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The adjoint equation method for constructing first integrals of difference equations

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Abstract
A new method for finding first integrals of discrete equations is presented. It can be used for discrete equations which do not possess a variational (Lagrangian or Hamiltonian) formulation. The method is based on a newly established identity which links symmetries of the underlying discrete equations, solutions of the discrete adjoint equations and first integrals. The method is applied to an invariant mapping and to discretizations of second order and third order ordinary differential equations. In examples the set of independent first integrals makes it possible to find the general solution of the discrete equations.

Keywords: Lie symmetry, first integral, discrete equations

1. Introduction

Considerable progress has been made over the last 25 years in the applications of Lie group theory to difference equations (for reviews see [14, 32, 44] and for original papers [7, 8, 10–13, 15, 17–22, 24, 30, 31, 33–35, 39, 42]). The overall aim of the program is to turn Lie group theory into a tool for solving discrete equations that is as efficient as it is for differential ones.

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For ordinary differential equations (ODEs) one of the important applications of Lie group theory is to reduce the order of the equation and ideally to solve it analytically and explicitly. Essentially there are two ways of doing this, once a non-trivial Lie point symmetry group of the equation is found. One is to perform a transformation of the independent and dependent variables that takes the Lie algebra into a convenient form. This also transforms the equation to a form in which the reduction of the order becomes obvious.

An alternative method is to use the Lie point symmetry group to construct first integrals of the equation that are of lower order than the equation (or system of equations) itself. This can be done if a Lagrangian exists and the symmetries are variational ones. If a sufficient number of first integrals can be obtained using the Noether theorem, then the derivatives can be eliminated from the set of first integrals. This provides a solution of the original equation by purely algebraic operations, without any changes of variables or any integration.

If no invariant Lagrangian exists alternative methods of constructing lower order first integrals have been proposed in [2, 3, 5, 6] and in [26–28]. They make use of the so-called adjoint equations, solutions of which one uses to construct the required first integrals. We shall call this the 'adjoint equation method'. The same method for differential equations was called the method of symmetry–adjoint-symmetry pairs [5]. We prefer to shorten the terminology, especially for the case of difference equations. We mention that solutions of the adjoint equations have been called adjoint symmetries in [5] and cosymmetries in [36].

The integration methods based on transformations of coordinates have not been adapted to difference equations. The algebraic methods based on invariant Lagrangians and Hamiltonians have been adapted and successfully applied to solve three point difference schemes in [13–15, 21, 22] and [18–20], respectively. A research note on adapting the 'adjoint equation method' to difference equations has been published in [45].

The purpose of this paper is to present and justify the adjoint equation method for difference systems with an arbitrary number of variables and also to document its usefulness on examples. The paper is organized as follows. In section 2 we present a brief summary of the adjoint equation method for an arbitrary system of partial or ordinary differential equations (PDEs or ODEs). Section 3 specializes the theory sketched in section 2 to the case of one scalar ODE. The adjoint equation method for discrete systems is presented in section 4. This theory is specialized to the case of scalar discrete equations (mappings) and discretizations of scalar ODEs in sections 5 and 6, respectively. Finally, section 7 provides concluding remarks.

2. Adjoint equation method for constructing conservation laws for differential equations

Let us consider a system of nth order PDEs

$$F_\beta\left(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, \underset{2}{\mathbf{u}}, \ldots, \underset{n}{\mathbf{u}}\right) = 0, \qquad \beta = 1, \ldots, r, \tag{2.1}$$

where $\mathbf{x} = (x^1, \ldots, x^p)$, $\mathbf{u} = (u^1, \ldots, u^q)$,

$$\underset{1}{\mathbf{u}} := \left\{ u_i^k \right\} = \left\{ \frac{\partial u^k}{\partial x^i} \right\}, \quad \ldots, \quad \underset{s}{\mathbf{u}} := \{ u_{i_1 \ldots i_s} \} = \left\{ \frac{\partial^s u^k}{\partial x^{i_1} \ldots \partial x^{i_s}} \right\}, \quad \ldots,$$

$i = 1, \ldots, p$, $k = 1, \ldots, q$.

Let $L_{\alpha\beta}$ be a linear operator

$$L_{\alpha\beta} = \sum_{k=0}^{\infty} F_{\beta,u^{\alpha}_{i1\ldots ik}} D_{i_k} \ldots D_{i_1}, \qquad F_{\beta,u^{\alpha}_{i1\ldots ik}} = \frac{\partial F_\beta}{\partial u^{\alpha}_{i_1\ldots i_k}}, \tag{2.2}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u^k_i \frac{\partial}{\partial u^k} + v^k_i \frac{\partial}{\partial v^k} + u^k_{ji} \frac{\partial}{\partial u^k_j} + v^k_{ji} \frac{\partial}{\partial v^k_j} + u^k_{jli} \frac{\partial}{\partial u^k_{jl}} + v^k_{jli} \frac{\partial}{\partial v^k_{jl}} + \ldots$$

is the total derivative operator. The *adjoint equations* $F^*_\alpha = 0$ and the *adjoint operators* $L^*_{\alpha\beta}$ are given by the variational derivatives (or Euler–Lagrange operators):

$$F^*_\alpha = L^*_{\alpha\beta} v^\beta = \frac{\delta}{\delta u^\alpha}\left(v^\beta F_\beta\right) = \sum_{k=0}^{\infty} (-1)^k D_{i_1} \ldots D_{i_k}\left(v^\beta F_{\beta,u^{\alpha}_{i1\ldots ik}}\right) = 0, \quad \alpha = 1, \ldots, q. \tag{2.3}$$

We assume summation over repeated indices. Notice that the adjoint equations are always linear equations for $\mathbf{v} = (v^1, \ldots, v^r)$ with coefficients that in general depend upon $\mathbf{u}$ (a solution of (2.1)).

Equation (2.3) is the classical adjoint equation in the case when the original equation (2.1) is linear. When equation (2.1) is nonlinear the action of the operator $L_{\alpha\beta}$ yields a *linearization* of the original equation (the same result can be obtained by applying the Frechet derivative to equation (2.1)). In that case the adjoint equation becomes a nonclassical adjoint equation for the original nonlinear equation (for a discussion of this point see [1]).

The *basic operator identity* is the following

$$v^\beta L_{\alpha\beta} w^\alpha - w^\alpha L^*_{\alpha\beta} v^\beta = D_i C^i, \tag{2.4}$$

where v^β and w^α are arbitrary functions of $\mathbf{x}$, $\mathbf{u}$ and derivatives of $\mathbf{u}$. Here

$$C^i = \sum_{k=0}^{\infty} D_{i_k} \ldots D_{i_1}\left(w^\alpha\right) \frac{\delta}{\delta u^{\alpha}_{i_1\ldots i_k i}}\left(v^\beta F_\beta\right), \tag{2.5}$$

where

$$\frac{\delta}{\delta u^{\alpha}_{i_1\ldots i_k i}} = \sum_{s=0}^{\infty} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u^{\alpha}_{i_1\ldots i_k i j_1 \ldots j_s}} \tag{2.6}$$

are higher order variational operators (or higher order Euler–Lagrange operators). Since the scalar ($q = r = 1$) relation is probably due to Lagrange (see for example [9], equation (2.75) on p 80), we refer to (2.4) as the *Lagrange identity*. Identity (2.4) for the case of systems of ODEs ($p = 1$) already appeared in [3].

Remark 2.1. Applying higher order variational operators (2.6) as well as symmetry operators given below, we assume that they are extended to all possible mixed derivatives. Partial differentiation with respect to $u^{\alpha}_{i_1\ldots i_r}$ stands for differentiation with resect to this special mixed derivative with order of indexes $i_1, \ldots, i_r$.

We will be interested in Lie symmetries [25, 38, 41]

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} \zeta^{\alpha}_{i_1\ldots i_s} \frac{\partial}{\partial u^{\alpha}_{i_1\ldots i_s}}, \tag{2.7}$$

where ξ^i and η^α are some functions of $\mathbf{x}$, $\mathbf{u}$ and a finite number of derivatives of $\mathbf{u}$ and

$$\zeta^\alpha_{i_1\ldots i_s} = D_{i_s}\ldots D_{i_1}\left(\eta^\alpha - \xi^j u^\alpha_j\right) + \xi^i u^\alpha_{i_1\ldots i_s i}.$$

Note that for point symmetries ξ^i and η^α depend only on $\mathbf{x}$ and $\mathbf{u}$. To each symmetry (2.7) there corresponds the evolutionary (or canonical) symmetry

$$\bar{X} = \bar{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty} \bar{\zeta}^\alpha_{i_1\ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \tag{2.8}$$

where

$$\bar{\eta}^\alpha = \eta^\alpha - \xi^j u^\alpha_j, \qquad \bar{\zeta}^\alpha_{i_1\ldots i_s} = D_{i_s}\ldots D_{i_1}\left(\bar{\eta}^\alpha\right).$$

The identity (2.4) can be used to link symmetries of the differential equations (2.1), solutions of the corresponding adjoint equations (2.3) and conservation laws.

Choosing $w^\alpha = \bar{\eta}^\alpha = \eta^\alpha - \xi^j u^\alpha_j$, we obtain the identity

$$v^\beta \bar{X} F_\beta = \bar{\eta}^\alpha F^*_\alpha + D_i C^i \tag{2.9}$$

for evolutionary operators (2.8). For the Lie symmetry operator (2.7) we obtain

$$v^\beta X F_\beta = v^\beta \xi^i D_i F_\beta + \left(\eta^\alpha - \xi^j u^\alpha_j\right) F^*_\alpha + D_i C^i. \tag{2.10}$$

Here the quantities C^i are

$$C^i = \sum_{k=0}^{\infty} D_{i_k}\ldots D_{i_1}\left(\eta^\alpha - \xi^j u^\alpha_j\right) \frac{\delta}{\delta u^\alpha_{i_1\ldots i_k i}}\left(v^\beta F_\beta\right). \tag{2.11}$$

The following theorem is based on the Lagrange identity:

Theorem 2.2. *The system of equations (2.1) and its adjoint system (2.3) possess the following conservation law*

$$D_i C^i \Big|_{(2.1),(2.3)} = 0 \tag{2.12}$$

for each Lie symmetry (2.7) of the differential equations (2.1) and for each solution of the adjoint equations (2.3).

Proof. The result follows directly from equation (2.10). Indeed, $XF_\beta = 0$ because it is a symmetry criterion for equations (2.1), $D_i(F_\beta) = 0$ since it is a differential consequence of equations (2.1), and $F^*_\alpha = 0$ on a solution of adjoint equations (2.3). □

Since we are interested in equations (2.1) we need conservation laws for these equations alone, i.e., without using solutions of the adjoint equations (2.3).

One can get rid of the adjoint variables $\mathbf{v}$ figuring in the conservation law (2.12) and subsequent formulas. The identity (2.9) and the idea of solving the adjoint equations in terms of solutions of the original equations were explicitly presented in [2] (see also [6]). These ideas were also suggested and further developed in [26–28], where numerous examples for PDEs were worked out explicitly. The introduction of adjoint variables, of linear equations adjoint to nonlinear ones and the extension variational principles for equations without classical Lagrangians were also considered in [1, 29, 43] and others.

Theorem 2.3. *Let the adjoint equations (2.3) be satisfied for all solutions of the differential equations (2.1) upon a substitution*

$$\mathbf{v} = \boldsymbol{\varphi}\left(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, \underset{2}{\mathbf{u}}, \ldots\right), \qquad \boldsymbol{\varphi} \not\equiv \mathbf{0}. \tag{2.13}$$

Then, any Lie symmetry (2.7) of the equations (2.1) leads to the conservation law

$$D_i C^i \Big|_{(2.1)} = 0, \tag{2.14}$$

where $\mathbf{v}$ *and its derivatives should be eliminated via equation (2.13) and its differential consequences.*

Remark 2.4. For some symmetries and some solutions of the adjoint equations (2.3) the conservation laws may be trivial. For example, the quantities C_i may simply be numbers. See for instance example 3.1 below. Moreover the non-trivial first integrals are not necessarily functionally independent (see the same example).

Remark 2.5. The same operator identities (2.9) and (2.10) form the basis of the Noether theorem [37] for Lagrangian systems (see [25] for details). Indeed, consider the case $r = 1$, put $v = 1$ and apply it to a Lagrangian $F = \mathcal{L}(\mathbf{x}, \mathbf{u}, \underset{1}{\mathbf{u}}, \underset{2}{\mathbf{u}}, \ldots)$. Then we get the following identities

$$\bar{X}\mathcal{L} = \bar{\eta}^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i\left(\bar{N}^i \mathcal{L}\right),$$

$$\bar{N}^i = \sum_{k=0}^{\infty} D_{i_k} \ldots D_{i_1}\left(\bar{\eta}^\alpha\right) \frac{\delta}{\delta u^\alpha_{i_1 \ldots i_k i}}$$

and

$$X\mathcal{L} + \mathcal{L} D_i \xi^i = \left(\eta^\alpha - \xi^j u^\alpha_j\right) \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i\left(N^i \mathcal{L}\right),$$

$$N^i = \xi^i + \sum_{k=0}^{\infty} D_{i_k} \ldots D_{i_1}\left(\eta^\alpha - \xi^j u^\alpha_j\right) \frac{\delta}{\delta u^\alpha_{i_1 \ldots i_k i}},$$

which yield a conservation law

$$D_i \bar{C}^i = 0, \qquad \bar{C}^i = \bar{N}^i \mathcal{L}$$

and

$$D_i C^i = 0, \qquad C^i = N^i \mathcal{L},$$

correspondingly, for the Euler–Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \qquad \alpha = 1, \ldots, q$$

provided that the Lagrangian is invariant, i.e. satisfies

$$\bar{X}\mathcal{L} = 0 \qquad \text{and} \qquad X\mathcal{L} + \mathcal{L} D_i \xi^i = 0,$$

respectively. Operators N^i and $\bar{N}^i$ are called Noether operators.

3. The case of one ODE

In this section we restrict ourselves to scalar ODEs. It is a particular case of the general theory sketched in the previous section. We restrict ourselves to Lie point symmetries because later we will consider the discrete case, to which we wish to adapt the Lie point symmetry approach.

Let us consider a scalar ODE of order n

$$F\left(x, u, \dot{u}, \ddot{u}, \dots, u^{(n)}\right) = 0. \tag{3.1}$$

We will be interested in Lie point symmetries

$$X = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial \dot{u}} + \zeta_2\frac{\partial}{\partial \ddot{u}} + \dots + \zeta_k\frac{\partial}{\partial u^{(k)}} + \dots, \tag{3.2}$$

where

$$\zeta_k = D^k\left(\eta - \xi\dot{u}\right) + \xi u^{(k+1)}$$

and

$$D = \frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \ddot{v}\frac{\partial}{\partial \dot{v}} + \dots + u^{(k+1)}\frac{\partial}{\partial u^{(k)}} + v^{(k+1)}\frac{\partial}{\partial v^{(k)}} + \dots$$

is the total differentiation operator.

To each Lie point symmetry (3.2) there corresponds the symmetry in evolutionary form

$$\bar{X} = \bar{\eta}\frac{\partial}{\partial u} + \bar{\zeta}_1\frac{\partial}{\partial \dot{u}} + \bar{\zeta}_2\frac{\partial}{\partial \ddot{u}} + \dots + \bar{\zeta}_k\frac{\partial}{\partial u^{(k)}} + \dots, \tag{3.3}$$

where

$$\bar{\eta} = \eta(x, u) - \xi(x, u)\dot{u},$$

$$\bar{\zeta}_1 = D(\bar{\eta}), \qquad \dots, \qquad \bar{\zeta}_k = D^k(\bar{\eta}), \qquad \dots.$$

By means of the variational operator

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D\frac{\partial}{\partial \dot{u}} + D^2\frac{\partial}{\partial \ddot{u}} + \dots + (-1)^k D^k\frac{\partial}{\partial u^{(k)}} + \dots \tag{3.4}$$

we introduce the adjoint equation

$$F^* = \frac{\delta}{\delta u}(vF) = 0. \tag{3.5}$$

Thus (2.3) simplifies to

$$F^* = v\frac{\partial F}{\partial u} - D\left(v\frac{\partial F}{\partial \dot{u}}\right) + D^2\left(v\frac{\partial F}{\partial \ddot{u}}\right) + \dots + (-1)^n D^n\left(v\frac{\partial F}{\partial u^{(n)}}\right) = 0.$$

Let us define higher order variational (or Euler–Lagrange) operators

$$\frac{\delta}{\delta u^{(i)}} = \frac{\partial}{\partial u^{(i)}} - D\frac{\partial}{\partial u^{(i+1)}} + D^2\frac{\partial}{\partial u^{(i+2)}} + \dots + (-1)^k D^k\frac{\partial}{\partial u^{(i+k)}} + \dots. \tag{3.6}$$

Note that Euler–Lagrange operator (3.4) belongs to this set as

$$\frac{\delta}{\delta u} = \frac{\delta}{\delta u^{(0)}}.$$

Lemma 3.1 *(Main identity for scalar ODEs). The following identity holds:*

$$vXF = v\xi DF + \left(\eta - \xi \dot{u}\right)F^* + DI, \tag{3.7}$$

where

$$I = \sum_{i=0}^{n-1} D^i\left(\eta - \xi \dot{u}\right)\frac{\delta}{\delta u^{(i+1)}}(vF). \tag{3.8}$$

This is a special case of (2.10), (2.11).

We prefer identity (3.7) instead of the corresponding identity for the canonical operator

$$v\bar{X}F = \bar{\eta}F^* + DI. \tag{3.9}$$

In the discrete case the framework of Lie point symmetries is better developed in terms of standard vector fields (3.2) than evolutionary ones. The goal of this paper is to develop a discrete analog of the identity (3.7).

Let us examine the identity (3.7) on solutions of the ODE (3.1). The left-hand side is zero if operator X is a symmetry of the ODE. The first term on the right-hand side contains DF and drops out as a differential consequence of the ODE. We are left with

$$\left(\eta - \xi \dot{u}\right)F^*\Big|_{F=0} + DI\,|_{F=0} = 0.$$

If we can find a substitution for the function v providing $F^* = 0$, then we can get rid of the adjoint equation. Thus, we obtain a first integral of the ODE. Let us formulate this as the following theorem.

Theorem 3.2 *(Main theorem for scalar ODEs). Let the adjoint equation (3.5) be satisfied for all solutions of the original ODE (3.1) upon a substitution*

$$v = \varphi(x, u), \qquad \varphi \not\equiv 0. \tag{3.10}$$

Then, any Lie point symmetry (3.2) of the equation (3.1) leads to the first integral

$$I = \left[\sum_{i=0}^{n-1} D^i\left(\eta - \xi \dot{u}\right)\frac{\delta}{\delta u^{(i+1)}}(vF)\right]_{v=\varphi}, \tag{3.11}$$

where v and its derivatives should be eliminated via equation (3.10) and its differential consequences.

Remark 3.3. Theorem 3.2 has the same content as theorem 3.8.1-1 of reference [5]. We include it here to make this paper self-contained and to establish our notations and terminology.

Remark 3.4. First integrals I, given by (3.11), can depend on $u^{(n)}$ as well as higher derivatives. We will call such expressions *higher* first integrals. It is reasonable to use the ODE (3.1) and its differential consequences to express these first integrals as functions of the minimal set of variables, i.e., in the form

$$\tilde{I}\left(x, u, \dot{u}, \ldots, u^{(n-1)}\right) = I\left(x, u, \dot{u}, \ldots, u^{(n-1)}, \ldots\right)\Big|_{F=0}.$$

In the examples of this section we will bypass the higher first integrals I and provide only the final results $\tilde{I}$.

Remark 3.5. Theorem 3.2 can be extended from point-wise substitutions (3.10) to differential substitutions

$$v = \varphi(x, u, \dot{u}), \qquad \ldots, \qquad v = \varphi\left(x, u, \dot{u}, \ddot{u}, \ldots, u^{(n-1)}\right), \qquad \varphi \not\equiv 0.$$

Let us investigate the ODE

$$F = \frac{1}{\dot{u}^2}\left(\dot{u}\dddot{u} - \frac{3}{2}\ddot{u}^2\right) - f(x) = 0. \tag{3.12}$$

Its numerical solutions were considered in [7, 8] using a symmetry-preserving discretization.

The first term is the well-known Schwarzian derivative that has many important applications in mathematics, physics and even (originally) in cartography (for an interesting review see [40]).

In the general case this ODE admits the symmetry group $SL(2, \mathbb{R})$. Its Lie algebra is realized as

$$X_1 = \frac{\partial}{\partial u}, \qquad X_2 = u\frac{\partial}{\partial u}, \qquad X_3 = u^2\frac{\partial}{\partial u}, \tag{3.13}$$

for $f = M =$ const there is an additional symmetry

$$X_4 = \frac{\partial}{\partial x}, \tag{3.14}$$

and for $f = M = 0$ there are two further symmetries

$$X_5 = x\frac{\partial}{\partial x}, \qquad X_6 = x^2\frac{\partial}{\partial x}. \tag{3.15}$$

Example 3.1. The ODE that we shall consider is

$$F = \frac{1}{\dot{u}^2}\left(\dot{u}\dddot{u} - \frac{3}{2}\ddot{u}^2\right) - M = 0, \qquad M = \text{const}. \tag{3.16}$$

Let us solve ODE (3.16) using theorem 3.2. The idea is to find three independent first integrals of (3.16) and then to eliminate the derivatives $\dot{u}$ and $\ddot{u}$ from them (third order ODEs can have at most three independent first integrals).

The adjoint equation (3.5) takes the form

$$F^* = -\frac{1}{\dot{u}}\left(D^3 v + 2MDv\right) = 0. \tag{3.17}$$

Let us look for solutions of the form $v = v(x)$, which is the simplest ansatz. We obtain three independent solutions of the adjoint equation (3.17)

$$\begin{aligned} &M = 0\colon v_a = 1,\ v_b = x, && v_c = x^2; \\ &M > 0\colon v_a = 1,\ v_b = \cos(2\omega x), && v_c = \sin(2\omega x), \quad \omega = \sqrt{M/2}; \\ &M < 0\colon v_a = 1,\ v_b = \cosh(2\omega x), && v_c = \sinh(2\omega x), \quad \omega = \sqrt{-M/2}. \end{aligned} \tag{3.18}$$

We will use these solutions of the adjoint equation to find first integrals of the ODE (3.16).

Let us use symmetries (3.13)–(3.15) and solutions of the adjoint equation (3.17) to construct first integrals. The notation $\tilde{I}_{j\alpha}$ means that this integral corresponds to symmetry X_j and solution v_α of the adjoint equation.

For all values of the parameter M there is only one common solution of the adjoint equation, namely

$$v_a(x) = 1. \tag{3.19}$$

It provides us with the first integrals

$$\tilde{I}_{1a} = \frac{1}{2}\frac{\ddot{u}^2}{\dot{u}^3} + \frac{M}{\dot{u}}, \qquad \tilde{I}_{2a} = u\left(\frac{1}{2}\frac{\ddot{u}^2}{\dot{u}^3} + \frac{M}{\dot{u}}\right) - \frac{\ddot{u}}{\dot{u}},$$

$$\tilde{I}_{3a} = u^2\left(\frac{1}{2}\frac{\ddot{u}^2}{\dot{u}^3} + \frac{M}{\dot{u}}\right) - 2\,u\frac{\ddot{u}}{\dot{u}} + 2\dot{u}, \qquad \tilde{I}_{4a} \equiv -2M.$$

Two additional first integrals for $M = 0$ are trivial:

$$\tilde{I}_{5a} \equiv 0, \qquad \tilde{I}_{6a} \equiv -2.$$

The non-trivial first integrals obey the relation

$$\tilde{I}_{1a}\tilde{I}_{3a} - \tilde{I}_{2a}^2 = 2M.$$

Thus we have only two independent first integrals and it is not sufficient for the integration of the third order ODE. To find a sufficient number of first integrals we will consider solutions of the adjoint equation which are specific for particular cases of the parameter M. Let us go through different cases of the parameter.

Case: $M = 0$

This case was considered in [45]. The solution consists of the generic three-parameter solution

$$u(x) = \frac{1}{C_1 x + C_2} + C_3, \tag{3.20}$$

where $C_1 \neq 0$, C_2 and C_3 are constants, and the degenerate two-parameter solution

$$u(x) = C_1 x + C_2, \qquad C_1 \neq 0. \tag{3.21}$$

Case: $M > 0$

Special solutions of the adjoint equation are (3.19) and

$$v_b(x) = \cos(2\omega x) \qquad \text{and} \qquad v_c(x) = \sin(2\omega x), \qquad \omega = \sqrt{\frac{M}{2}}. \tag{3.22}$$

For v_b and symmetry X_1 we compute the first integral

$$\tilde{I}_{1b} = \cos(2\omega x)\left(\frac{1}{2}\frac{\ddot{u}^2}{\dot{u}^3} - \frac{M}{\dot{u}}\right) - 2\omega\,\sin(2\omega x)\left(\frac{\ddot{u}}{\dot{u}^2}\right).$$

We choose $\tilde{I}_{1a}$, $\tilde{I}_{2a}$ and $\tilde{I}_{1b}$ as three independent first integrals. The Jacobian is

$$J = \det\left(\frac{\partial\left(\tilde{I}_{1a}, \tilde{I}_{2a}, \tilde{I}_{1b}\right)}{\partial(u, \dot{u}, \ddot{u})}\right)$$

$$= \frac{\omega}{2} \frac{\ddot{u}^2 + 4\omega^2 \dot{u}^2}{\dot{u}^9} \Big(\sin(2\omega x)\ddot{u}^2 + 4\omega \cos(2\omega x)\dot{u}\ddot{u} - 4\omega^2 \sin(2\omega x)\dot{u}^2 \Big).$$

(1) For $J \neq 0$ we obtain the solutions

$$u(x) = C_1 \tan\big(\omega x + C_2\big) + C_3, \tag{3.23}$$

where $C_1 \neq 0$, $C_2 \neq \frac{\pi}{2}n$, $n \in \mathbb{Z}$ and C_3 are constants.

(2) The equality $J = 0$ can happen in two cases:

(a) The first case is

$$\ddot{u}^2 + 4\omega^2 \dot{u}^2 = 0, \qquad \dot{u} \neq 0$$

and it has no real solutions.

(b) The second case is

$$\sin(2\omega x)\ddot{u}^2 + 4\omega \cos(2\omega x)\dot{u}\ddot{u} - 4\omega^2 \sin(2\omega x)\dot{u}^2 = 0, \qquad \dot{u} \neq 0.$$

It can be solved for $\ddot{u}$:

$$\ddot{u} = 2\omega \tan(\omega x)\dot{u} \qquad \text{or} \qquad \ddot{u} = -2\omega \cot(\omega x)\dot{u}, \qquad \dot{u} \neq 0.$$

The general solutions of these equations are:

$$u(x) = C_1 \tan(\omega x) + C_2 \qquad \text{or} \qquad u(x) = C_1 \cot(\omega x) + C_2, \tag{3.24}$$

where $C_1 \neq 0$ and C_2 are integration constants.

Finally, we unite solutions (3.23) and (3.24) into the generic solution of the ODE

$$u(x) = C_1 \tan\big(\omega x + C_2\big) + C_3, \tag{3.25}$$

where $C_1 \neq 0$, C_2 and C_3 are integration constants.

Case: $M < 0$

This case was examined in [16], where we obtained the generic solutions

$$u(x) = C_1 \tanh\big(\omega x + C_2\big) + C_3, \qquad u(x) = C_1 \coth\big(\omega x + C_2\big) + C_3 \tag{3.26}$$

and the degenerate solutions

$$u(x) = C_1 \mathrm{e}^{\pm 2\omega x} + C_2, \tag{3.27}$$

where $C_1 \neq 0$, C_2 and C_3 are constants.

Example 3.2. The ODE

$$F = \frac{1}{\dot{u}^2}\left(\dot{u}\dddot{u} - \frac{3}{2}\ddot{u}^2\right) - \frac{1}{x^2} = 0, \tag{3.28}$$

admits symmetries

$$X_1 = \frac{\partial}{\partial u}, \qquad X_2 = u\frac{\partial}{\partial u}, \qquad X_3 = u^2\frac{\partial}{\partial u},$$

$$X_4 = x\frac{\partial}{\partial x}, \qquad X_5 = x \sin(\ln|x|)\frac{\partial}{\partial x}, \qquad X_6 = x \cos(\ln|x|)\frac{\partial}{\partial x}. \tag{3.29}$$

To find the solution of this equation one can use the adjoint equation method as it was illustrated in the previous example. The adjoint equation is

$$F^* = \frac{-x^3D^3v - 2xDv + 2v}{x^3\dot{u}} = 0. \tag{3.30}$$

Three independent solutions of (3.30) are

$$v_a = x, \qquad v_b = x \sin(\ln|x|) \qquad \text{and} \qquad v_c = x \cos(\ln|x|). \tag{3.31}$$

One can use these solutions of the adjoint equation and the symmetries (3.29) to find first integrals, which can be used to obtain the solution of the ODE.

Alternatively, we can exploit the change of the independent variable

$$x \to \tilde{x} = \tan\left(\ln\sqrt{|x|}\right), \tag{3.32}$$

which transforms ODE (3.28) into the ODE (3.16) with $M = 0$. Thus, we can use the results of the previous example, namely the solutions (3.20) and (3.21), to find the general solution of ODE (3.28) as

$$u(x) = C_1 \tan\left(\ln\sqrt{|x|} + C_2\right) + C_3, \tag{3.33}$$

where $C_1 \neq 0$, C_2 and C_3 are constants.$\diamond$

The direct method versus the adjoint equation method.

Let us compare the direct method [3, 5] with the adjoint equation method. We consider a scalar ODE (3.1) and assume that this ODE is solved with respect to the highest derivative $u^{(n)}$:

$$F = u^{(n)} - f\left(x, u, \dot{u}, \ddot{u}, \ldots, u^{(n-1)}\right) = 0. \tag{3.34}$$

We are interested in first integrals

$$I = I\left(x, u, \dot{u}, \ldots, u^{(n-1)}\right) \tag{3.35}$$

of this ODE such that

$$D(I) = \Lambda F, \tag{3.36}$$

holds identically in the whole space for some nonsingular function

$$\Lambda = \Lambda\left(x, u, \dot{u}, \ldots, u^{(n-1)}\right), \tag{3.37}$$

called an *integrating factor*. Since the left-hand side of relation (3.36) is a total derivative it is annihilated by the action of the variational operator. Therefore we obtain the equation

$$\frac{\delta}{\delta u}(\Lambda F) = 0. \tag{3.38}$$

Let us mention that every first integral (3.35) corresponds to some non-zero integrating factor (3.37), which is a solution of equation (3.38).

Let us compare with the adjoint equation

$$\left.\frac{\delta}{\delta u}(vF)\right|_{F=0} = 0, \qquad v = \varphi\left(x, u, \dot{u}, \ldots, u^{(n-1)}\right). \tag{3.39}$$

If we consider Λ and φ which depend on the same variables, we obtain the following result.

Proposition 3.6. *An integrating factor is always a solution of the adjoint equation independently of whether a symmetry of the underlying equation exists. The inverse statement is not true.*

Even though an integration factor is always a solution of the adjoint equation we cannot expect that the direct method and the adjoint equation method will always provide the same first integral.

Using integrating factors, we have to solve the equation (3.36) in order to find a first integral I. Explicit line integral formulas which provide first integrals were given in [5]. The approach based on the solution of the adjoint equation yields the first integral by formula (3.11), which does not require any integration.

It was observed on the examples of this section that the pair consisting of a symmetry X and a solution of the adjoint equation v can generate a trivial first integral. To the contrary, an integrating factor Λ provides a non-trivial first integral. On the other hand, it was observed in [16] that the approach based on theorem 3.2 has the advantage that we can use a simpler ansatz for φ then for Λ.

4. Adjoint equation method for mappings

In this section we will consider mappings (discrete equations) and develop a theory analogous to the continuous case results reviewed in section 3. It should be noted that mappings may not possess continuous limits. Such mappings have no relation to discretizations of ODEs.

Let us consider mappings with the dependent variable

$$\mathbf{u}_m = \left(u_m^1, \ldots, u_m^q\right), \qquad m \in \mathbb{Z}.$$

Discrete systems of order n can be presented as equations involving $n + 1$ points

$$F_\beta(m, \mathbf{u}_m, \mathbf{u}_{m+1}, \mathbf{u}_{m+2}, \ldots, \mathbf{u}_{m+n}) = 0, \qquad \beta = 1, \ldots, r. \tag{4.1}$$

We will assume that these equations can be resolved for $\mathbf{u}_m$ and $\mathbf{u}_{m+n}$. This assumption is necessary to solve the Cauchy problem to the left and to the right from the lattice points containing initial values.

We consider Lie point symmetries

$$X = \eta^\alpha(\mathbf{u})\frac{\partial}{\partial u^\alpha}. \tag{4.2}$$

As in the previous sections we will assume summation over repeated indices. For application to functions on lattice points we consider symmetry operators which are extended to all points involved in equations (4.1)

$$X = \eta_m^\alpha \frac{\partial}{\partial u_m^\alpha} + \eta_{m+1}^\alpha \frac{\partial}{\partial u_{m+1}^\alpha} + \ldots + \eta_{m+n}^\alpha \frac{\partial}{\partial u_{m+n}^\alpha}, \qquad \eta_l^\alpha = \eta^\alpha(\mathbf{u}_l). \tag{4.3}$$

It is helpful to introduce forward and backward shift operators S_+ and S_-:

$$S_+ m = m + 1, \quad S_+ \mathbf{u}_m = \mathbf{u}_{m+1},$$

$$S_- m = m - 1, \quad S_- \mathbf{u}_m = \mathbf{u}_{m-1}.$$

Discrete variational operators are defined by the relation

$$\delta \sum_m \mathcal{F}(m, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_{m+n})$$
$$= \sum_m \delta u_m^\alpha \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+k}^\alpha} \mathcal{F}(m, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_{m+n}).$$

We suppose $\mathcal{F} \to 0$ sufficiently fast when $m \to \pm\infty$ so that the discrete functional is well defined. The relation provides us with discrete variational operators

$$\frac{\delta}{\delta u_m^\alpha} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+k}^\alpha} = \frac{\partial}{\partial u_m^\alpha} + S_- \frac{\partial}{\partial u_{m+1}^\alpha} + \ldots + S_-^k \frac{\partial}{\partial u_{m+k}^\alpha} + \ldots. \tag{4.4}$$

Note that these operators are defined for the system (4.1) of arbitrary order n.

We will make use of adjoint variables $\mathbf{v}_m = (v_m^1, \ldots, v_m^r)$ and adjoint equations

$$F_\alpha^* = \frac{\delta}{\delta u_m^\alpha}\left(v_m^\beta F_\beta\right) = 0, \qquad \alpha = 1, \ldots, q, \tag{4.5}$$

which are always linear for the adjoint variables $\mathbf{v}_m$. These equations can be presented as

$$F_\alpha^* = v_m^\beta \frac{\partial F_\beta}{\partial u_m^\alpha} + v_{m-1}^\beta S_-\left(\frac{\partial F_\beta}{\partial u_{m+1}^\alpha}\right) + \ldots$$
$$+ v_{m-k}^\beta S_-^k\left(\frac{\partial F_\beta}{\partial u_{m+k}^\alpha}\right) + \ldots + v_{m-n}^\beta S_-^n\left(\frac{\partial F_\beta}{\partial u_{m+n}^\alpha}\right) = 0.$$

Now we will obtain the main identity which will be used to find first integrals.

Let us fix the value of index m, which corresponds to the left point in the equations (4.1), and define *higher* order discrete Euler–Lagrange operators

$$\frac{\delta}{\delta u_{m(j)}^\alpha} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+j+k}^\alpha} = \frac{\partial}{\partial u_{m+j}^\alpha} + S_- \frac{\partial}{\partial u_{m+j+1}^\alpha} + \ldots + S_-^k \frac{\partial}{\partial u_{m+j+k}^\alpha} + \ldots. \tag{4.6}$$

We note that variational operators (4.4) belong to this family:

$$\frac{\delta}{\delta u_m^\alpha} = \frac{\delta}{\delta u_{m(0)}^\alpha}.$$

Lemma 4.1 *(Main identity for mappings). The following identity holds:*

$$v_m^\beta X F_\beta = \eta_m^\alpha F_\alpha^* + \left(1 - S_-\right) J, \tag{4.7}$$

where

$$J = \sum_{j=1}^{n} \eta_{m+j}^\alpha \frac{\delta}{\delta u_{m(j)}^\alpha}\left(v_m^\beta F_\beta\right). \tag{4.8}$$

Proof. The identity can be obtained by a direct calculation. □

An alternative derivation of the main identity (4.7) can be based on the following operator identity.

Lemma 4.2. *The following operator identity (no summation over α) holds:*

$$\sum_{k=0}^{\infty} \eta_{m+k}^\alpha \frac{\partial}{\partial u_{m+k}^\alpha} = \eta_m^\alpha \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+k}^\alpha} + \left(1 - S_-\right) \sum_{j=1}^{\infty} \eta_{m+j}^\alpha \frac{\delta}{\delta u_{m(j)}^\alpha}. \tag{4.9}$$

If we take summation of the identities (4.9) for all $\alpha = 1, \ldots, q$ and apply the resulting operator identity to the quantity $v_m^\beta F_\beta$, we get the identity (4.7).

Let us adapt the results of the continuous case, given in section 3, to the discrete case.

Theorem 4.3 *(Main theorem for mappings). Let the adjoint equations (4.5) be satisfied for all solutions of the original equations (4.1) upon a substitution*

$$\mathbf{v}_m = \boldsymbol{\varphi}(m, \mathbf{u}_m), \qquad \boldsymbol{\varphi} \not\equiv 0. \tag{4.10}$$

Then, any Lie point symmetry (4.2) of the equations (4.1) leads to the first integral

$$J = \left[\sum_{j=1}^{n} \eta_{m+j}^{\alpha} \frac{\delta}{\delta u_{m(j)}^{\alpha}} \left(v_m^\beta F_\beta \right) \right]_{\mathbf{v}_m = \boldsymbol{\varphi}}, \tag{4.11}$$

where values $\mathbf{v}_m, \ldots, \mathbf{v}_{m-n}$ should be eliminated by means of the equations (4.10) and their shifts to the left.

Proof. The result follows from the identity (4.7) just as theorem 3.2 follows form the identity (3.7). □

Theorem 4.3 is the discrete analog of theorem 3.2 and the operator $1 - S_-$ is the discrete analog of the total derivative D, i.e., discrete first integrals satisfy the equation

$$\left(1 - S_-\right) J = 0$$

on the solutions of the discrete equations.

Remark 4.4. Generally first integrals J, given by (4.11), can depend on more than n points. We will call such expressions *higher* first integrals. Using equations (4.1), we can always reduce this number of points to minimal set, for example, to points $m, \ m+1, \ \ldots, \ m+n-1$, i.e.,

$$\begin{aligned} &\tilde{J}(m, \mathbf{u}_m, \mathbf{u}_{m+1}, \mathbf{u}_{m+2}, \ldots, \mathbf{u}_{m+n-1}) \\ &\quad = J(m, \mathbf{u}_m, \mathbf{u}_{m+1}, \mathbf{u}_{m+2}, \ldots, \mathbf{u}_{m+n-1}, \ldots)\big|_{F_\beta = 0, \ \beta = 1, \ldots, r}. \end{aligned}$$

Remark 4.5. Instead of point substitutions (4.10) we can use generalized substitutions which involve neighbouring points. For systems (4.1) we can consider substitutions like

$$\mathbf{v}_m = \boldsymbol{\varphi}(m, \mathbf{u}_m, \mathbf{u}_{m+1}), \qquad \ldots, \qquad \mathbf{v}_m = \boldsymbol{\varphi}(m, \mathbf{u}_m, \mathbf{u}_{m+1}, \ldots, \mathbf{u}_{m+n-1}).$$

Remark 4.6. The requirement that the substitution (4.10) annihilates the adjoint equation (4.5) on the solutions of the original equations, used in theorem 4.3, can be replaced by a weaker condition

$$\eta_m^\alpha F_\alpha^* = 0.$$

This should hold for a given symmetry X of the system (4.1) on the solutions of this system. This is a weaker condition than the requirement of the theorem that all equations $F_\alpha^* = 0$ hold individually.

In the following sections we will consider applications of these results.

5. Case of mapping involving a single dependent variable

In this section we will consider scalar mappings of order n

$$F(m, u_m, u_{m+1}, u_{m+2}, \ldots, u_{m+n}) = 0 \tag{5.1}$$

admitting symmetries of the form

$$X = \eta(u)\frac{\partial}{\partial u}. \tag{5.2}$$

Such symmetries are expanded as

$$X = \eta_m \frac{\partial}{\partial u_m} + \eta_{m+1}\frac{\partial}{\partial u_{m+1}} + \ldots + \eta_{m+n}\frac{\partial}{\partial u_{m+n}}, \qquad \eta_l = \eta(u_l) \tag{5.3}$$

to all points involved in the equation (5.1).

The corresponding adjoint equation (4.5) has the form

$$F^* = \frac{\delta}{\delta u_m}\left(v_m F\right) = 0, \tag{5.4}$$

where

$$\frac{\delta}{\delta u_m} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+k}} = \frac{\partial}{\partial u_m} + S_- \frac{\partial}{\partial u_{m+1}} + \ldots + S_-^k \frac{\partial}{\partial u_{m+k}} + \ldots \tag{5.5}$$

is the discrete variational operator. Explicitly we have

$$\begin{aligned} F^* = v_m \frac{\partial F}{\partial u_m} + v_{m-1} S_- \left(\frac{\partial F}{\partial u_{m+1}}\right) + \ldots \\ + v_{m-k} S_-^k \left(\frac{\partial F}{\partial u_{m+k}}\right) + \ldots + v_{m-n} S_-^n \left(\frac{\partial F}{\partial u_{m+n}}\right) = 0. \end{aligned}$$

Theorem 4.3 restricted to the case of this section states the following.

Theorem 5.1 *(Main theorem for scalar mappings). Let the adjoint equation (5.4) be satisfied for all solutions of the original equation (5.1) upon a substitution*

$$v_m = \varphi(m, u_m), \qquad \varphi \not\equiv 0. \tag{5.6}$$

Then, any Lie point symmetry (5.2) of the equation (5.1) leads to a first integral

$$J = \left[\sum_{j=1}^{n} \eta_{m+j} \frac{\delta}{\delta u_{m(j)}}\left(v_m F\right)\right]_{v_m = \varphi}, \tag{5.7}$$

where

$$\frac{\delta}{\delta u_{m(j)}} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+j+k}} = \frac{\partial}{\partial u_{m+j}} + S_- \frac{\partial}{\partial u_{m+j+1}} + \ldots + S_-^k \frac{\partial}{\partial u_{m+j+k}} + \ldots$$

and values $v_m, \ldots, v_{m-n}$ should be eliminated by means of the equation (5.6) and its shifts to the left.

Example 5.1. Let us consider the four-point mapping

$$F = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - K = 0, \qquad K \neq 0. \tag{5.8}$$

This equation was considered in [7, 8] as a part of the system (6.29), which will be examined below. We exclude the case $K = 0$ because for $K = 0$ this equation is equivalent to the system

$$u_{m+2} - u_m = 0,$$
$$u_{m+1} - u_m \neq 0,$$

which can be easily solved as

$$u_m = A(-1)^m + B, \qquad A \neq 0.$$

The equation (5.8) admits symmetries

$$X_1 = \frac{\partial}{\partial u}, \qquad X_2 = u\frac{\partial}{\partial u}, \qquad X_3 = u^2\frac{\partial}{\partial u}. \tag{5.9}$$

The adjoint equation (5.4) (after use of the original equation $F = 0$) is

$$F^* = \frac{K(u_{m+2} - u_{m+1})}{(u_{m+2} - u_m)(u_{m+1} - u_m)}\left(v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3}\right) = 0. \tag{5.10}$$

This simplifies to a linear mapping

$$v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3} = 0.$$

It is easy to find solutions $v_m = v_m(m)$. We obtain three independent solutions of the adjoint equation

$$\begin{aligned}
&K = 4: && v_m^a = 1,\ v_m^b = m, && v_m^c = m^2;\\
&0 < K < 4: && v_m^a = 1,\ v_m^b = \cos(2\phi m), && v_m^c = \sin(2\phi m);\\
&0 < K \quad \text{or} \quad K > 4: && v_m^a = 1,\ v_m^b = {\mu_1}^m, && v_m^c = {\mu_2}^m;
\end{aligned} \tag{5.11}$$

where

$$\phi = \arccos\left(\frac{\sqrt{K}}{2}\right) \qquad \text{and} \qquad \mu_{1,2} = \frac{(K-2) \pm \sqrt{K^2 - 4K}}{2}.$$

First of all we consider the solution of the adjoint equation $v_m^a = 1$, which is common for all values $K \neq 0$. Applying theorem 5.1 with this solution and symmetries X_1, X_2 and X_3 and simplifying the obtained first integrals as described in remark 4.4, we get the first integrals

$$\tilde{J}_{1a} = 2\left(\frac{K}{u_{m+2} - u_m} - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m}\right),$$
$$\tilde{J}_{2a} = \frac{K(u_{m+2} + u_m)}{u_{m+2} - u_m} - \frac{2u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{2u_{m+1}}{u_{m+1} - u_m},$$
$$\tilde{J}_{3a} = 2\left(\frac{Ku_{m+2}u_m}{u_{m+2} - u_m} - \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}^2}{u_{m+1} - u_m}\right),$$

respectively. These three first integrals, which hold for all $K \neq 0$, are not independent. They satisfy the relation

$$\tilde{J}_{1a}\tilde{J}_{3a} - \left(\tilde{J}_{2a}\right)^2 = 4K - K^2.$$

To integrate the mapping (5.8) we need one more independent first integral (it should be a first integral which involves m). As in the continuous case we need to consider different cases of the parameter K separately.

Case: $K = 4$

This case was treated in [45], where we found the generic solution

$$u_m = \frac{1}{C_1\, m + C_2} + C_3, \qquad C_1 \neq 0 \tag{5.12}$$

and the degenerate solution

$$u_m = C_1\, m + C_2, \qquad C_1 \neq 0. \tag{5.13}$$

Case: $0 < K < 4$

In this case we obtain two specific solutions of the adjoint equation (5.10)

$$v_m^b = \cos(2\phi m) \qquad \text{and} \qquad v_m^c = \sin(2\phi m), \qquad \phi = \arccos\left(\frac{\sqrt{K}}{2}\right).$$

Application of theorem 5.1 with symmetry X_1 and solution v_m^b gives us the first integral

$$\begin{aligned}\tilde{J}_{1b} = \cos(2\phi m)&\left[\frac{K}{u_{m+2} - u_m} - \frac{K}{u_{m+1} - u_m}\right] + \cos(2\phi(m-1))\\ &\times\left[K\left(\frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m}\right) - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m}\right]\\ &- \cos(2\phi(m-2))\left[\frac{1}{u_{m+2} - u_{m+1}} + \frac{1}{u_{m+1} - u_m}\right].\end{aligned}$$

We can chose first integrals $\tilde{J}_{1a}$, $\tilde{J}_{2a}$ and $\tilde{J}_{1b}$ as three independent first integrals. The Jacobian is

$$\begin{aligned}J &= \det\left(\frac{\partial\left(\tilde{J}_{1a}, \tilde{J}_{2a}, \tilde{J}_{1b}\right)}{\partial(u_m, u_{m+1}, u_{m+2})}\right)\\ &= \frac{K\left(u_{m+2} - 2u_{m+1} + u_m\right)^2 + (4-K)(u_{m+2} - u_m)^2}{(u_{m+1} - u_m)^3(u_{m+2} - u_m)^3(u_{m+2} - u_{m+1})^3}\,\frac{KR_1R_2}{\cos(2\phi(m+1)) - \cos(2\phi m)},\end{aligned}$$

where

$$\begin{aligned}R_1 &= \alpha\left(u_{m+2} - 2u_{m+1} + u_m\right) + \beta(u_{m+2} - u_m),\\ \alpha &= \sin 2\phi(\sin(2\phi m) + \sin\phi), \qquad \beta = (1 - \cos 2\phi)(\cos(2\phi m) - \cos\phi)\end{aligned}$$

and

$$\begin{aligned}R_2 &= \gamma\left(u_{m+2} - 2u_{m+1} + u_m\right) + \delta(u_{m+2} - u_m),\\ \gamma &= \sin 2\phi(\sin(2\phi m) - \sin\phi), \qquad \delta = (1 - \cos 2\phi)(\cos(2\phi m) + \cos\phi).\end{aligned}$$

(1) In the case $J \neq 0$ we set these first integrals equal to constants and obtain the generic solution

$$u_m = C_1 \tan\left(\phi m + C_2\right) + C_3, \tag{5.14}$$

where $C_1 \neq 0$, $C_2 \neq -\frac{3}{2}\phi + \frac{\pi}{2}k$, $k \in \mathbb{Z}$ and C_3 are constants.

(2) Analysis of the case $J = 0$ splits into three subcases.

(a) The case

$$K\left(u_{m+2} - 2u_{m+1} + u_m\right)^2 + (4 - K)(u_{m+2} - u_m)^2 = 0,$$
$$u_{m+1} \neq u_m, \qquad u_{m+2} \neq u_m$$

has no real solutions.

(b) The case

$$R_1 = 0,$$
$$u_{m+1} \neq u_m, \qquad u_{m+2} \neq u_m$$

has solutions

$$u_m = C_1 \tan\left(\phi m + C_2\right) + C_3, \qquad C_1 \neq 0, \quad C_2 = -\frac{3}{2}\phi + \pi k. \tag{5.15}$$

Verification shows that these functions are solutions of the equation (5.8).

(c) The case

$$R_2 = 0,$$
$$u_{m+1} \neq u_m, \qquad u_{m+2} \neq u_m$$

has solutions

$$u_m = C_1 \tan\left(\phi m + C_2\right) + C_3, \qquad C_1 \neq 0, \quad C_2 = -\frac{3}{2}\phi + \frac{\pi}{2} + \pi k, \tag{5.16}$$

which are solutions of the equation.

Finally, we unite the obtained solutions into the generic solution of the form

$$u_m = C_1 \tan\left(\phi m + C_2\right) + C_3, \tag{5.17}$$

where $C_1 \neq 0$, C_2 and C_3 are constants.

Case: $K < 0$ **or** $K > 4$

This case was given in [16]. The solution consists of

$$u_m = C_1 \frac{(4 - K)(\mu_2 - \mu_1) - C_2\left(1 - \mu_1\right)^2 \mu_2^m + \dfrac{K}{C_2}\left(1 - \mu_2\right)^2 \mu_1^m}{K(\mu_2 - \mu_1) - C_2\left(1 - \mu_1^2\right)\mu_2^m + \dfrac{K}{C_2}\left(1 - \mu_2^2\right)\mu_1^m} + C_3, \tag{5.18}$$

where $C_1 \neq 0$, $C_2 \neq 0$ and C_3 are constants, and

$$u_m = C_1 \mu_1^m + C_2 \qquad \text{and} \qquad u_m = C_1 \mu_2^m + C_2, \qquad C_1 \neq 0. \tag{5.19}$$

The generic solution (5.18) can be conveniently rewritten as follows:

- $K > 4$:

$$u_m = C_1 \tanh\left(\psi m + C_2\right) + C_3 \tag{5.20}$$

or

$$u_m = C_1 \coth\left(\psi m + C_2\right) + C_3 \tag{5.21}$$

- $K < 0$:

$$u_m = \begin{cases} C_1 \tanh\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is even,} \\ C_1 \coth\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is odd,} \end{cases} \tag{5.22}$$

or

$$u_m = \begin{cases} C_1 \coth\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is even,} \\ C_1 \tanh\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is odd.} \end{cases} \tag{5.23}$$

Here

$$\psi = \frac{1}{2} \ln \left|\mu_1\right| = \frac{1}{2} \ln \left| \frac{K - 2 + \sqrt{K^2 - 4K}}{2} \right|$$

and $C_1 \neq 0$, C_2 and C_3 are constants.

In addition to the generic solutions we have the degenerate solutions (5.19), which can be rewritten as

$$u_m = C_1 (\operatorname{sgn} K)^m \mathrm{e}^{\pm 2\psi m} + C_2. \tag{5.24}$$

6. Discretizations of a scalar ODE

In this section we are interested in discretizations of a scalar ODE. For the discretization of an ODE of order n we need a difference stencil with at least $n + 1$ points. We will use precisely $n + 1$ points, namely, points $x_m, \ldots, x_{m+n}$. These points are not specified in advance and will be defined by an additional mesh equation [14].

As a discretization we will consider a discrete equation on $n + 1$ points

$$F\left(x_m, u_m, x_{m+1}, u_{m+1}, \ldots, x_{m+n}, u_{m+n}\right) = 0, \tag{6.1}$$

on a mesh

$$\Omega\left(x_m, u_m, x_{m+1}, u_{m+1}, \ldots, x_{m+n}, u_{m+n}\right) = 0. \tag{6.2}$$

These two equations form the difference system to be used. In the continuous limit the first equation goes into the original ODE and the second equation turns into an identity (for example, $0 = 0$).

The Lie point symmetry

$$X = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u} \tag{6.3}$$

gets expanded to the points of the difference stencil as

$$X = \xi_m\frac{\partial}{\partial x_m} + \eta_m\frac{\partial}{\partial u_m} + \ldots + \xi_{m+n}\frac{\partial}{\partial x_{m+n}} + \eta_{m+n}\frac{\partial}{\partial u_{m+n}}, \tag{6.4}$$

$$\xi_l = \xi(x_l, u_l), \qquad \eta_l = \eta(x_l, u_l).$$

The discrete variational operators (4.4) take the form

$$\frac{\delta}{\delta u_m} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+k}} = \frac{\partial}{\partial u_m} + S_-\frac{\partial}{\partial u_{m+1}} + \ldots + S_-^k\frac{\partial}{\partial u_{m+k}} + \ldots, \tag{6.5}$$

$$\frac{\delta}{\delta x_m} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial x_{m+k}} = \frac{\partial}{\partial x_m} + S_-\frac{\partial}{\partial x_{m+1}} + \ldots + S_-^k\frac{\partial}{\partial x_{m+k}} + \ldots. \tag{6.6}$$

The adjoint equations corresponding to the system of difference equations (6.1) and (6.2) are

$$F^* = \frac{\delta}{\delta u_m}\left(v_m F + w_m \Omega\right) = 0 \tag{6.7}$$

and

$$\Omega^* = \frac{\delta}{\delta x_m}\left(v_m F + w_m \Omega\right) = 0, \tag{6.8}$$

where v_m and w_m are adjoint variables. In detail they are

$$\begin{aligned} F^* = v_m\frac{\partial F}{\partial u_m} &+ v_{m-1}S_-\left(\frac{\partial F}{\partial u_{m+1}}\right) + \ldots + v_{m-k}S_-^k\left(\frac{\partial F}{\partial u_{m+k}}\right) + \ldots + v_{m-n}S_-^n\left(\frac{\partial F}{\partial u_{m+n}}\right) \\ &+ w_m\frac{\partial \Omega}{\partial u_m} + w_{m-1}S_-\left(\frac{\partial \Omega}{\partial u_{m+1}}\right) + \ldots + w_{m-k}S_-^k\left(\frac{\partial \Omega}{\partial u_{m+k}}\right) + \ldots \\ &+ w_{m-n}S_-^n\left(\frac{\partial \Omega}{\partial u_{m+n}}\right) = 0 \end{aligned}$$

and

$$\begin{aligned} \Omega^* = v_m\frac{\partial F}{\partial x_m} &+ v_{m-1}S_-\left(\frac{\partial F}{\partial x_{m+1}}\right) + \ldots + v_{m-k}S_-^k\left(\frac{\partial F}{\partial x_{m+k}}\right) + \ldots + v_{m-n}S_-^n\left(\frac{\partial F}{\partial x_{m+n}}\right) \\ &+ w_m\frac{\partial \Omega}{\partial x_m} + w_{m-1}S_-\left(\frac{\partial \Omega}{\partial x_{m+1}}\right) + \ldots + w_{m-k}S_-^k\left(\frac{\partial \Omega}{\partial x_{m+k}}\right) + \ldots \\ &+ w_{m-n}S_-^n\left(\frac{\partial \Omega}{\partial x_{m+n}}\right) = 0. \end{aligned}$$

In this setting theorem 4.3 takes the following form.

Theorem 6.1 *(Main theorem for discretized ODE). Let the adjoint equations (6.7) and (6.8) be satisfied for all solutions of the original equations (6.1) and (6.2) upon a substitution*

$$v_m = \varphi_1(m, x_m, u_m), \qquad \varphi_1 \not\equiv 0 \quad \text{or} \quad \varphi_2 \not\equiv 0. \tag{6.9}$$
$$w_m = \varphi_2(m, x_m, u_m),$$

Then, any Lie point symmetry (6.3) of the equations (6.1) and (6.2) leads to the first integral

$$J = \left[\sum_{j=1}^{n}\left(\xi_{m+j}\frac{\delta}{\delta x_{m(j)}} + \eta_{m+j}\frac{\delta}{\delta u_{m(j)}}\right)\left(v_m F + w_m \Omega\right)\right]_{v_m=\varphi_1,\ w_m=\varphi_2}, \tag{6.10}$$

where

$$\frac{\delta}{\delta u_{m(j)}} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial u_{m+j+k}} = \frac{\partial}{\partial u_{m+j}} + S_- \frac{\partial}{\partial u_{m+j+1}} + \ldots + S_-^k \frac{\partial}{\partial u_{m+j+k}} + \ldots \tag{6.11}$$

and

$$\frac{\delta}{\delta x_{m(j)}} = \sum_{k=0}^{\infty} S_-^k \frac{\partial}{\partial x_{m+j+k}} = \frac{\partial}{\partial x_{m+j}} + S_- \frac{\partial}{\partial x_{m+j+1}} + \ldots + S_-^k \frac{\partial}{\partial x_{m+j+k}} + \ldots \tag{6.12}$$

are higher order discrete Euler–Lagrange operators and v_m, w_m, …, v_{m-n}, w_{m-n} should be eliminated by means of equations (6.9) and their shifts to the left.

As in the general case of theorem 4.3 the first integral J satisfies the equation

$$(1 - S_-)J = 0$$

on the solutions of the difference scheme.

Remark 6.2. As in the general case the condition that the adjoint equations are satisfied, i.e., $F^* = \Omega^* = 0$, can be substituted by a weaker condition

$$\xi_m \Omega^* + \eta_m F^* = 0,$$

which should hold for a given symmetry X of the system (6.1), (6.2) on the solutions of this system.

Example 6.1. Let us consider the one-dimensional harmonic oscillator

$$\ddot{u} + u = 0. \tag{6.13}$$

As a discretization we choose the scheme

$$\frac{2}{x_{m+2} - x_m}\left(\frac{u_{m+2} - u_{m+1}}{x_{m+2} - x_{m+1}} - \frac{u_{m+1} - u_m}{x_{m+1} - x_m}\right) + \frac{u_{m+2} + 2u_{m+1} + u_m}{4} = 0 \tag{6.14}$$

on the uniform mesh

$$x_{m+2} - x_{m+1} = x_{m+1} - x_m. \tag{6.15}$$

This discretization of the harmonic oscillator was considered in [20].

Let us rewrite the scheme in an equivalent form

$$F = \frac{u_{m+2} - u_{m+1}}{x_{m+2} - x_{m+1}} - \frac{u_{m+1} - u_m}{x_{m+1} - x_m} + \frac{x_{m+2} - x_m}{2}\frac{u_{m+2} + 2u_{m+1} + u_m}{4} = 0,$$
$$\Omega = (x_{m+2} - x_{m+1}) - (x_{m+1} - x_m) = 0. \tag{6.16}$$

It is not difficult to verify that the difference system (6.16) admits the symmetries generated by the operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \sin(\omega x)\frac{\partial}{\partial u}, \qquad X_3 = \cos(\omega x)\frac{\partial}{\partial u}, \qquad X_4 = u\frac{\partial}{\partial u}, \tag{6.17}$$

where

$$\omega = \frac{\arctan(h/2)}{h/2}, \qquad h = x_{m+2} - x_{m+1} = x_{m+1} - x_m.$$

The adjoint equations are

$$F^* = v_m\left(\frac{1}{x_{m+1} - x_m} + \frac{x_{m+2} - x_m}{8}\right) + v_{m-1}\left(-\frac{1}{x_{m+1} - x_m} - \frac{1}{x_m - x_{m-1}} + \frac{x_{m+1} - x_{m-1}}{4}\right) + v_{m-2}\left(\frac{1}{x_m - x_{m-1}} + \frac{x_m - x_{m-2}}{8}\right) = 0 \tag{6.18}$$

and

$$\Omega^* = v_m\left(-\frac{u_{m+1} - u_m}{(x_{m+1} - x_m)^2} - \frac{u_{m+2} + 2u_{m+1} + u_m}{8}\right) + v_{m-1}\left(\frac{u_{m+1} - u_m}{(x_{m+1} - x_m)^2} + \frac{u_m - u_{m-1}}{(x_m - x_{m-1})^2}\right) + v_{m-2}\left(-\frac{u_m - u_{m-1}}{(x_m - x_{m-1})^2} + \frac{u_m + 2u_{m-1} + u_{m-2}}{8}\right) + w_m - 2w_{m-1} + w_{m-2} = 0, \tag{6.19}$$

considered on the solutions of the equations (6.16).

On the solutions of the equations (6.16) the adjoint equations (6.18) and (6.19) have the particular solution

$$v_m^a = 0, \qquad w_m^a = x_m. \tag{6.20}$$

For symmetries (6.3) with $\xi = 0$ we can consider the equation (6.18) instead of the system (6.18), (6.19) (see remark 6.2). In this case we find the special solution

$$v_m^b = u_m, \qquad w_m^b = 0. \tag{6.21}$$

Let us use these solutions to find first integrals with the help of theorem 6.1 and symmetries (6.17). We will bypass the higher first integrals and provide only the final results for both pairs (6.20) and (6.21).

- $v_m^a = 0$, $w_m^a = x_m$.

Application of the theorem with symmetry X_1 gives the first integral

$$\tilde{J}_1^a = x_m - x_{m+1} = -h. \tag{6.22}$$

The other symmetries provide trivial first integrals.

- $v_m^b = u_m,\ w_m^b = 0.$

For symmetries X_2, X_3 and X_4 we obtain the first integrals

$$\tilde{J}_2^b = \left(\frac{1}{h} + \frac{h}{4}\right)\big(-u_{m+1}\sin(\omega x_{m+1}) + u_m \sin(\omega x_{m+2})\big), \tag{6.23}$$

$$\tilde{J}_3^b = \left(\frac{1}{h} + \frac{h}{4}\right)\big(-u_{m+1}\cos(\omega x_{m+1}) + u_m \cos(\omega x_{m+2})\big), \tag{6.24}$$

$$\tilde{J}_4^b = -h\left[\left(\frac{u_{m+1} - u_m}{h}\right)^2 + \left(\frac{u_{m+1} + u_m}{2}\right)^2\right], \tag{6.25}$$

where we used $h = x_{m+1} - x_m$ and $x_{m+2} = x_{m+1} + h$.

Using values of the first integrals $\tilde{J}_1^a$ $\tilde{J}_2^b$ and $\tilde{J}_3^b$, we can express the solution of the difference system in the form

$$u_m = A\cos(\omega x_m) + B\sin(\omega x_m). \tag{6.26}$$

The mesh for this solution

$$x_m = x_0 + mh, \qquad m = 0, \pm 1, \pm 2, \ldots \tag{6.27}$$

can be obtained by integration of the linear equation (6.22). Here A, B, $h > 0$ and x_0 are constants. Note that x_0 appears from the integration of the linear equation (6.22).

Example 6.2. Let us return to the ODE

$$F = \frac{1}{\dot{u}^2}\left(\dot{u}\dddot{u} - \frac{3}{2}\ddot{u}^2\right) - M = 0, \tag{6.28}$$

which we examined in the example 3.1. We recall that in the general case it admits symmetries (3.13) and (3.14). For $M = 0$ there are additional symmetries (3.15). We will consider these two cases separately.

Case: $M = 0$

As a discretization we consider the invariant scheme

$$F = \frac{u_{m+3} - u_{m+1}}{x_{m+3} - x_{m+1}}\,\frac{u_{m+2} - u_m}{x_{m+2} - x_m} - \frac{u_{m+3} - u_{m+2}}{x_{m+3} - x_{m+2}}\,\frac{u_{m+1} - u_m}{x_{m+1} - x_m} = 0,$$
$$\Omega = \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)} - K = 0, \qquad K \neq 0, \tag{6.29}$$

which was introduced in [7, 8]. It admits all six symmetries (3.13), (3.14) and (3.15).

The adjoint system for the presented scheme is

$$F^* = \frac{\alpha(u_{m+2} - u_{m+1})}{(u_{m+2} - u_m)(u_{m+1} - u_m)}\big(v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3}\big) = 0$$

and

$$\Omega^* = -\frac{\alpha(x_{m+2} - x_{m+1})}{(x_{m+2} - x_m)(x_{m+1} - x_m)}\big(v_m + (1-K)v_{m-1} + (K-1)v_{m-2} - v_{m-3}\big)$$
$$+ \frac{K(x_{m+2} - x_{m+1})}{(x_{m+2} - x_m)(x_{m+1} - x_m)}\big(w_m + (1-K)w_{m-1} + (K-1)w_{m-2} - w_{m-3}\big) = 0,$$

where

$$\alpha = \frac{u_{m+3} - u_{m+1}}{x_{m+3} - x_{m+1}}\frac{u_{m+2} - u_m}{x_{m+2} - x_m} = \frac{u_{m+3} - u_{m+2}}{x_{m+3} - x_{m+2}}\frac{u_{m+1} - u_m}{x_{m+1} - x_m}.$$

Variables u_{m+3} and x_{m+3} in the coefficient α should be expressed in terms of the other variables involved in the scheme.

The adjoint equations lead to the system of linear mappings

$$v_m + (1-K)v_{m-1} + (K-1)v_{m-2} - v_{m-3} = 0,$$
$$w_m + (1-K)w_{m-1} + (K-1)w_{m-2} - w_{m-3} = 0.$$

One can use pairs (v_m, w_m) which solve this system to find first integrals and employ these first integrals to find the solution of the scheme.

However, it is more convenient to rewrite the scheme (6.29) in the equivalent form

$$\tilde{F} = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - K = 0,$$
$$\Omega = \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)} - K = 0. \tag{6.30}$$

Note that the system is symmetric under the interchange of u and x. We can use the results obtained for mapping (5.8) to integrate this scheme. We need to consider different subcases for different values of K.

(1) $K = 4$. We obtain the solution

$$u_m = \frac{1}{C_1 m + C_2} + C_3 \qquad \text{or} \qquad u_m = C_1 m + C_2 \tag{6.31}$$

on the mesh

$$x_m = \frac{1}{C_4 m + C_5} + C_6 \qquad \text{or} \qquad x_m = C_4 m + C_5, \tag{6.32}$$

where $C_1 \neq 0$, C_2, C_3, $C_4 \neq 0$, C_5 and C_6 are constants.

(2) $0 < K < 4$. We obtain the solution

$$u_m = C_1 \tan\big(\phi m + C_2\big) + C_3 \tag{6.33}$$

on the mesh

$$x_m = C_4 \tan\big(\phi m + C_5\big) + C_6, \tag{6.34}$$

where $C_1 \neq 0$, C_2, C_3, $C_4 \neq 0$, C_5 and C_6 are constants. Here

$$\phi = \arccos\left(\frac{\sqrt{K}}{2}\right). \tag{6.35}$$

(3) $K > 4$. We obtain the solution

$$u_m = C_1 \tanh\left(\psi m + C_2\right) + C_3 \tag{6.36}$$

or

$$u_m = C_1 \coth\left(\psi m + C_2\right) + C_3 \tag{6.37}$$

or

$$u_m = C_1 \mu_{1,2}^m + C_2 = C_1 \mathrm{e}^{\pm 2\psi m} + C_2 \tag{6.38}$$

on the mesh

$$x_m = C_4 \tanh\left(\psi m + C_5\right) + C_6 \tag{6.39}$$

or

$$x_m = C_4 \coth\left(\psi m + C_5\right) + C_6 \tag{6.40}$$

or

$$x_m = C_4 \mu_{1,2}^m + C_5 = C_4 \mathrm{e}^{\pm 2\psi m} + C_5, \tag{6.41}$$

where $C_1 \neq 0$, C_2, C_3, $C_4 \neq 0$, C_5 and C_6 are constants. Here

$$\mu_{1,2} = \frac{K - 2 \pm \sqrt{K^2 - 4K}}{2} \tag{6.42}$$

and

$$\psi = \frac{1}{2} \ln \left|\mu_1\right| = \frac{1}{2} \ln \left| \frac{K - 2 + \sqrt{K^2 - 4K}}{2} \right|. \tag{6.43}$$

(4) $K < 0$. We obtain the solution

$$u_m = \begin{cases} C_1 \tanh\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is even,} \\ C_1 \coth\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is odd,} \end{cases} \tag{6.44}$$

or

$$u_m = \begin{cases} C_1 \coth\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is even,} \\ C_1 \tanh\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is odd,} \end{cases} \tag{6.45}$$

or

$$u_m = C_1 \mu_{1,2}^m + C_2 = C_1 (-1)^m \mathrm{e}^{\pm 2\psi m} + C_2 \tag{6.46}$$

on the mesh

$$x_m = \begin{cases} C_4 \tanh\left(\psi m + C_5\right) + C_6 & \text{if } m \text{ is even,} \\ C_4 \coth\left(\psi m + C_5\right) + C_6 & \text{if } m \text{ is odd,} \end{cases} \tag{6.47}$$

or

$$x_m = \begin{cases} C_1 \coth\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is even,} \\ C_1 \tanh\left(\psi m + C_2\right) + C_3 & \text{if } m \text{ is odd,} \end{cases} \tag{6.48}$$

or

$$x_m = C_4 \mu_{1,2}^m + C_5 = C_4(-1)^m e^{\pm 2\psi m} + C_5, \tag{6.49}$$

where $C_1 \neq 0$, C_2, C_3, $C_4 \neq 0$, C_5 and C_6 are constants. Here $\mu_{1,2}$ and ψ are given by (6.42) and (6.43), respectively.

Remark 6.3. Let us note that all these solutions for any $K \neq 0$ can be presented in the unified form

$$u_m = \frac{1}{\alpha x_m + \beta} + \gamma \qquad \text{or} \qquad u_m = \alpha x_m + \beta, \tag{6.50}$$

where $\alpha \neq 0$, β and γ are constants. They should be considered on the corresponding meshes, which are different for different values of the parameter K. Thus, the discretization (6.29) provides the exact solution of the ODE (6.28) for $M = 0$. For the case $K = 4$ this was observed in [7, 8], where the result was essentially guessed, then verified. Here we obtained it systematically, using the adjoint equation method.

Remark 6.4. It should be noted that for some cases we do not get monotonicity for mesh points x_m. A monotone sequence of mesh points satisfies the natural requirement

$$\frac{x_{m+2} - x_{m+1}}{x_{m+1} - x_m} > 0. \tag{6.51}$$

We omit the detailed examination of the monotonicity of the mesh points because we would have to consider many different cases. Let us stress that in all cases we obtain the exact solution of the ODE in the mesh points.

Remark 6.5. The problem of non-monotone meshes also occurs when using adaptive meshes [23] and there exist various ways of dealing with it in numerical analysis. One of them is to restrict the analysis to parts of the mesh where (6.51) holds. For other possibilities see [4] and references therein.

Case: $M \neq 0$

As a discretization we consider the invariant scheme

$$\begin{gathered} F = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)} \\ \times\left(1 - \frac{M}{6}(x_{m+3} - x_m)(x_{m+2} - x_{m+1})\right) = 0, \\ \Omega(x_{m+3} - x_{m+2}, x_{m+2} - x_{m+1}, x_{m+1} - x_m) = 0. \end{gathered} \tag{6.52}$$

It admits the four symmetries (3.13) and (3.14). To find solutions we specify the mesh as a regular one

$$\Omega = x_{m+1} - x_m - h = 0, \tag{6.53}$$

where $h > 0$ is a constant. The first equation will take the form

$$F = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - \bar{K} = 0, \tag{6.54}$$

where

$$\bar{K} = 4\left(1 - \frac{M}{2}h^2\right). \tag{6.55}$$

For the equation (6.54) we can use results obtained for the mapping (5.8) in example 5.1. Since $h \neq 0$ we have $\bar{K} \neq 4$. For non-trivial cases $\bar{K} \neq 0$ there can be three possibilities.

(1) $0 < \bar{K} < 4$ ($M > 0$, $0 < h < \sqrt{2/M}$). We obtain the solution

$$u_m = C_1 \tan\left(\bar{\phi} m + C_2\right) + C_3, \tag{6.56}$$

where $C_1 \neq 0$, C_2 and C_3 are constants, on the mesh

$$x_m = x_0 + hm. \tag{6.57}$$

Here

$$\bar{\phi} = \arccos\left(\frac{\sqrt{\bar{K}}}{2}\right). \tag{6.58}$$

(2) $\bar{K} > 4$ ($M < 0$). We obtain the solution

$$u_m = C_1 \tanh\left(\bar{\psi} m + C_2\right) + C_3 \tag{6.59}$$

or

$$u_m = C_1 \coth\left(\bar{\psi} m + C_2\right) + C_3 \tag{6.60}$$

or

$$u_m = C_1 \bar{\mu}_{1,2}^m + C_2 = C_1 \mathrm{e}^{\pm 2\bar{\psi} m} + C_2, \tag{6.61}$$

where $C_1 \neq 0$, C_2 and C_3 are constants, on the regular mesh (6.57). Here

$$\bar{\mu}_{1,2} = \frac{\bar{K} - 2 \pm \sqrt{\bar{K}^2 - 4\bar{K}}}{2} \tag{6.62}$$

and

$$\bar{\psi} = \frac{1}{2} \ln \left|\bar{\mu}_1\right| = \frac{1}{2} \ln \left|\frac{\bar{K} - 2 + \sqrt{\bar{K}^2 - 4\bar{K}}}{2}\right|. \tag{6.63}$$

(3) $\bar{K} < 0$ ($M > 0$, $h > \sqrt{2/M}$). We obtain the solution

$$u_m = \begin{cases} C_1 \tanh\left(\bar{\psi} m + C_2\right) + C_3 & \text{if } m \text{ is even}, \\ C_1 \coth\left(\bar{\psi} m + C_2\right) + C_3 & \text{if } m \text{ is odd}, \end{cases} \tag{6.64}$$

or

$$u_m = \begin{cases} C_1 \coth\left(\bar{\psi}m + C_2\right) + C_3 & \text{if } m \text{ is even,} \\ C_1 \tanh\left(\bar{\psi}m + C_2\right) + C_3 & \text{if } m \text{ is odd,} \end{cases} \tag{6.65}$$

or

$$u_m = C_1 \bar{\mu}_{1,2}^m + C_2 = C_1(-1)^m \mathrm{e}^{\pm 2\bar{\psi}m} + C_2, \tag{6.66}$$

where $C_1 \neq 0$, C_2 and C_3 are constants, on the regular mesh (6.57). Here $\bar{\mu}_{1,2}$ and $\bar{\psi}$ are given by (6.62) and (6.63). Note that because of the steplength restriction $h > \sqrt{2/M}$ we do not obtain a consistent discretization of the ODE in this case (the mesh cannot be refined).

It should be noted that for sufficiently small steplengths $h \ll 1$ we will always have $\bar{K} > 0$ (see (6.55)) and thus avoid 'jumping solutions' of the last case. As in the continuous case the solution has several branches which correspond to different initial conditions.

Remark 6.6. We recall that in the case $M = 0$ the scheme (6.29) provided us with the exact solution of the ODE (6.28). In the present case $M \neq 0$ the scheme (6.52) with regular mesh specification (6.53) provides the exact solutions of the ODE (6.28) if we apply the scheme to the modified equation

$$F_{\text{mod}} = \frac{1}{\dot{u}^2}\left(\dot{u}\dddot{u} - \frac{3}{2}\ddot{u}^2\right) - M_{\text{mod}} = 0. \tag{6.67}$$

The original equation parameter $M > 0$ should be changed to the modified value

$$M_{\text{mod}} = \frac{2}{h^2}\sin^2\left(\sqrt{\frac{M}{2}}h\right), \qquad 0 < h < \sqrt{2/M}$$

and the parameter $M < 0$ should be changed to the modified value

$$M_{\text{mod}} = -\frac{2}{h^2}\sinh^2\left(\sqrt{-\frac{M}{2}}h\right).$$

Application of the scheme (6.52) to the modified equation (6.67) with the modified constant M_{mod} gives exact solution of the ODE (6.28) with constant M. Note that in both cases $M_{\text{mod}} \to M$ as $h \to 0$.

We note that modification of the constant M can be interpreted as scaling of the independent variable x.$\diamond$

Example 6.3. Now we turn to the ODE (3.28) admitting symmetries (3.29). For the discretization we take the invariant scheme

$$F = \frac{u_{m+3} - u_{m+1}}{\sin\left(\ln\sqrt{\left|\frac{x_{m+3}}{x_{m+1}}\right|}\right)} \frac{u_{m+2} - u_m}{\sin\left(\ln\sqrt{\left|\frac{x_{m+2}}{x_m}\right|}\right)} - \frac{u_{m+3} - u_{m+2}}{\sin\left(\ln\sqrt{\left|\frac{x_{m+3}}{x_{m+2}}\right|}\right)} \frac{u_{m+1} - u_m}{\sin\left(\ln\sqrt{\left|\frac{x_{m+1}}{x_m}\right|}\right)} = 0, \tag{6.68}$$

$$\Omega = \frac{\sin\left(\ln\sqrt{\left|\frac{x_{m+3}}{x_{m+1}}\right|}\right)\sin\left(\ln\sqrt{\left|\frac{x_{m+2}}{x_m}\right|}\right)}{\sin\left(\ln\sqrt{\left|\frac{x_{m+3}}{x_{m+2}}\right|}\right)\sin\left(\ln\sqrt{\left|\frac{x_{m+1}}{x_m}\right|}\right)} - K = 0, \quad K \neq 0,$$

which admits the same six symmetries as the underlying ODE. One can approach this system by finding the adjoint equations, using their solutions and symmetries of the scheme to obtain first integrals and exploiting these first integrals to obtain the solution of the scheme. This method is lengthy and requires complicated computations.

Noting that this scheme is transformed into scheme (6.29) by the change of variable (3.32), we can use the results of the previous example to write down the solution. We get the solution

$$u_m = C_1 \tan\left(\ln\sqrt{|x_m|} + C_2\right) + C_3, \tag{6.69}$$

where $C_1 \neq 0$, C_2 and C_3 are constants, on the mesh which depends on the value of K as follows:

(1) $K = 4$

$$x_m = \pm \mathrm{e}^{2\arctan\left(\frac{1}{C_4 m + C_5} + C_6\right)} \tag{6.70}$$

or

$$x_m = \pm \mathrm{e}^{2\arctan\left(C_4 m + C_5\right)} \tag{6.71}$$

(2) $0 < K < 4$

$$x_m = \pm \mathrm{e}^{2\arctan\left(C_4 \tan\left(\phi m + C_5\right) + C_6\right)}, \tag{6.72}$$

where ϕ is given by (6.35).

(3) $K > 4$

$$x_m = \pm \mathrm{e}^{2\arctan\left(C_4 \tanh\left(\psi m + C_5\right) + C_6\right)} \tag{6.73}$$

or

$$x_m = \pm \mathrm{e}^{2\arctan\left(C_4 \coth\left(\psi m + C_5\right) + C_6\right)} \tag{6.74}$$

or

$$x_m = \pm \mathrm{e}^{2\arctan\left(C_4 \mu_{1,2}^m + C_5\right)} = \pm \mathrm{e}^{2\arctan\left(C_4 \mathrm{e}^{\pm 2\psi m} + C_5\right)}, \tag{6.75}$$

where $\mu_{1,2}$ and ψ are given by (6.42) and (6.43), respectively.

(4) $K < 0$

$$x_m = \begin{cases} \pm \mathrm{e}^{2\arctan\left(C_4 \tanh\left(\psi m + C_5\right) + C_6\right)} & \text{if } m \text{ is even,} \\ \pm \mathrm{e}^{2\arctan\left(C_4 \coth\left(\psi m + C_5\right) + C_6\right)} & \text{if } m \text{ is odd,} \end{cases} \tag{6.76}$$

or

$$x_m = \begin{cases} \pm e^{2\,\arctan\left(C_1 \coth\left(\psi m + C_2\right) + C_3\right)} & \text{if } m \text{ is even}, \\ \pm e^{2\,\arctan\left(C_1 \tanh\left(\psi m + C_2\right) + C_3\right)} & \text{if } m \text{ is odd}, \end{cases} \tag{6.77}$$

or

$$x_m = \pm e^{2\,\arctan\left(C_4 \mu_{1,2}^m + C_5\right)} = \pm e^{2\,\arctan\left(C_4^{(-1)m} e^{\pm 2\psi m} + C_5\right)}, \tag{6.78}$$

where $\mu_{1,2}$ and ψ are given by (6.42) and (6.43). ⋄

In all cases $C_4 \neq 0$, C_5 and C_6 are constants. Note that the scheme gives the exact solution of the ODE (3.28). Let us note that for some cases we do not get monotonicity of the mesh points x_m (see remark 6.4).

7. Conclusion

This paper consists of two parts. The first is a brief review the 'adjoint equation method' (sections 2 and 3). It is particularly useful either when no Lagrangian exists, or when the symmetries of the equation are not Lagrangian ones and the Noether theorem cannot be applied. The method is valid both for ODEs and PDEs. We apply the method to obtain first integrals and general solutions of third order nonlinear ODEs (the Schwarzian equations (3.16) and (3.28)).

The second part is an adaptation of the adjoint equation method first to mappings, then to discretizations of ODEs. The mappings are equations involving several discrete points. The discretizations are difference equations on lattices that arise e.g. when differential equations are solved numerically. In both cases (see sections 5 and 6, respectively) we apply the discretized adjoint equation method to a specific four-point equation, respectively four-point difference systems. These systems have the Schwarzian ODEs (3.16) and (3.28) as continuous limits and and share their Lie point symmetry groups. We have also treated a simpler example, namely a discrete linear harmonic oscillator. The results for the examples can be summed up as follows:

(1) The adjoint equation method makes it possible to obtain complete sets of functionally independent first integrals of the differential equations and difference systems. These in turn provide the general solutions of these equations. If the number of integrals is not sufficient to provide the solutions, the integrals can be used to lower the order of the difference system, i.e., decrease the number of points involved.

(2) The invariant discretizations of ODE (3.16) with $M = 0$ and ODE (3.28) considered here are exact. The solutions of the difference system coincide with the solutions of the original ODEs. The invariant discretizations of the other continuous ODEs considered here, namely of the harmonic oscillator and ODE (3.16) with $M \neq 0$, can be made exact if we allow a parameter modification.

(3) The adjoint equation method is entirely constructive. To use it we need to know the symmetry algebra of the discrete equations that we are studying and some particular solutions of the adjoint equations. Nowhere did we use the knowledge of solutions of the original differential equations.

(4) For the ODE (3.28) we for brevity used the transformation (3.32) that transforms equation (3.28) into (3.16) with $M = 0$. It also transforms the invariant scheme (6.68) into (6.29). This equivalence of two schemes is not crucial. The invariant scheme (6.68) could have been obtained and solved directly though with considerably more calculational effort.

In the paper we restricted ourselves to ordinary difference equations. However, the presented approach can be extended to differential–difference equations as well as to partial difference equations.

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