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Lie-point symmetries of the discrete Liouville equation

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Abstract

The Liouville equation is well known to be linearizable by a point transformation. It has an infinite dimensional Lie point symmetry algebra isomorphic to a direct sum of two Virasoro algebras. We show that it is not possible to discretize the equation keeping the entire symmetry algebra as point symmetries. We do however construct a difference system approximating the Liouville equation that is invariant under the maximal finite subgroup $SL_x(2, \mathbb{R}) \otimes SL_y(2, \mathbb{R})$. The invariant scheme is an explicit one and provides a much better approximation of exact solutions than a comparable standard (noninvariant) scheme and also than a scheme invariant under an infinite dimensional group of generalized symmetries.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The purpose of this article is to investigate the possibility of discretizing the Liouville equation

$$z_{xy} = e^{z}, \tag{1.1}$$

or its algebraic version

$$uu_{xy} - u_x\, u_y = u^3, \qquad u = e^z, \tag{1.2}$$

while preserving all of its Lie point symmetries. This is quite a challenge, since the Lie point symmetry group of these equations is infinite dimensional. We shall call (1.2) the *algebraic Liouville equation*.

The article is part of a general program on the study of continuous symmetries of discrete equations [8–10, 13, 17–24, 32–35] and on the symmetry preserving discretization of differential equations [2–7, 15, 16, 30, 31]. This program has several aspects, each possibly requiring different approaches. Among them we mention:

1. In relativistic and nonrelativistic quantum mechanics or field theory on a discrete space–time, a problem is to discretize the continuous theory while preserving continuous symmetries such as rotational, Lorentz, Galilei or conformal invariance. One possible way of doing this is the way explored in the present article, namely to not use a preconceived constant lattice. Instead one can construct an invariant set of equations defining both the lattice and system of difference equations. The lattice thus appears as part of a solution of a set of discrete equations and the symmetry group acts on the solutions of the equation and on the lattice.
2. The second aspect of this program fits into the general field of geometrical integration [11, 14, 26, 27]. The basic idea is to improve numerical methods of solving specific ordinary and partial differential equations (PDEs), by incorporating important qualitative features of these equations into their discretization. Such features may be integrability, linearizability, Lagrangian or Hamiltonian formulation, or some other features.

We concentrate on the preservation of Lie point symmetries. In our case the idea is to take an ordinary or partial differential equation (ODE or PDE) with a known Lie point symmetry algebra $\mathcal{L}$ realized by vector-fields. The differential equation is then approximated by a difference system with the same symmetry algebra. The difference system consists of a set of difference equations, describing both the approximation of the ODE (PDE) and the lattice. The difference system is constructed out of the invariants of the Lie point symmetry group $\mathcal{G}$ of the original ODE (PDE). The Lie algebra $\mathcal{L}$ of $\mathcal{G}$ is realized by the same vector fields as for the continuous equation, however its action is prolonged to all points of the lattice, rather than to derivatives.

In section 2 we present the Lie point symmetry algebra of the continuous algebraic Liouville equation and the corresponding vector fields depending on two arbitrary functions of one variable each. The symmetry algebra is isomorphic to the direct sum of two Virasoro algebras (with no central extension). We also give the two second order differential invariants of the maximal finite-dimensional subgroup $SL_x(2, \mathbb{R}) \otimes SL_y(2, \mathbb{R})$ of the corresponding infinite dimensional symmetry group. Section 3 is devoted to a brief exposition of the method of discretizing differential equations while preserving their point symmetries. In section 4 we discretize the Liouville equation on a four-point stencil. The discretization is invariant under the maximal finite dimensional subgroup, not however under the entire infinite-dimensional group. An alternative symmetry preserving discretization of the Liouville equation due to Rebelo and Valiquette [32] is discussed in section 5. They have succeeded in preserving the entire symmetry group but as generalized symmetries rather than point ones (only translations and dilations remain as point symmetries). Section 6 is devoted to numerical experiments. We choose three different exact solutions of the continuous Liouville equation and formulate a boundary value problem that leads to these solutions. The boundary value problem is then solved numerically, using three different discretizations: standard, Rebelo–Valiquette and $SL_x(2, \mathbb{R}) \otimes SL_y(2, \mathbb{R})$ invariant, respectively. The results are compared to the analytic solutions. In all three cases our invariant discretization is shown to perform considerably better than the other ones. The last section 7 is devoted to conclusions and comments on the other discretizations of the Liouville equation. A linearizable discretization and its symmetries are discussed in the appendix.

2. Lie point symmetries of the continuous Liouville equation

The Liouville system (1.1) is a remarkable equation that has already been thoroughly investigated. It was shown by Liouville himself [25] that it is linearized into the linear wave equation by the transformation

$$z = \ln\left[2\frac{\phi_x\,\phi_y}{\phi^2}\right], \qquad \phi_{x\,y} = 0. \tag{2.1}$$

Putting $\phi(x, y) = \phi_1(x) + \phi_2(y)$, where ϕ_i, $i = 1, 2$ are arbitrary functions, we get a very general class of solutions of (1.1) (and (1.2)), namely

$$z = \ln\left[2\frac{\phi_{1,x}\,\phi_{2,y}}{\left(\phi_1 + \phi_2\right)^2}\right]. \tag{2.2}$$

In view of (2.1) the Liouville equation is linearizable and it is not surprising that its symmetry algebra is infinite dimensional, as was already known in 1898 [28]. The symmetry algebra of the algebraic Liouville equation (1.2) is given by the vector fields

$$X(f(x)) = f(x)\partial_x - f_x(x)\,u\,\partial_u, \qquad Y(g(y)) = g(y)\partial_y - g_y(y)\,u\,\partial_u, \tag{2.3}$$

where $f = f(x)$ and $g = g(y)$ are arbitrary smooth functions. The nonzero commutation relations of the vector fields (2.3) are

$$\begin{aligned} &\left[X(f), X\left(\tilde{f}\right)\right] = X\left(f\tilde{f}_x - \tilde{f}f_x\right),\\ &\left[Y(g), Y(\tilde{g})\right] = Y\left(g\,\tilde{g}_y - \tilde{g}\,g_y\right),\\ &[X(f), Y(g)] = 0. \end{aligned} \tag{2.4}$$

The algebra (2.3)–(2.4) is isomorphic to the direct sum of two Virasoro algebras. We denote it $L = vir_x \oplus vir_y$. Its maximal finite dimensional subalgebra is $sl_x(2, \mathbb{R}) \oplus sl_y(2, \mathbb{R})$, obtained by restricting $f(x)$ and $g(y)$ to be second order polynomials. Limiting ourselves to a neighborhood of the origin, the above vector fields can be expanded in the basis $\left\{X\left(x^n\right)\right\}_{n\in\mathbb{N}}$ and $\left\{Y\left(y^n\right)\right\}_{n\in\mathbb{N}}$, which leads to the commutation relations

$$\begin{aligned} &\left[X\left(x^m\right), X\left(x^n\right)\right] = (n - m)X\left(x^{m+n-1}\right),\\ &\left[Y\left(y^m\right), Y\left(y^n\right)\right] = (n - m)Y\left(y^{m+n-1}\right),\\ &\left[X\left(x^m\right), Y\left(y^n\right)\right] = 0. \end{aligned} \tag{2.5}$$

As said above, the maximal finite subalgebra corresponds to the basis elements with $m, n = 0, 1, 2$.

Let us find the most general second order expression of the form $I(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ invariant under the group corresponding to the algebra (2.3). The second order prolongation of $X(f)$ is

$$\mathrm{pr}^{(2)} X(f) = f\partial_x - f'\Big[u\partial_u + 2u_x\partial_{u_x} + u_y\,\partial_{u_y} + 2u_{xy}\partial_{u_{xy}} + 3u_{xx}\partial_{u_{xx}} + u_{yy}\partial_{u_{yy}}\Big] - f''\Big[u\partial_{u_x} + u_y\partial_{u_{xy}} + 3u_x\partial_{u_{xx}}\Big] - f'''u\partial_{u_{xx}} \tag{2.6}$$

and similarly for $Y(g)$. We see that the last term in (2.6) is absent in the subalgebra.

The group $SL_x(2, \mathbb{R}) \otimes SL_y(2, \mathbb{R})$ allows two functionally independent 'strong' invariants, namely

$$I_1 = \frac{uu_{xy} - u_x\,u_y}{u^3}, \qquad I_2 = \frac{\left(2uu_{xx} - 3u_x^2\right)\left(2uu_{yy} - 3u_y^2\right)}{u^6}. \tag{2.7}$$

We have

$$\mathrm{pr}^{(2)} X(f)\, I_1 = \mathrm{pr}^{(2)}\, Y(g)\, I_1 = 0 \tag{2.8}$$

for arbitrary f and g, but

$$\mathrm{pr}^{(2)} X(f)\, I_2 = \frac{2f_{xxx}\left(3u_y^2 - 2uu_{yy}\right)}{u^4}, \qquad \mathrm{pr}^{(2)}\, Y(g)\, I_2 = \frac{2g_{yyy}\left(3u_x^2 - 2uu_{xx}\right)}{u^4}. \tag{2.9}$$

Thus, I_1 is invariant under the direct product the two Virasoro groups $VIR(x) \otimes VIR(y)$. The PDE $I_1 = A$, for any real constant A, is invariant under this group. For $A \neq 0$ we scale to A=1 and obtain the equation (1.2). For $A = 0$ we obtain an equation equivalent to the linear wave equation $z_{xy} = 0$, namely

$$uu_{xy} - u_x u_y = 0. \tag{2.10}$$

On the other hand I_2 is invariant only for $f_{xxx} = g_{yyy} = 0$, i.e. it is only invariant under $SL_x(2, \mathbb{R}) \otimes SL_y(2, \mathbb{R})$. Even the equation $I_2 = 0$ is only invariant on the manifold satisfying the system

$$2uu_{xx} - 3u_x^2 = 0, \qquad 2uu_{yy} - 3u_y^2 = 0, \tag{2.11}$$

i.e. on a very restricted class of solutions, namely

$$u = (axy + bx + cy + d)^{-2}, \tag{2.12}$$

for arbitrary constants a ,..., d.

3. Symmetry preserving discretization of partial difference equations

The basic idea of the invariant discretization of a PDE is to replace it by a system of difference equations, formed out of invariants of the action of the symmetry group of the PDE. This difference system (ΔS) describes both the original PDE and a lattice [8, 9, 22, 34, 35].

To be specific, let us restrict to the case of one scalar PDE involving two independent variables (x, y) and one dependent one $u(x, y)$. The PDE is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \cdots) = 0 \tag{3.1}$$

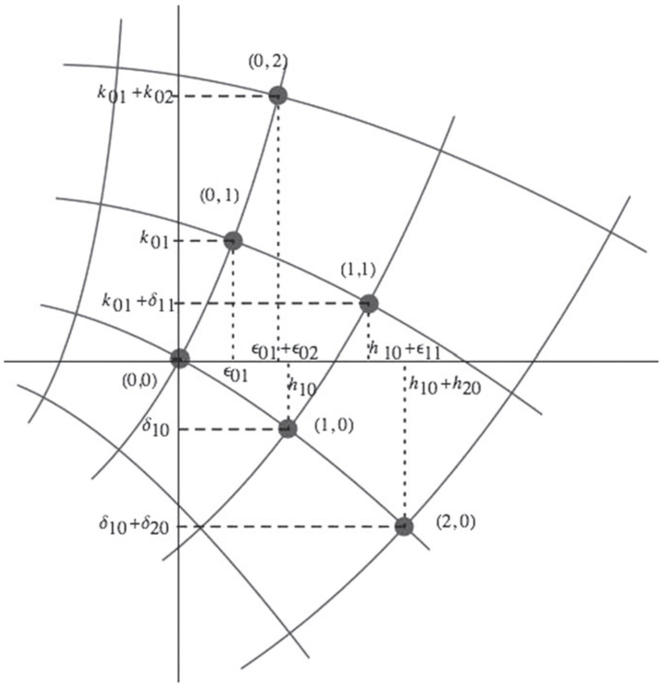

Figure 1. Points on a general lattice, e.g. $x_{0,0} = x$, $x_{1,0} = x + h_{1,0}$, $x_{0,1} = x + \epsilon_{0,1}$, $x_{1,1} = x + h_{1,0} + \epsilon_{1,1}$, $x_{2,0} = x + h_{1,0} + h_{2,0}$, $x_{0,2} = x + \epsilon_{0,1} + \epsilon_{0,2}$, $y_{0,0} = y$, $y_{0,1} = y + k_{0,1}$, $y_{1,0} = y + \delta_{1,0}$, $y_{1,1} = y + k_{0,1} + \delta_{1,1}$, $y_{0,2} = y + k_{0,1} + k_{0,2}$, $y_{2,0} = y + \delta_{1,0} + \delta_{2,0}$.

and its Lie point symmetry group $\mathcal{G}$ is assumed to be known, together with its symmetry algebra $\mathcal{L}$. The ΔS describing (3.1) will have the form

$$E_\alpha\left(x_{m+i,n+j}, y_{m+i,n+j}, u_{m+i,n+j}\right) = 0,$$
$$\alpha = 1,\ldots,N, \quad i_{\min} \leqslant i \leqslant i_{\max}, \quad j_{\min} \leqslant j \leqslant j_{\max}. \tag{3.2}$$

On figure 1 we depict a general lattice, a priori extending indefinitely in all directions. An orthogonal lattice (not necessarily uniform) is obtained by setting $\epsilon_{ik} = 0$, $\delta_{ik} = 0$ (ϵ_{ik} and δ_{ik} are defined in figure 1). The difference system (3.2) is written on a *stencil*: a finite number N of adjacent points, sufficient to reproduce, in the continuous limit, all derivatives figuring in the differential equation (3.1). For instance, for a first order PDE the minimal number of points on a stencil is three: (m, n) $(m + 1, n)$ $(m, n + 1)$. Since the system (3.2) is *autonomous*, i.e. the labels (m, n) do not figure in the ΔS (3.2) explicitly, we can shift the stencil around on the lattice arbitrarily. For convenience we will choose the reference point to be $(m, n) = (0, 0)$ and build the stencil around it. Thus, in (3.2) we start with $m = n = 0$ and then shift as needed.

For a first order initial value problem

$$F(x, y, u, u_x, u_y) = 0, \quad u(x, 0) = \phi(x) \tag{3.3}$$

it would be sufficient to choose $N = 3$ in (3.2) and give as initial data $x_{m,0}$, $y_{m,0}$, $u_{m,0}$ for all m.

On the first stencil we know $x_{0,0},\ x_{1,0},\ y_{0,0},\ y_{1,0},\ u_{0,0},\ u_{1,0}$ and calculate $x_{0,1},\ y_{0,1},\ u_{0,1}$ from (3.2). Then we shift the stencil one step in any direction and calculate further values till we fill the entire lattice.

To facilitate the calculations of the continuous limit we perform a transformation of variables on the stencil, introducing differences between coordinates and discrete partial derivatives [18, 19, 22, 23]. The new coordinates are $\{x_{0,0},\ y_{0,0},\ u_{0,0},\ h_{1,0},\ \epsilon_{0,1},\ k_{0,1},\ \delta_{1,0},\ u_x^d,\ u_y^d\}$, with

$$h_{1,0} = x_{1,0} - x_{0,0},\ \ k_{0,1} = y_{0,1} - y_{0,0},\ \ \delta_{1,0} = y_{1,0} - y_{0,0},\ \ \epsilon_{0,1} = x_{0,1} - x_{0,0}, \tag{3.4}$$

$$\begin{aligned} u_x^d &= \frac{1}{D}\Big[(y_{1,0} - y_{0,0})(u_{0,1} - u_{0,0}) - (y_{0,1} - y_{0,0})(u_{1,0} - u_{0,0})\Big],\\ u_y^d &= \frac{1}{D}\Big[(x_{0,1} - x_{0,0})(u_{1,0} - u_{0,0}) - (x_{1,0} - x_{0,0})(u_{0,1} - u_{0,0})\Big],\\ D &= \epsilon_{0,1}\delta_{1,0} - h_{1,0}k_{0,1} \neq 0. \end{aligned} \tag{3.5}$$

To describe an arbitrary second order PDE we need a stencil consisting of at least six points. A possible choice is to take points $\{(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (0, 2)\}$. For PDEs of the type

$$u_{xy} = F(x, y, u, u_x, u_y), \tag{3.6}$$

i.e. not involving u_{xx}, u_{yy}, it may be sufficient to take four points: $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$.

An element of the symmetry algebra $\mathcal{L}$ of the PDE (3.1) will have the form

$$\hat{Z} = \xi(x, y, u)\partial_x + \eta(x, y, u)\partial_y + \phi(x, y, u)\partial_u, \tag{3.7}$$

where the smooth functions ξ, η and ϕ are known (obtained by a standard algorithm for PDEs [29]).

In order to obtain an *invariant* ΔS (3.2) we must construct it out of difference invariants of the group $\mathcal{G}$, the Lie point symmetry group of the PDE (3.1). To calculate these invariants we consider the action of the vector field $\hat{Z}$ at some reference point $\{x_{0,0},\ y_{0,0},\ u_{0,0}\}$ and prolong it to all points figuring on a chosen stencil. This amounts to a prolongation to the discrete jet space:

$$\mathrm{pr}\hat{Z} = \sum_{i,j}\Big(\xi_{i,j}\partial_{x_{i,j}} + \eta_{i,j}\partial_{y_{i,j}} + \phi_{i,j}\partial_{u_{i,j}}\Big). \tag{3.8}$$

As in the continuous case, we can use both *strong* and *weak* invariants. The strong and weak invariants satisfy

$$\mathrm{pr}\hat{Z}I_s = 0, \tag{3.9}$$

$$\mathrm{pr}\hat{Z}I_w \Big|_{I_w=0} = 0, \tag{3.10}$$

respectively. To determine both types of invariants we choose a basis $\Big\{\hat{Z}_1, \cdots, \hat{Z}_A\Big\}$ ($A = \dim\mathcal{L}$) for the Lie algebra $\mathcal{L}$ and solve the set of equations

$$\mathrm{pr}\hat{Z}_a I\Big(x_{i,j}, y_{i,j}, u_{i,j}\Big) = 0, \quad a = 1, \cdots, A. \tag{3.11}$$

For strong invariants the rank r of the matrix of coefficients in (3.11) is maximal and the same for all points $(m + j, n + k)$. Invariants exist if we have $r = A < N$. Weak invariants are only invariant on some manifold in the space of points, obtained by requiring that the rank of coefficients in (3.11) be less than maximal. Thus, there may be more weak invariants than strong ones (strong invariants satisfy both (3.9) and (3.10)). The number of strong invariants is $n = N - A$.

4. Invariant discretization of the algebraic Liouville equation on a four-point stencil

We choose the four-points stencil $\mathfrak{s}_4^0 \equiv \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ on figure 1 and can translate them to any stencil $\mathfrak{s}_4^{m,n} = \{(m, n), (m + 1, n), (m, n + 1), (m + 1, n + 1)\}$ on the (x, y) plane. The vector fields (2.3) of the symmetry algebra $\mathcal{L}$ can be discretized and prolonged to all points of the stencil:

$$\begin{aligned} X^D(f) &= \mathrm{pr}\, X(f) = \sum_{(m,n)\in\mathfrak{s}_4^{m,n}} \left[f(x_{m,n})\partial_{x_{mn}} - f'(x_{m,n})\, u_{mn}\, \partial_{u_{mn}}\right], \\ Y^D(g) &= \mathrm{pr}\, Y(g) = \sum_{(m,n)\in\mathfrak{s}_4^{m,n}} \left[g\left(y_{m,n}\right)\partial_{y_{mn}} - g'\left(y_{m,n}\right)\, u_{mn}\, \partial_{u_{mn}}\right]. \end{aligned} \tag{4.1}$$

The prime and the dot denote (continuous) derivatives with respect to x and y, respectively.

Let us first restrict to the maximal finite-dimensional subalgebra $sl_x(2, \mathbb{R}) \oplus sl_y(2, \mathbb{R})$. The corresponding group acts transitively on the space of the continuous variables $(x, y, u) \in \mathbb{R}^3$, and sweeps out an orbit of codimension 6 on the 12-dimensional direct product $\mathbb{R}^3 \otimes \mathfrak{s}_4$. Hence we obtain six functionally independent invariants. A simple basis for these invariants is given by

$$\begin{aligned} \xi_1 &= \frac{(x_{0,1} - x_{0,0})(x_{1,1} - x_{1,0})}{(x_{0,0} - x_{1,0})(x_{0,1} - x_{1,1})} = \frac{\epsilon_{0,1}\epsilon_{1,1}}{h_{1,0}\left(h_{1,0} + \epsilon_{1,1} - \epsilon_{0,1}\right)}, \\ \eta_1 &= \frac{\left(y_{0,0} - y_{1,0}\right)\left(y_{0,1} - y_{1,1}\right)}{\left(y_{0,1} - y_{0,0}\right)\left(y_{1,1} - y_{1,0}\right)} = \frac{\delta_{1,0}\delta_{1,1}}{k_{0,1}\left(k_{0,1} + \delta_{1,1} - \delta_{1,0}\right)}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} H_1 &= u_{0,0}u_{0,1}\epsilon_{0,1}^2 k_{0,1}^2, \\ H_2 &= u_{1,0}u_{1,1}\epsilon_{1,1}^2\left(k_{0,1} + \delta_{1,1} - \delta_{1,0}\right)^2, \\ H_3 &= \frac{u_{1,0}\left(h_{1,0} - \epsilon_{0,1}\right)^2\left(k_{0,1} - \delta_{1,0}\right)^2}{u_{0,0}\, \epsilon_{0,1}^2\, k_{0,1}^2}, \\ H_4 &= \frac{u_{1,1}\epsilon_{1,1}^2\left(k_{0,1} + \delta_{1,1} - \delta_{1,0}\right)^2}{u_{0,0}\, h_{1,0}^2\, \delta_{1,0}^2}\ . \end{aligned} \tag{4.3}$$

The quantities $h_{1,0}$, $k_{0,1}$, $\epsilon_{0,1}$, $\epsilon_{1,1}$, $\delta_{1,0}$ and $\delta_{1,1}$ are defined in figure 1. The invariants ξ_1 and η_1 can be conveniently used to define an invariant lattice, e.g. by putting $\xi_1 = A$, $\eta_1 = B$, where A and B are constants. We choose the simplest possibility, namely

$$\xi_1 = 0, \qquad \eta_1 = 0. \tag{4.4}$$

This implies that e.g. $x_{0,1} - x_{0,0} = \epsilon_{0,1} = 0$ and also as a consequence $x_{1,1} - x_{1,0} = \epsilon_{1,1} = 0$. Similarly $\delta_{1,0} = \delta_{1,1} = 0$. Thus we have

$$x_{m,n} = x_m, \qquad y_{m,n} = y_n, \tag{4.5}$$

i.e. $x_{m,n}$ depends only on the first index, $y_{m,n}$ only on the second one. We thus obtain an orthogonal lattice (in an invariant manner). The quantities ξ_1 and η_1 are only invariant under $SL_x(2) \otimes SL_y(2)$, however we have

$$\begin{aligned}&\hat{X}^D\left(x^3\right)\xi_1 = (x_{1,1} - x_{0,0})(x_{1,0} - x_{0,1})\xi_1 \Big|_{\xi_1=0} = 0\\ &\hat{X}^D\left(x^3\right)\eta_1 = 0.\end{aligned} \tag{4.6}$$

It follows from the commutation relations (2.4) that a quantity annihilated by $\hat{X}^D(x^3)$ is also annihilated by $\hat{X}^D(x^n)$ for any n. Thus the lattice condition (4.4) is invariant under $VIR(x) \otimes VIR(y)$. On the other hand the equations $\xi_1 = A$ and $\eta_1 = B$ are not Virasoro invariant, if A and B are nonzero constants. We conclude that an orthogonal lattice is obligatory if we define it in terms of ξ_1 and η_1 alone. Conditions (4.4) and (4.5) are compatible with choosing a uniform orthogonal lattice

$$x_m = hm + x_0, \qquad y_n = kn + y_0, \tag{4.7}$$

where $h > 0$, $k > 0$, x_0, y_0 are constants, but the choice (4.7) is not obligatory.

The invariants $H_1, \cdots, H_4$ of (4.3) are not suitable on the lattice (4.4) since they all vanish or become infinite on the lattice. Before specifying the lattice we must choose new invariants (functions of those in (4.2) and (4.3)) which remain finite and nonzero for $\epsilon_{i,j} = \delta_{i,j} = 0$. Only two such $SL_x(2) \otimes SL_y(2)$ invariants exist, namely:

$$J_1 = H_1 H_3 = u_{0,1} u_{1,0} h_{1,\,0}^2 k_{0,\,1}^2, \tag{4.8}$$

$$J_2 = \frac{1}{\xi_1^2}\frac{H_2}{H_3} = u_{0,0} u_{1,1} h_{1,\,0}^2 k_{0,\,1}^2. \tag{4.9}$$

Neither of them is strongly invariant under the Virasoro group, since we have

$$\hat{X}^D\left(x^3\right)J_1 = -\, h_{1,\,0}^2 J_1, \qquad \hat{X}^D\left(x^3\right)J_2 = -\, h_{1,\,0}^2 J_2. \tag{4.10}$$

The equation $J_2 - J_1 = 0$ is Virasoro invariant (on its solution set) and this equation is a discretization of $uu_{xy} - u_x u_y = 0$ (equivalent to the wave equation $z_{xy} = 0$).

Putting $u_{0,0} = u(x, y)$, $u_{1,0} = u(x + h_{1,0}, y)$, $u_{0,1} = u(x, y + k_{0,1})$ and $u_{1,1} = u(x + h_{1,0}, y + k_{0,1})$, expanding in a Taylor series and keeping only the lowest order terms, we find

$$J_2 - J_1 = h_{1,\,0}^3 k_{0,\,1}^3 (uu_{xy} - u_x u_y). \tag{4.11}$$

The Liouville equation is approximated by the difference scheme

$$\begin{aligned}&J_2 - J_1 = a\left| J_1 \right|^{3/2} + bJ_1\left| J_2 \right|^{1/2} + c\left| J_1 \right|^{1/2} J_2 + d\left| J_2 \right|^{3/2},\\ &\xi_1 = 0, \quad \eta_1 = 0, \qquad a + b + c + d = 1.\end{aligned} \tag{4.12}$$

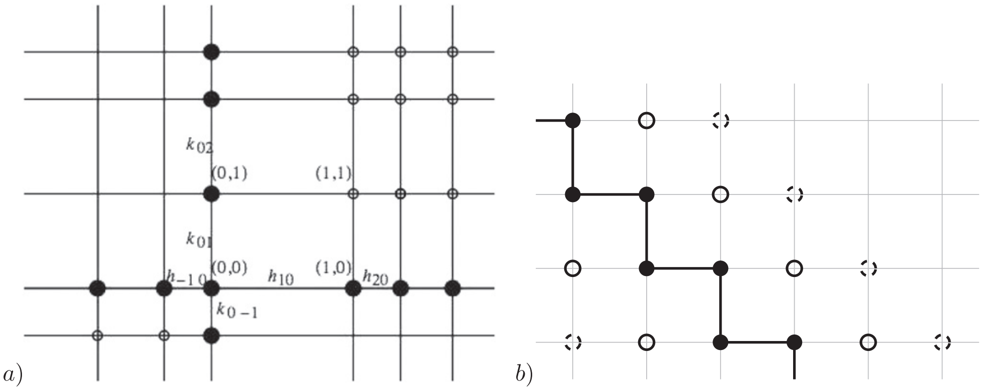

Figure 2. In (a) the boundary values for the discretized Liouville equation are given on two sets of points (black disks) along the x and y axes. The value in (1, 1) is obtained by the equation (4.14) and so are all the other points (empty circles) upwards on the right. On the opposite quadrant, one has to use the same equation, but solved for $u_{0,0}$. In (b) the boundary data are given on the downward staircase (black discs). The two diagonal lines of open discs are calculated using (4.14) for $u_{1,1}$ (line above the original staircase) or for $u_{0,0}$ (line below staircase). The two lines of dashed open discs represent the next level of calculations.

Indeed the Taylor expansion yields

$$J_2 - J_1 - \left[aJ_1^{3/2} + bJ_1 I_2^{1/2} + cJ_1^{1/2} J_2 + dJ_2^{3/2} \right] = h_{1,\,0}^3 k_{0,\,1}^3 \\ \times \left[uu_{xy} - u_x u_y - u^3 \right] + h_{1,\,0}^4 k_{0,\,1}^3 \left[\frac{1}{2} u_y u_{xx} (u - 1) - \frac{3}{2} u^2 u_x \right] \\ + h_{1,\,0}^3 k_{0,\,1}^4 \left[\frac{1}{2} u_x u_{yy} (u - 1) - \frac{3}{2} u^2 u_y \right] + \mathcal{O}\left(h_{1,\,0}^4 k_{0,\,1}^4 \right), \tag{4.13}$$

where the constants $a,\ b,\ c,\ d$ only appear in the $O(h_{1,\,0}^4 k_{0,\,1}^4)$ terms. The differential scheme (4.12) is $SL_x(2) \otimes SL_y(2)$ invariant, not however Virasoro invariant. The scheme is suitable for solving various types of boundary value problems, giving u_{ik} on the axes, or on upward or downward staircases (see figures 2(a) and (b) for examples).

Equation (4.12) must be solved for the values $u_{i,k}$ at one vertex of the rectangle $\mathfrak{s}_4^0$ in terms of the three others. In view of the rhs of the (4.12), this would in general require that a cubic equation has to be solved at each stage of the algorithm. This can be avoided by a convenient choice of the free constants $a,\ldots,\ d$. Let us consider two different possibilities:

1. $b = d = 0$ and $c = 1 - a$ for $a \in \mathbb{R}$;
2. $a = c = 0$ and $d = 1 - b$ for $b \in \mathbb{R}$.

The case 1 allows us to calculate $u_{1,1}$ (or $u_{0,0}$) linearly and we have e.g. the 1-parameter family of recursion formulae

$$u_{1,1} = \frac{u_{0,1} u_{1,0} \left(a h_{1,0} k_{0,1} \sqrt{\left| u_{0,1} u_{1,0} \right|} + 1 \right)}{u_{0,0} \left((a - 1) h_{1,0} k_{0,1} \sqrt{\left| u_{0,1} u_{1,0} \right|} + 1 \right)}. \tag{4.14}$$

This allows us to start from boundary conditions given on the horizontal and vertical axes and proceed upwards to the right, thus filling up the entire first quadrant. Interchanging $u_{1,1} \leftrightarrow u_{0,0}$ in (4.14) we can fill up the entire third quadrant (see figure 2(a)). Similarly, starting from the downward staircase of figure 2(b), we can use (4.14) to fill up all points above and also below the staircase.

In the case 2 we obtain

$$u_{1,0} = \frac{u_{0,0}u_{1,1}\left(1 - dh_{1,0}k_{0,1}\sqrt{\left|u_{0,0}u_{1,1}\right|}\right)}{u_{0,1}\left(1 - (1-d)h_{1,0}k_{0,1}\sqrt{\left|u_{0,0}u_{1,1}\right|}\right)}, \tag{4.15}$$

which allows to fill out the second and the fourth quadrants. If boundary conditions are given on an upward staircase we can fill in the area below and above it.

5. Symmetries of the Rebelo–Valiquette discretized Liouville equation

In [32] Rebelo and Valiquette considered a symmetry preserving discretization of the Liouville equation (1.2) namely:

$$\begin{aligned} L_{RV}^{D} = u_{1,1}u_{0,0} - u_{1,0}u_{0,1} - u_{0,0}u_{0,1}u_{1,0}(x_{1,0} - x_{0,0})\left(y_{0,1} - y_{0,0}\right) = 0,\\ x_{0,1} = x_{0,0}, \quad y_{1,0} = y_{0,0}. \end{aligned} \tag{5.1}$$

The equation for the lattice clearly states that $x_{i,j} = x_i$ and $y_{i,j} = y_j$, so the lattice coincides with the one we used above. They constructed (5.1) from the invariance with respect to the pseudo-group

$$\tilde{x}_i = F(x_i), \quad \tilde{y}_j = G(y_j), \quad \tilde{u}_{i,j} = \frac{u_{i,j}}{\frac{F(x_{i+1}) - F(x_i)}{x_{i+1} - x_i}\frac{G\left(y_{j+1}\right) - G\left(y_j\right)}{y_{j+1} - y_j}} \tag{5.2}$$

for arbitrary regular F and G.

First, let us notice that the equation (5.1) is not invariant with respect the algebra $sl_x(2, \mathbb{R}) \oplus sl_y(2, \mathbb{R})$ considered in the previous sections. In fact it results that

$$X^D\left(x^2\right)L_{RV}^D\Big|_{L_{RV}^D=0} = u_{0,0}u_{0,1}u_{1,0}(x_{1,0} - x_{0,0})^2(y_{0,1} - y_{0,0}) \tag{5.3}$$

and similarly for $Y^D\left(y^2\right)$.

Let us look here for infinitesimal symmetries of (5.1) of the form

$$\begin{aligned} \hat{X} = Q_{ij}^{(1)}\left(x_{i,j}, y_{i,j}, u_{i,j}\right)\partial_{x_{i,j}} + Q_{ij}^{(2)}\left(x_{i,j}, y_{i,j}, u_{i,j}\right)\partial_{y_{i,j}}\\ + Q_{ij}^{(3)}\left(x_{i,j}, x_{i+1,j}, y_{i,j}, y_{i,j+1}, u_{i,j}\right)\partial_{u_{i,j}}. \end{aligned} \tag{5.4}$$

The determining equations are:

$$Q_{01}^{(1)} = Q_{00}^{(1)}, \tag{5.5}$$

$$Q_{10}^{(2)} = Q_{00}^{(2)}, \tag{5.6}$$

$$
\begin{aligned}
&Q_{11}^{(3)}u_{0,0} + u_{1,1}Q_{00}^{(3)} - u_{1,0}Q_{01}^{(3)} - u_{0,1}Q_{10}^{(3)}\\
&= Q_{00}^{(3)}u_{0,1}u_{1,0}(x_{1,0} - x_{0,0})(y_{0,1} - y_{0,0})\\
&+ Q_{10}^{(3)}u_{0,1}u_{0,0}\big(x_{1,0} - x_{0,0})(y_{0,1} - y_{0,0})\\
&+ Q_{01}^{(3)}u_{0,0}u_{1,0}(x_{1,0} - x_{0,0})(y_{0,1} - y_{0,0})\\
&+ u_{0,0}u_{0,1}u_{1,0}(Q_{10}^{(1)} - Q_{00}^{(1)})(y_{0,1} - y_{0,0})\\
&+ u_{0,0}u_{0,1}u_{1,0}(x_{1,0} - x_{0,0})(Q_{01}^{(2)} - Q_{00}^{(2)}).
\end{aligned}
\tag{5.7}
$$

We put $x_{0,1} = x_{0,0}$, $y_{1,0} = y_{0,0}$ and $u_{1,1} = u_{0,1}u_{1,0}[\frac{1}{u_{0,0}} + (x_{1,0} - x_{0,0})(y_{0,1} - y_{0,0})]$ so that $x_{0,0}$, $y_{0,0}$, $y_{0,1}$, $x_{1,0}$, $u_{0,0}$, $u_{0,1}$ and $u_{1,0}$ are independent variables in the determining equations. From (5.5) we deduce that $Q_{ij}^{(1)} = f(x_i)$ and from (5.6) $Q_{ij}^{(2)} = g(y_j)$ where f and g are arbitrary functions of their arguments. Dividing (5.7) by $u_{0,0}$ and applying the operator $A = u_{1,0}\partial_{u_{1,0}} - u_{0,1}\partial_{u_{0,1}}$ (we have $A\phi(u_{1,1}) = 0$ for any function ϕ) and we get

$$
\frac{Q_{01}^{(3)}}{u_{0,1}} - \frac{\partial Q_{01}^{(3)}}{\partial u_{0,1}} = \frac{Q_{10}^{(3)}}{u_{1,0}} - \frac{\partial Q_{10}^{(3)}}{\partial u_{1,0}},
\tag{5.8}
$$

i.e. the quantity $\frac{Q_{ij}^{(3)}}{u_{i,j}} - \frac{\partial Q_{ij}^{(3)}}{\partial u_{i,j}} = h(i + j)$. So

$$
Q_{ij}^{(3)} = u_{i,j}\Big[h(i + j)\log_e(u_{i,j}) + A_{ij}\big(x_{i,j}, x_{i+1,j}, y_{i,j}, y_{i,j+1}\big)\Big].
\tag{5.9}
$$

Introducing this result into (5.7) and taking into account that $\log_e(u_{1,1}) = \log_e(u_{1,0}) + \log_e(u_{0,1}) + \log_e[\frac{1}{u_{0,0}} + (x_{1,0} - x_{0,0})(y_{0,1} - y_{0,0})]$ we find from the coefficient of $\log_e[\frac{1}{u_{0,0}} + (x_{1,0} - x_{0,0})(y_{0,1} - y_{0,0})]$ that $h(i + j) = 0$. Thus $Q_{ij}^{(3)} = u_{i,j}A_{ij}(x_{i,j}, x_{i+1,j}, y_{i,j}, y_{i,j+1})$. Introducing this last result into (5.7) we find two equations for $A_{ij}(x_{i,j}, x_{i+1,j}, y_{i,j}, y_{i,j+1})$

$$
A_{00} + A_{11} - A_{01} - A_{10} = 0,
\tag{5.10}
$$

$$
A_{ij} = -\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} - \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}.
\tag{5.11}
$$

Equation (5.10) is identically satisfied by the result obtained in (5.11) and as a consequence the symmetry algebra of the Liouville equation presented by Rebelo and Valiquette is indeed the sum of two Virasoro algebras determined by the two functions f and g:

$$
\hat{X}(f, g) = f(x_i)\partial_{x_i} + g\big(y_j\big)\partial_{y_j} - \left[\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} + \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}\right]\partial_{u_{i,j}}.
\tag{5.12}
$$

The main difference between these generators and those in (4.1) is that in (4.1) the coefficients of $\partial_{u_{i,j}}$ are locally dependent on the space points, while two points are involved in (5.12). Thus, the expression (5.4) has to be understood as a summation over all points of the lattice. On the contrary (4.1) contains only finite sums over the stencil points. Thus the Rebelo–Valiquette discretization of the Liouville equation is invariant under $VIR(x) \otimes VIR(y)$, but these are generalized symmetries that reduce to point ones only if $f(x)$ and $g(y)$ are linear (rather then quadratic) functions. These are actually very special generalized symmetries: The Lie algebra (5.12) can be integrated to the finite transformations (5.2). These finite transformations were actually the starting point in the Rebelo–Valiquette approach.

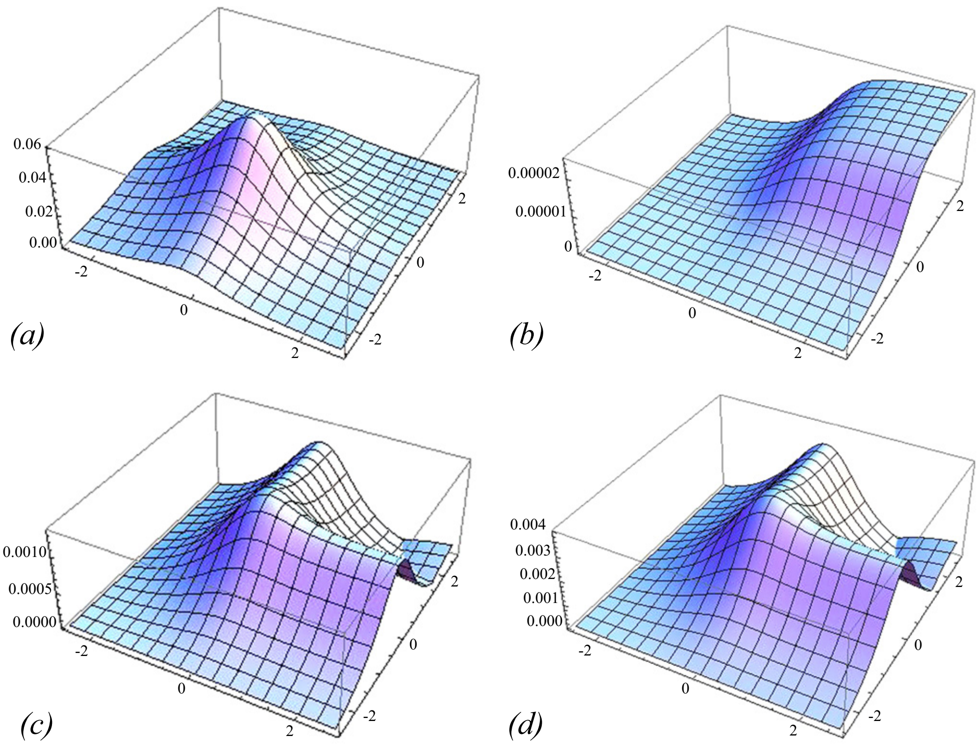

Figure 3. The solution s_1 with the choice of parameters $\alpha = 6,\ \beta = 1,\ \gamma = 1,\ \delta = 1$ is numerically computed giving a boundary value problem on a lattice with corner point $(x_{00}, y_{00}) = (-2.5, -2.5)$ and steps of equal length $h = k = 0.02$ for a lattice of 260×260 points. In (a) we show the graph of the analytic expression and in (b) the relative error of the numerical results using the invariant formula (4.14) with respect to the analytic solution. Analogously, the errors of the numerical results for the Rebelo–Valiquette representation are reported in (c), while in (d) we present the corresponding relative errors of the numerical results for the standard formula (6.1). Despite the generic similarities of the two results, the difference of two orders of magnitude in the relative errors is remarkable.

6. Numerical results and analysis

In order to test the efficiency of the numerical algorithms based on the invariant difference scheme (4.12), we will solve a set of boundary value problems for the Liouville equation on a uniform lattice $h_{m,n} = h,\ \ k_{m,n} = k$.

We will compare the results with the analytic solutions and with numerical solutions obtained by the standard (*naive*) finite difference approximation

$$u_{1,1}u_{0,0} - u_{0,1}u_{1,0} = hk\ u_{0,\,0}^3 \tag{6.1}$$

and by the Rebelo–Valiquette discretization

$$u_{1,1}u_{0,0} - u_{1,0}u_{0,1} = h\ k\ u_{0,0}u_{0,1}u_{1,0}. \tag{6.2}$$

The equations (4.12), (6.1) and (6.2) all relate the values of $u_{i,k}$ at the corners of a rectangle on the mesh with sides of length h and k, respectively. We will solve boundary value problems

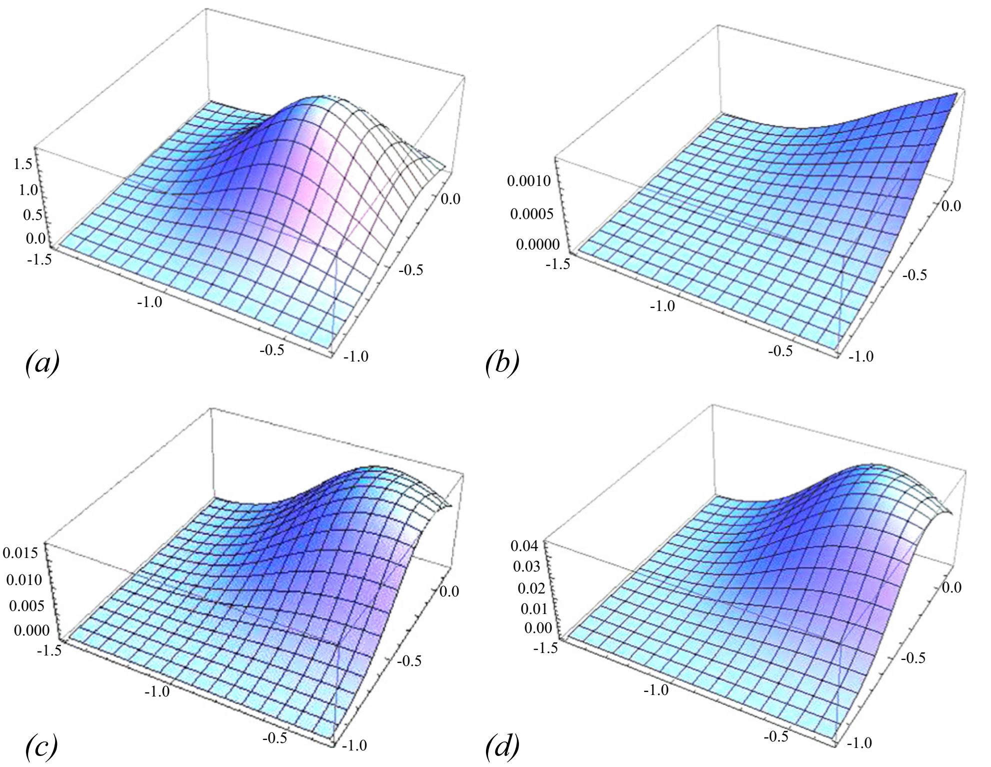

Figure 4. The same analysis as for s_1 is carried out for s_2 on a lattice with corner point $(x_{00}, y_{00}) = (-1.5, -1.0)$ and steps of equal length $h = k = 0.02$ for a lattice of 60×60 points.

with boundaries along the vertical and horizontal axes and we will calculate values of $u_{i,k}$ in the first quadrant. Thus, $u_{m,0}$ and $u_{n,0}$ will be given for $m \geqslant 0$ and $n \geqslant 0$ (see figure 2(a)), in the computational basis. We will use (4.12), (6.1) and (6.2) to compute the top right values of $u_{i,k}$. More precisely, equation (4.14) is a 1-parameter family of recursion relations. For the actual calculations we use the symmetric case $a = c = \frac{1}{2}$. We could also study the dependence on the parameter a to get the best numerical fits. We have not done this and do not consider it important at this stage, since equation (4.13) shows that the dependence on $a, \ldots, d$ only appears in higher order terms (order hk rather than h or k).

We analyzed several exact solutions of the Liouville equations. We reproduce the results for three of them, namely

$$s_1 = \frac{2\beta\gamma\delta}{\left(\beta^2x^2 + 1\right)\left(\delta^2y^2 + 1\right)\left(\tan^{-1}(\beta x) + \gamma\tan^{-1}(\mathrm{d}y) + \alpha\right)^2}, \tag{6.3}$$

$$s_2 = \frac{8\left(1 - 4\left(x + \frac{1}{2}\right)\right)(1 - 4y)\exp\left(-4\left(x + \frac{1}{2}\right)^2 + 2\left(x + \frac{1}{2}\right) - 4y^2 + 2y\right)}{\left(\mathrm{e}^{2\left(x+\frac{1}{2}\right)-4\left(x+\frac{1}{2}\right)^2} + \mathrm{e}^{2y-4y^2} + 1\right)^2}, \tag{6.4}$$

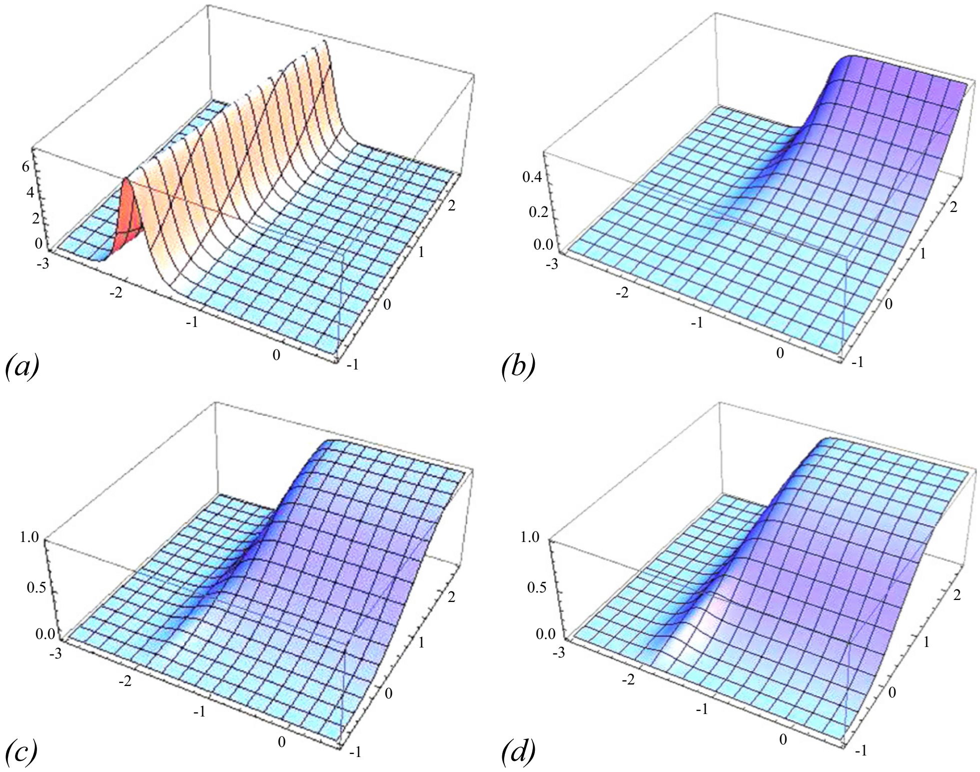

Figure 5. The same analysis as for s_1 is carried out for s_3 for the choice of parameters $A = 12.8397$, $p = 3.862\,33$ on a lattice with corner point $(x_{00}, y_{00}) = (-3, -1)$ and steps of equal length $h = k = 0.02$ for a lattice of 180×180 points.

$$s_3 = \frac{2\,Ap^2 \mathrm{e}^{p(x+y)}}{\left(A\mathrm{e}^{py} + \mathrm{e}^{px}\right)^2}, \tag{6.5}$$

for certain values of the constants A, p, α, β, γ, δ specified in figures 3–5 .

To proceed numerically we specify the lattice constants h and k and place the corner point (0, 0) on the lattice. Using the analytic solutions we compute the values of the solutions on the boundary. These values are then used as initial data for the numerical calculations. We used the boundaries on the axes as on figure 2(a). The three different discretizations used here were the $SL_x(2, \mathbb{R}) \otimes SL_y(2, \mathbb{R})$ invariant one (4.14), the Rebelo–Valiquette invariant one (6.2) and the naive non invariant one (6.1). On figures 3–5 we show the results for solutions s_1, s_2 and s_3, respectively.

On each figure the picture (a) represents the exact solution plotted from (6.3), (6.4) or (6.5), respectively. The pictures (b)–(d) show the errors of the numerical results using the $SL_x(2, \mathbb{R}) \otimes SL_y(2, \mathbb{R})$ invariant formula (4.14), the invariant Rebelo–Valiquette formula (6.2) and the naive non invariant one (6.1), respectively.

In all cases the errors are small so that visually all numerical results almost coincide with the exact one. The differences between the individual discretizations are best seen in table 1. There we give the mean square averages (the normalized $L^2_{\mathbb{R}^2}$ metric) of the distances between the analytic solutions and the three numerical ones. We see that for all three solutions we have

Table 1. Mean square average differences between the analytic solutions and the numerical ones.

	χ_{Inv}	χ_{RV}	χ_{stand}
s_1	4.6×10^{-11}	8.0×10^{-7}	7.2×10^{-5}
s_2	1.6×10^{-7}	1.3×10^{-4}	7.0×10^{-1}
s_3	1.7×10^{-2}	3.2×10^{-1}	6.0×10^{-1}

$\chi_{\text{Inv}} < \chi_{\text{RV}} < \chi_{\text{stand}}$, i.e. the invariant method performs better than the other two. The Rebelo–Valiquette method that preserves the infinite dimensional symmetry group as generalized symmetries also performs better than the standard one in agreement with [32].

7. Conclusions

We have shown that at least on a four-point lattice it is not possible to discretize the Liouville equation (1.2) (nor (1.1)) while preserving $VIR(x) \otimes VIR(y)$ as the Lie point symmetry group. On the other hand, Rebelo and Valiquette [32] have introduced a special type of generalized symmetries that leave their discretization of the algebraic Liouville equation invariant. In the continuous case these symmetries reduce to point ones. In the discrete case they are special in that the vector fields can be integrated to group transformations acting on the equation and on the lattice. This is somewhat similar to the case of the symmetries of the Toda hierarchy [12] where some generalized symmetries *contract* to point ones in the continuous limit.

From the point of view of numerical methods for the three exact solutions considered the discretization preserving the maximal finite subgroup of the infinite dimensional point symmetry group, performs better than the one that transforms point symmetries into generalized ones.

As stated in the Introduction, the main purpose of this article is to investigate how continuous physical theories can be discretized while preserving their continuous Lie point symmetries. For the Liouville equation we have shown that in a complete discretization it is possible to preserve invariance under the maximal finite subgroup. The infinite dimensional Lie pseudogroup does not survive as a group of point symmetries. Rebelo and Valiquette have shown that the entire Virasoro pseudogroup does survive in a different discretization [32], but as generalized symmetries. We also see that preserving the maximal finite sub algebra as point symmetries is incompatible with preserving the entire symmetry group as generalized symmetries.

In section 5 we have tested the quality of our invariant discretization as a numerical method. We have shown that it actually performs very well. We are of course aware that what we here call ‘standard’ methods can be improved in many other ways. The use of point symmetries in numerical solutions of PDEs deserves a further detailed analysis, in particular for other classes of solutions of the Liouville equation.

Another interesting point is that the linearizable discretization of Adler and Startsev preserves no point symmetries, see appendix. It is thus important to decide which features of a continuous theory one wishes to preserve in a discretization. In this case linearizability is incompatible with the preservation of point symmetries.

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Appendix. Lie point symmetries of a linearizable discrete Liouville equation

Adler and Startsev [1] have presented a discretization of the algebraic Liouville equation (1.2) on a four-point lattice, namely

$$u_{i+1,j+1}\left(1+\frac{1}{u_{i+1,j}}\right)\left(1+\frac{1}{u_{i,j+1}}\right)u_{i,j}=1. \tag{A.1}$$

This equation is linearizable by the substitution

$$u_{i,j}=-\frac{(v_{i+1,j}-v_{i,j})(v_{i,j+1}-v_{i,j})}{v_{i+1,j}v_{i,j+1}}, \tag{A.2}$$

where $v_{i,j}$ satisfies the linear equation

$$v_{i+1,j+1}-v_{i+1,j}-v_{i,j+1}+v_{i,j}=0. \tag{A.3}$$

Hence the general solution of (A.1) is

$$u_{i,j}=-\frac{(c_{i+1}-c_i)\left(k_{j+1}-k_j\right)}{\left(c_{i+1}+k_j\right)\left(c_i+k_{j+1}\right)}, \tag{A.4}$$

where c_i, k_j are arbitrary functions of one index each.

The continuous limit of (A.1) is taken in two steps. First we define $x=\epsilon j$ and $u_{i,j}=-\epsilon v_i(x)$. When $\epsilon\to 0$ one gets the differential—difference Liouville equation [1]

$$v_{i+1,x}\,v_i-v_{i+1}\,v_{i,x}=v_{i+1}\,v_i(v_{i+1}+v_i). \tag{A.5}$$

The continuous limit of the last equation is obtained by setting $y=\mu i$ and $v_i(x)=\mu w(x,y)$. When $\mu\to 0$ one gets the algebraic Liouville equation as

$$(\log w)_{x\,y}=2w. \tag{A.6}$$

We restrict (A.1) to the stencil with $i=j=0$, i.e.

$$E=u_{1,1}\left(u_{1,0}+1\right)\left(u_{0,1}+1\right)u_{0,0}-u_{1,0}u_{0,1}=0, \tag{A.7}$$

and calculate the Lie point symmetries of this equation. The equation is autonomous, the lattice is fixed (orthogonal and uniform). Hence the symmetry algebra is generated by vector fields of the form

$$\hat{X}_e = Q_{ij}(u_{i,j})\partial_{u_{i,j}}, \tag{A.8}$$

satisfying

$$\hat{X}E\Big|_{E=0} = 0. \tag{A.9}$$

We obtain

$$Q_{11}\big(u_{0,1}+1\big)\big(u_{1,0}+1\big)u_{0,0} + Q_{10}u_{1,1}\big(u_{0,1}+1\big)u_{0,0} + Q_{01}u_{1,1}\big(u_{1,0}+1\big)u_{0,0} + Q_{00}u_{1,1}\big(u_{0,1}+1\big)\big(u_{1,0}+1\big) = Q_{10}u_{0,1} + Q_{01}u_{1,0}. \tag{A.10}$$

We eliminate $u_{1,1}$ from (A.10) using (A.7), then differentiate with respect to $u_{0,0}$ and obtain

$$\frac{Q_{11}}{u_{1,1}} - \frac{\mathrm{d}Q_{11}}{\mathrm{d}u_{1,1}} = \frac{Q_{00}}{u_{0,0}} - \frac{\mathrm{d}Q_{00}}{\mathrm{d}u_{0,0}}. \tag{A.11}$$

The general solution of (A.11) is

$$Q_{ij} = u_{i,j}\Big[g_{ij} + f(i-j)\log_e(u_{i,j})\Big], \tag{A.12}$$

where g_{ij} and $f(i-j)$ are functions of i and j. Substituting (A.12) into (A.10) we find $g(i, j) = f(i-j) = 0$. It follows that the linearizable discrete Liouville equation has no continuous point symmetries at all! Two comments are in order.

1. The equation (A.1) is linearizable and hence must have generalized symmetries.
2. [1] also contains the linearizable differential–difference Liouville equation (A.5). It can be shown using the formalism presented in [24] that (A.5) does have an infinite dimensional Lie point symmetry algebra, isomorphic to the Virasoro algebra. The algebra is realized by evolutionary vector fields of the form

$$\hat{X}_e = Q_i(x, v_i, \dot{v}_i)\partial_{v_i}, \quad Q_i = f(x)\dot{v}_i + \dot{f}(x)v_i. \tag{A.13}$$

This corresponds to the standard factor fields

$$\hat{X} = f(x)\partial_x - \dot{f}(x)v\partial_v. \tag{A.14}$$

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