aligned}$$

The algebra has a Casimir operator that is a 4th order polynomial in the Hamiltonian H (with constant coefficients)

The representation theory of the algebra (3.19) and its realization in terms of a deformed oscillator algebra is used to calculate the energy spectrum and wave functions of the system. A connection with "higher order supersymmetry" also gives the wave functions.

One obtains 3 series of states with energies

$$\begin{aligned}
 E_1 &= \hbar \left(p + \frac{\epsilon + 3}{3} - \frac{\hbar K_1}{6\omega} \right) \\
 E_2 &= \hbar \left(p + \frac{-\epsilon + 6}{6} + \frac{\hbar K_1}{12\omega} + \sqrt{\frac{-K_2}{2}} \right), \quad K_2 < 0 \\
 E_3 &= \hbar \left(p + \frac{-\epsilon + 6}{6} + \frac{\hbar K_1}{2\omega} - \sqrt{\frac{-K_2}{2}} \right), \quad p = 0, 1, 2, 3, \dots \quad (3.20)
 \end{aligned}$$

and 3 "ground states", all in terms of the Painlevé transcendent \mathcal{P}_{IV} .

Second order "polar" integral

We have

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2) + R(r) + \frac{S(\theta)}{r^2} \\ Y &= L_3^2 + 2S(\theta) \end{aligned} \quad (3.21)$$

The compatibility condition (3.8) can be rewritten in polar coordinates and then solved for $R(r)$. The general form of $R(r)$ is

$$\begin{aligned} R(r) = \frac{a_1}{r^4} + \frac{a_2}{r^3} + \frac{a_3}{r} + a_4 r^2 + a_5 r^4 + a_6 \log r + \frac{a_7}{\sqrt{A + Cr^2}} \\ + \frac{a_8}{\sqrt{A + Cr^2}} \log \left(\frac{\sqrt{A} + \sqrt{A + Cr^2}}{r} \right) \end{aligned} \quad (3.22)$$

however, the system (3.3) , . . . , (3.7) implies $a_1 = a_2 = a_5 = a_6 = a_7 = a_8 = 0$.

Finally, new superintegrable systems that are obtained satisfy $R(r) = 0$ and are :

$$V(r, \theta) = \frac{\alpha}{r^2 \sin^2 3\theta} \quad (3.23)$$

This is a special case of the 3 body Calogero model. Potential (3.23) is superintegrable both in classical and quantum mechanics.

In the quantum case, we have a further potential, namely

$$V(r, \theta) \frac{\hbar^2}{4r^2 \sin^2 \theta} \left[4 \sin^2 \theta W' \mp 8 \cos \theta W - 4\beta - 1 \right]$$

$$W = W(x_{1,2}) \quad x_1 = \sin^2 \frac{\theta}{2} \quad x_2 = \cos^2 \frac{\theta}{2} \quad (3.24)$$

where $W(x)$ is expressed in terms of the Painlevé transcendent \mathcal{P}_6 as

$$W(x) = \frac{x^2(x-1)^2}{4\mathcal{P}_6(\mathcal{P}_6-1)(\mathcal{P}_6-x)} \left[\mathcal{P}'_6 - \frac{\mathcal{P}_6(\mathcal{P}_6-1)}{x(x-1)} \right]^2 + \frac{1}{8}(1-\sqrt{2\gamma_1})^2(1-2\mathcal{P}_6)$$

$$- \frac{1}{4}\gamma_2 \left(1 - \frac{2x}{\mathcal{P}_6} \right) - \frac{1}{4}\gamma_3 \left(1 - \frac{2(x-1)}{\mathcal{P}_6-1} \right) + \left(\frac{1}{8} - \frac{\gamma_4}{4} \right) \left(1 - \frac{2x(\mathcal{P}_6-1)}{\mathcal{P}_6-x} \right) \quad (3.25)$$

where $\mathcal{P}_6 = \mathcal{P}_6(x, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and the constants γ_i must satisfy $\gamma_2 + \gamma_3 = 0$ or $\gamma_1 + \gamma_4 - \sqrt{2\gamma_1} = 0$

The third order integral in this case has the general form

$$X = L_3^2 p_1 + p_1 L_3^2 + \{g_1, p_1\} + \{g_2, p_2\} \quad (3.26)$$

where g_1 and g_2 are expressed in terms of \mathcal{P}_6 .

Infinite families of superintegrable potentials with integrals of arbitrary order

In a recent (2009) article, Tremblay, Turbiner and Winternitz introduced the quantum mechanical Hamiltonian

$$H_k(r, \varphi; \omega, \alpha, \beta) = -\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\varphi^2 + \omega^2 r^2 + \frac{\alpha k^2}{r^2 \cos^2 k\varphi} + \frac{\beta k^2}{r^2 \sin^2 k\varphi}, \quad (4.1)$$

where α, β, ω and $k \neq 0$ are parameters, $\alpha > -\frac{1}{4k^2}, \beta > -\frac{1}{4k^2}$ and r, φ are polar coordinates.

This Hamiltonian is integrable since it allows the integral of motion

$$\mathcal{X}_k(\alpha, \beta) = -L_3^2 + \frac{\alpha k^2}{\cos^2 k\varphi} + \frac{\beta k^2}{\sin^2 k\varphi}. \quad (4.2)$$

For $k = 1$ it reduces to one of the superintegrable systems found in [5], for $k = 2$ it is the rational BC_2 model, for $k = 3$ the Wolfes model or G_2 model. The configuration space is the sector

$$0 \leq \varphi \leq \frac{\pi}{2k}, \quad 0 \leq r \leq \infty \quad (4.3)$$

The Hamiltonian is exactly solvable and its energy spectrum and wave functions are

$$\begin{aligned}
 E_{N,n} &= 2\omega[2N + (2n + a + b)k + 1] \\
 \Psi_{N,n} &= r^{2nk} R_N(r^2) P_n^{(a-1/2, b-1/2)}(2\sin^2 k\varphi - 1) \Psi_0 \\
 \Psi_0 &= r^{(a+b)k} \cos^a k\varphi \sin^b k\varphi e^{-\frac{\omega r^2}{2}}
 \end{aligned} \tag{4.4}$$

where R_N and $P_n^{(\alpha, \beta)}$ are Laguerre and Jacobi polynomials, respectively. We see that (4.1) is an isospectral deformation of the isotropic harmonic oscillator, however for k integer (or rational), the degeneracy of the energy levels is given by the number of integer solutions of the equation $N + kn = \text{integer}$. This corresponds to the degeneracy of the levels of an anisotropic harmonic oscillator.

It was shown that the system is superintegrable for $k = 1, 2, 3$ and 4 and conjectured that it is superintegrable for all integer $k \geq 1$. Moreover it was conjectured that the additional (to H_k and \mathcal{X}_k) integral Y_k is of order $2k$ (and has rational coefficients).

The conjecture was based on the existence of an underlying "hidden" algebra of operators generated by

$$\begin{aligned}
 J^1 &= \partial_t \\
 J_{\mathcal{N}}^2 &= t\partial_t - \frac{\mathcal{N}}{3} \\
 J_{\mathcal{N}}^3 &= su\partial_u - \frac{\mathcal{N}}{3} \\
 J_{\mathcal{N}}^4 &= t^2\partial_t + stu\partial_u - \mathcal{N}t \\
 R_i &= t^i\partial_u, \quad i = 0, 1, \dots, s \\
 T_s &= u(\partial_t)^s
 \end{aligned} \tag{4.5}$$

with $t = r^2$, $u = r^{2k} \sin^2 k\varphi$ and s an integer (related to k). The Hamiltonian and integral \mathcal{X}_k lie in the enveloping algebra of (4.5) for all k , the additional one Y_k was shown to lie in this enveloping algebra for $k = 1, \dots, 4$.

The same system was then considered in classical mechanics and it was shown that all bounded trajectories are periodic for all rational values of k : a clear indication of superintegrability.

The conjecture was later proven and thus turned into a theorem by C. Quesne and W. Miller et al. A second family of superintegrable systems in a plane was proposed and studied in [57].

The potential is

$$V = -\frac{Q}{r} + \frac{\alpha k^2}{4r^2 \cos^2(\frac{k}{2}\phi)} + \frac{\beta k^2}{4r^2 \sin^2(\frac{k}{2}\phi)} \quad (4.6)$$

which turns out to be an isospectral deformation of the Coulomb potential. It was shown that by the operation of coupling constant metamorphosis (also known as a Stäckel transform) and a subsequent change of variables $r = \frac{\rho^2}{2}$, $\phi = 2\theta$, the Hamiltonian and both integrals of motion of the TTW system (4.1) can be transformed into the deformed Coulomb system with potential (4.6).

The results reviewed in this section open the path to finding other families of superintegrable systems in higher dimensions and other spaces.

Superintegrability with spin

First, we present a method for generating superintegrable systems with a spin-orbital interaction in three-dimensional Euclidean space \mathbb{E}_3 from superintegrable scalar systems in \mathbb{E}_2 . The method starts with a Hamiltonian of the form

$$H^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + V(\rho), \quad \rho = \sqrt{x_1^2 + x_2^2} \quad (5.1)$$

and uses coalgebra symmetry to generate systems of the form

$$H^{(3)} = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V_0(r) + V_1(r)(\vec{\sigma}, \vec{L}) \quad (5.2)$$

where

$$p_k = -i\hbar \frac{\partial}{\partial x_k}, \quad L_k = \epsilon_{kab} x_a p_b, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (5.3)$$

and σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5.4)$$

Secondly, we use this coalgebra method to derive a maximally superintegrable system with the Hamiltonian

$$H = \frac{-\hbar^2}{2} \nabla^2 + \frac{2\gamma}{r^2} \vec{S} \cdot \vec{L} - \frac{\alpha}{r} + \frac{\hbar^2 \gamma (\gamma + 1)}{2r^2}, \quad \vec{S} = \frac{\hbar}{2} \vec{\sigma} \quad (5.5)$$

where α and γ are arbitrary constants. This Hamiltonian is integrable because it is spherically symmetric and hence the angular momentum

$$\vec{J} = \vec{L} + \vec{S} \quad (5.6)$$

is an integral of the motion.

We shall show below that the system is also superintegrable. The additional integrals of motion are the components of a vector that is in general a third order polynomial in the momenta (a third order Hermitian operator). The Hamiltonian (5.5) can be viewed as describing the Coulomb interaction of a particle with spin $\frac{1}{2}$ with another of spin 0

Finally let us give an explicit representation for the constant of motion \vec{X} which can be regarded as the generalization of the Laplace-Runge-Lenz vector for the hydrogen atom

$$\vec{X} = \frac{1}{2}((\vec{L} \cdot \sigma)\vec{A} + \vec{A}(\vec{L} \cdot \sigma)) + \hbar\vec{A} \quad (5.7)$$

$$\hat{\mathcal{A}} \equiv \frac{1}{2}(\hat{\vec{p}} \wedge \hat{\vec{L}} - \hat{\vec{L}} \wedge \hat{\vec{p}}) + 2\gamma\vec{p} \wedge \vec{S} + \frac{1}{2}(\vec{x}\hat{\nu} + \hat{\nu}\vec{x}) \quad (5.8)$$

$$\hat{\nu} \equiv -\frac{\alpha}{r} + \frac{2\gamma}{r^2}\vec{L} \cdot \vec{S} + \frac{\hbar^2\gamma(2\gamma+1)}{2r^2} \quad (5.9)$$

The algebra generated by the set of constants of the motion for $\hat{H}_G^{(3)}$ defines a closed polynomial algebra under the operation of commutation.

We obtain the following polynomial symmetry algebra

$$\begin{aligned}
 [\mathcal{J}_i, \mathcal{J}_j] &= i\hbar\epsilon_{ijk}\mathcal{J}_k \\
 [X_i, \mathcal{J}_j] &= i\hbar\epsilon_{ijk}X_k \\
 [Y_i, \mathcal{J}_j] &= i\hbar\epsilon_{ijk}Y_k \\
 [X_i, X_j] &= -i\hbar\epsilon_{ijk}\mathcal{J}_k\mathcal{F}(H, L \cdot \sigma) \\
 [Y_i, Y_j] &= -i\hbar\epsilon_{ijk}\mathcal{J}_k\mathcal{F}(H, L \cdot \sigma) \\
 [X_i, Y_j] &= i\hbar(L \cdot \sigma + \hbar(\gamma + \frac{1}{2}))(\mathcal{J}_i\mathcal{J}_j + \mathcal{J}_j\mathcal{J}_i)H + \delta_{ij}\mathcal{G}(H, L \cdot \sigma, \mathcal{J}^2) \\
 [X_i, L \cdot \sigma] &= -i\hbar Y_i \\
 [Y_i, L \cdot \sigma] &= i\hbar X_i
 \end{aligned}$$

where

$$\mathcal{F}(H, L \cdot \sigma) = \alpha^2 + H \left(4(L \cdot \sigma)^2 + \hbar(L \cdot \sigma)(6\gamma + 5) + 2\hbar^2(\gamma + 1)^2 \right),$$

$$\begin{aligned} \mathcal{G} = & \frac{-i\hbar}{2}(2\alpha^2(L \cdot \sigma + \hbar) + H(4(L \cdot \sigma)(\mathcal{J}^2 + (L \cdot \sigma)^2) + \\ & 2\hbar(\mathcal{J}^2(1 + 2\gamma) + 4(L \cdot \sigma)^2(1 + \gamma)) + 4\hbar^2(L \cdot \sigma)(1 + \gamma)(2 + \gamma) \\ & + \hbar^3(3 + 6\gamma + 4\gamma^2)) \end{aligned}$$

All commutators not shown above vanish. The basis elements of the algebra are $\{H, \mathcal{J}_i, X_i, Y_i, (\vec{\sigma}, \vec{L}), 1\}$ and the right hand sides are at most fourth order polynomials in the basis elements.

Exact bound states solutions of the Schrödinger-Pauli equation

Let us conclude the analysis of this Hamiltonian system by evaluating explicitly its eigenfunctions and its spectrum for bound states. We construct the wavefunction as a complete set of commutative operators

$$H\psi(r, \theta, \phi)_{q,n,j,k} = E\psi(r, \theta, \phi)_{q,n,j,k} \quad (5.10)$$

$$\hat{J}^2\Omega(\theta, \phi)_{q,j,k} = j(j+1)\Omega(\theta, \phi)_{q,j,k} \quad (5.11)$$

$$\hat{J}_3\Omega(\theta, \phi)_{q,j,k} = k\Omega(\theta, \phi)_{q,j,k} \quad (5.12)$$

$$\vec{L} \cdot \vec{S}\Omega(\theta, \phi)_{q,j,k} = \frac{q}{2}\Omega(\theta, \phi)_{q,j,k}; \quad q = \begin{cases} l \\ -l-1 \end{cases} \quad (5.13)$$

$$\hat{L}^2\Omega(\theta, \phi)_{q,j,k} = q(q+1)\Omega(\theta, \phi)_{q,j,k} \quad (5.14)$$

$$\psi(r, \theta, \phi)_{q,n,j,k} = \rho(r)_{q,n,j} \Omega(\theta, \phi)_{q,j,k} \quad (5.15)$$

$$\Omega(\theta, \phi)_{l,j,k} = \frac{1}{\sqrt{2j}} \begin{pmatrix} \sqrt{j+k} Y_{j-\frac{1}{2}, k-\frac{1}{2}}(\theta, \phi) \\ \sqrt{j-k} Y_{j-\frac{1}{2}, k+\frac{1}{2}}(\theta, \phi) \end{pmatrix} \quad (5.16)$$

$$\Omega(\theta, \phi)_{-l-1,j,k} = \frac{1}{\sqrt{2j+2}} \begin{pmatrix} \sqrt{j-k+1} Y_{j+\frac{1}{2}, k-\frac{1}{2}}(\theta, \phi) \\ \sqrt{j+k+1} Y_{j+\frac{1}{2}, k+\frac{1}{2}}(\theta, \phi) \end{pmatrix} \quad (5.17)$$

and the functions $Y_{l,m}(\theta, \phi)$ are the usual spherical harmonic functions:

$$\hat{L}^2 Y_{l,m}(\theta, \phi) = l(l+1) Y_{l,m}(\theta, \phi) \quad (5.18)$$

$$\hat{L}_3 Y_{l,m}(\theta, \phi) = m Y_{l,m}(\theta, \phi). \quad (5.19)$$

In view of (5.11) - (5.17) we can reduce the 3-dimensional Hamiltonian operator \hat{H} to the following radial one:

$$\hat{\mathcal{H}} = \langle \Omega(\theta, \phi)_{q,j,k} | \hat{H} | \Omega(\theta, \phi)_{q,j,k} \rangle = \quad (5.20)$$

$$= \begin{cases} q = l \rightarrow -\frac{\hbar^2}{2} \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{(l+\gamma)(l+\gamma+1)}{r^2} \right) - \frac{\alpha}{r} \\ q = -l-1 \rightarrow -\frac{\hbar^2}{2} \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{(l-\gamma)(l-\gamma+1)}{r^2} \right) - \frac{\alpha}{r}. \end{cases} \quad (5.21)$$

It is straightforward to get the explicit expression for the bound state eigenfunctions of $\hat{\mathcal{H}}$

$$\rho_{l,n,j} \propto r^{j+\gamma-\frac{1}{2}} e^{-\frac{\alpha}{\hbar^2(n+\gamma+j+\frac{1}{2})}r} L_n^{2j+2\gamma} \left(\frac{2\alpha r}{\hbar^2(n+\gamma+j+\frac{1}{2})} \right) \quad (5.22)$$

$$\rho_{-l-1,n,j} \propto r^{j-\gamma+\frac{1}{2}} e^{-\frac{\alpha}{\hbar^2(n-\gamma+j+\frac{3}{2})}r} L_n^{2j-2\gamma+2} \left(\frac{2\alpha r}{\hbar^2(n-\gamma+j+\frac{3}{2})} \right) \quad (5.23)$$

$L_n^k(x)$ are Laguerre polynomials.

Finally we have

$$\hat{\mathcal{H}}\rho_{l,n,j}(r) = -\frac{\alpha^2}{2\hbar^2(n+j+\gamma+\frac{1}{2})^2}\rho_{l,n,j}(r) \quad (5.24)$$

$$\hat{\mathcal{H}}\rho_{-l-1,n,j}(r) = -\frac{\alpha^2}{2\hbar^2(n+j-\gamma+\frac{1}{2})^2}\rho_{-l-1,n,j}(r). \quad (5.25)$$

Conclusion and open problems

The situation in the field of classical and quantum superintegrability can be summed up as follows.

- Quadratic superintegrability for Hamiltonians of the form (1.1) is well understood and is related to the separation of variables in configuration space and to quadratic algebras of the integrals of motion.
- Recently interesting infinite families of superintegrable systems with integrals that are polynomial of higher order in the momenta, or even rational functions.
- The integrals form higher order polynomial algebras and possibly more general algebras.
- The known systems are mainly in E_2 but can easily be extended to E_3 and other spaces.
- Other types of superintegrable systems are being studied, namely those involving velocity dependent potentials, particles with spin, or relativistic particles.

Among the open problems, let us just mention some conceptual ones.

1. Is the TTW conjecture correct, i. e. does maximally superintegrability imply exact solvability? Are the Hamiltonians and integrals of motion always in the enveloping algebra of an underlying "hidden" Lie algebra (a finite or infinite one)?
2. To provide an abstract classification of polynomial algebras and their representation theory. Comparison: "Graded Lie algebras" in physics and "superalgebras" (V. Kac) or "current algebra" and Kac-Moody algebras.
3. The role of Painlevé transcendents in quantum theory.
4. Integrability, superintegrability and exact solvability in discrete quantum mechanics umbral calculus.
5. Are there superintegrable many body problems in 1 dimension involving Painlevé transcendents? The three-body rational, trigonometric and elliptic problems have been generalized to n particles. How about n body Painlevé potentials?

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