Symmetry preserving discretization of ordinary differential equations. Large symmetry groups and higher order equations.

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Abstract. Ordinary differential equations (ODEs) and ordinary difference systems (OΔSs) invariant under the actions of the Lie groups $SL_x(2)$, $SL_y(2)$ and $SL_x(2) \times SL_y(2)$ of projective transformations of the independent variables x and dependent variables y are constructed. The ODEs are continuous limits of the OΔSs, or conversely, the OΔSs are invariant discretizations of the ODEs. The invariant OΔSs are used to calculate numerical solutions of the invariant ODEs of order up to five. The solutions of the invariant numerical schemes are compared to numerical solutions obtained by standard Runge-Kutta methods and to exact solutions, when available. The invariant method performs at least as well as standard ones and much better in the vicinity of singularities of solutions.

1. Introduction

The application of Lie groups to the study of difference equations is a relatively new topic that has been actively pursued for the last 30 years or so. For recent reviews we refer the reader to [8, 9, 10, 19, 24, 30, 42, 43]. Some of the original articles pertinent for this study are [7, 11, 12, 13, 14, 17, 29, 39, 37, 38].

This line of research has several aspects. From the point of view of physics, one aim is to preserve such fundamental symmetry properties as Lorentz, Galilei and conformal invariance in a discrete space-time. From the point of view of mathematics, both pure and applied, the aim is to turn Lie group theory into an efficient tool for studying the solution space of difference equations as it has long been for differential ones [2, 31, 32, 34]. From the point of view of computing, this approach belongs into the field of geometrical integration [18, 21, 20, 36]. The aim is to improve the qualitative and quantitative features of numerical solutions of differential equations by introducing difference systems that have the same Lie point symmetry groups as their continuous limits (invariant discretization).

The original idea [7] has been applied to both ordinary differential equations (ODEs) [3, 12, 13, 14, 15, 27, 39, 40] and partial differential equations (PDEs) [1, 4, 12, 15, 22, 23, 25, 26, 28, 37, 38, 41].

For first order ODEs the method provides exact discretizations, i.e. differential systems that have the same solutions as the ODEs [40]. For second and higher order ODEs invariant discretization often provides difference schemes that can be solved analytically, using either a Lagrangian [12, 14] or the adjoint equation method [11]. It was shown for numerous second and third order ODEs that invariant discretization provides qualitatively better fits to solutions than standard methods, specially in the neighborhoods of singularities [3, 39].

The purpose of this article is to extend the method of invariant discretization of differential equations to larger Lie groups and higher order ODEs than have been treated so far. More specifically, we consider the direct product group $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$, where x and y are the independent and dependent variables, respectively and treat ODEs up to order 5.

In Section 2 we briefly outline the general method of invariant discretization for ODEs. In Section 3, we sum up the differential invariants of $\mathrm{SL}_y(2)$ (to all orders), and of $\mathrm{SL}_x(2)$ and $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$ (up to order 5). The main results of the article are presented in Section 4. Thus we derive complete sets of difference invariants up to order 5 (using 6 points on a stencil) for all 3 groups under consideration and show how to obtain the differential invariants in the continuous limit. Section 5 is devoted to numerical examples in which we compare results using the invariant discretization with standard numerical methods.

2. Differential and difference invariants of a Lie group

Let us consider a Lie group of local point transformations acting on a Euclidean plane with Cartesian coordinates (x, y) generated by a Lie algebra of vector fields of the form

$$X = \xi(x, y)\partial_x + \phi(x, y)\partial_y \qquad (1)$$

We can view the Lie group as acting either on solutions $(u = f(x))$ of an ODE:

$$E \equiv y^{(N)} - F(x, y, y', y'', \ldots, y^{(N-1)}) = 0 \qquad (2)$$

or on solutions of an ordinary difference scheme OΔS

$$E_a \equiv E_a(\{x_k, y_k\}_{k=0}^N) = 0, \quad a = 1, 2. \qquad (3)$$

The difference scheme (3) that we use consists of two equations, each connecting the $N + 1$ points, satisfying

$$\frac{\partial(E_1, E_2)}{\partial(x_N, y_N)} \neq 0 \qquad (4)$$

so that we can calculate (x_N, y_N) if the previous points (x_k, y_k) are known. In the continuous limit we put

$$x_n - x_{n-1} = h_n, \quad h_n = \alpha_n \epsilon, \quad \epsilon \to 0 \qquad (5)$$

where α_n are some constants of the order $\alpha_n \sim 1$ and we require

$$\{E_1 = 0\} \to \{E = 0\}, \quad \{E_2 = 0\} \to \{0 = 0\}, \quad \epsilon \to 0 \qquad (6)$$

Thus the lattice equation goes into an identity and the difference scheme goes into the target ODE.

We will take a given group G and find a basis for the differential invariants up to a certain order (i.e. fix the order N of the highest derivative) and for the difference invariants up to the corresponding order (i.e. fix the number of points of the lattice to be used as $N+1$). The ODEs and the OΔSs will be constructed out of the respective invariants, $I_1, \ldots, I_K$ or $I_1^D, \ldots, I_L^D$ respectively and we have $K < L$.

Equations written in terms of these invariant will be "strongly invariant". Other equations may be "weakly invariant", i.e. invariant on some submanifold. We will encounter such cases below for both ODEs and OΔSs.

Basically two methods exist for calculating invariants of a given group action on a homogeneous manifold. One is the infinitesimal method based on the prolongations of the vector fields representing the Lie algebra of the group [31, 32]. The other method is a global one, called the method of moving frames [6, 16, 33]. In the second method it is necessary to express the group parameters in terms of the values of a sufficient number of the transformed variables on some section of the generic orbits. For simple and semisimple groups this typically leads to algebraic equations to solve. In particular for the $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$ action studied in this article that leads to a third order algebraic equation. We find the infinitesimal method more convenient for the problem at hand and we use it throughout the article.

The group G and the vector fields X of (1) act on the variables (x, y) and on functions $y = f(x)$ in the same manner, whether we are considering differential equations or difference systems. However in the continuous case we prolong to actions on derivatives in a a standard manner [32]. In the discrete case we write X at some point x_k of the one dimensional lattice and then sum over all points involved in the OΔS [9, 30]:

$$\mathrm{pr}^D X_n = \sum_k \left(\xi(x_{n+k}, y_{n+k}) \partial_{x_{n+k}} + \phi(x_{n+k}, y_{n+k}) \partial_{y_{n+k}} \right) \tag{7}$$

The summation over k is over all points on one stencil. The index n labels the position of the stencil used in the calculation.

In both cases we find the invariant by solving the system of determining equations following from the invariance condition

$$\mathrm{pr} X \Phi(x, y, y', \ldots, y^{(N)}) = 0 \tag{8}$$

or

$$\mathrm{pr}^D X_n \Phi(x_{n+k}, y_{n+k}) = 0 \tag{9}$$

There will be more functionally independent difference invariants than differential ones. We will divide the difference ones into two sets; those that go into differential ones in the continuous limit and those that vanish in this limit.

The connection between difference and differential invariants is established by using Taylor expansions of the discrete quantities. We restrict ourselves to a single stencil, i.e. points $(x_0, y_0), \ldots, (x_N, y_N)$, and choose a point about which to develop, say x_0. All other points are expressed as:

$$x_k = x_0 + \sum_{l=1}^{k} h_l, \quad y_k = y(x_k), \quad 1 \le k \le N \tag{10}$$

and we expand all discrete invariants using the truncated Taylor series:

$$y_k = \sum_{j=0}^{N} \frac{1}{j!} y_0^{(j)} \left(\sum_{l=1}^{k} h_l \right)^j \tag{11}$$

The continuous limit is taken as in (5). The result will be expressed in terms of the basis of differential invariants. In general the limit may depend on the constants α_n in (5). These will be specific numbers once the lattice is chosen. Detailed examples will be given in Section 4.

3. Differential invariants under $\mathrm{SL}_y(2)$, $\mathrm{SL}_x(2)$ and $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$

In this section we restrict the group G to be $\mathrm{SL}_y(2)$, $\mathrm{SL}_x(2)$ and $\mathrm{SL}_x(2) \times \mathrm{SL}_x(2)$ respectively, and will present bases for all differential invariants up to order 5, though it would be easy to proceed to higher orders. The reason for this choice is that $N = 5$ is the lowest order at which an $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$ invariant exists.

3.1. Invariants of $\mathrm{SL}_y(2)$

The Lie algebra of the group $\mathrm{SL}_y(2)$ is generated by vector fields ∂_y, $y\partial_y$, $y^2\partial_y$ with prolongations:

$$
\begin{aligned}
\mathrm{pr}^{(N)}\partial_y \quad &= \partial_y \\
\mathrm{pr}^{(N)}y\partial_y &= y\partial_y + \sum_{k=1}^{N} y^{(k)}\partial_{y^{(k)}} \\
\mathrm{pr}^{(N)}y^2\partial_y &= y^2\partial_y + \sum_{k=1}^{N} (y^2)^{(k)}\partial_{y^{(k)}},
\end{aligned}
\tag{12}
$$

Solving the corresponding PDEs (8) for $N = 5$ we find the lowest order differential invariant

$$
J_3 = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2
\tag{13}
$$

The third and fourth invariants can also be calculated directly.

Alternatively, since the variable x is invariant we can start from the lowest order invariant involving derivatives of y, namely J_3 and generate a different basis of $\mathrm{SL}_y(2)$ invariants, by invariant differentiation:

$$
J_{k+3} = \frac{\mathrm{d}^k}{\mathrm{d}x^k}J_3, \quad k = 1, 2, \ldots
\tag{14}
$$

All $\mathrm{SL}_y(2)$ differential invariants of order up to N will be functions of

$$
\{x, J_3, J_4, J_5, \ldots, J_N\}
\tag{15}
$$

We mention that J_3 is the well known Schwarzian derivative with many interesting applications [35]. We will use in this work the first three invariants, J_3, J_4 and J_5. The explicit form of J_4 and J_5 are:

$$
J_4 \equiv J_3' = \frac{y^{(4)}}{y'} - 4\frac{y''y'''}{y'^2} + 3\frac{y''^3}{y'^3}
\tag{16}
$$

$$
J_5 \equiv J_3'' = \frac{y^{(5)}}{y'} - 5\frac{y''y^{(4)}}{y'^2} + 17\frac{y''^2y'''}{y'^3} - 4\frac{y'''^2}{y'^2} - 9\frac{y''^4}{y'^4}
\tag{17}
$$

We could also use a simplified fifth order invariant, adding a multiple of J_3^2

$$
\tilde{J}_5 \equiv J_5 + 4J_3^2 = \frac{y^{(5)}}{y'} - 5\frac{y''y^{(4)}}{y'^2} + 5\frac{y''^2y'''}{y'^3}
\tag{18}
$$

3.2. Differential invariants of $\mathrm{SL}_x(2)$

The Lie algebra of $\mathrm{SL}_x(2)$ has the basis ∂_x, $x\partial_x$, $x^2\partial_x$ with prolongations

$$
\begin{aligned}
\mathrm{pr}^{(N)}\partial_x \quad &= \partial_x \\
\mathrm{pr}^{(N)}x\partial_x \quad &= x\partial_x - \sum_{k=1}^{N} ky^{(k)}\partial_{y^{(k)}} \\
\mathrm{pr}^{(N)}x^2\partial_x &= x^2\partial_x - \sum_{k=1}^{N} k\big((k-1)y^{(k-1)} + 2xy^{(k)}\big)\partial_{y^{(k)}}
\end{aligned}
\tag{19}
$$

It is of course equivalent to $\mathrm{SL}_y(2)$ and the two are transformed into each other by a hodograph transformation. We treat $\mathrm{SL}_x(2)$ separately here since we are interested mainly in the action of $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$.

The three lowest order invariants are (higher order ones are also computed in a straightforward way [5])

$$
\begin{aligned}
K_3 &= \frac{1}{(y')^2}\left(\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2\right), \\
K_4 &= \frac{y^{(4)}}{y'^4} - 6\frac{y'''y''}{y'^5} + 6\frac{y''^3}{y'^6}, \\
K_5 &= \frac{y^{(5)}}{y'^5} - 10\frac{y^{(4)}y''}{y'^6} - 4\frac{y'''^2}{y'^6} + 42\frac{y'''y''^2}{y'^7} - \frac{63}{2}\frac{y''^4}{y'^8}
\end{aligned}
\tag{20}
$$

and, of course, y itself is an invariant.

It is interesting to consider the behavior of the $\mathrm{SL}_y(2)$ invariant J_3. We have

$$
(\mathrm{pr}^{(3)}\partial_x)J_3 = 0, \quad (\mathrm{pr}^{(3)}x\partial_x)J_3 = -2J_3, \quad (\mathrm{pr}^{(3)}x^2\partial_x)J_3 = -4xJ_3 \tag{21}
$$

Thus, while J_3 is not invariant under $\mathrm{SL}_x(2)$, the equation

$$
J_3 = 0 \tag{22}
$$

determines an invariant manifold and the equation (22) is "weakly invariant" under the entire group $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$. The same of course holds for $K_3 = 0$.

3.3. Differential invariants of $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$

We start from the invariants of $\mathrm{SL}_y(2)$, $I(J_3, J_4, J_5,)$ and require that this function be annihilated by the vector fields (19). We find that there are no differential invariants of order $N < 5$ and only one of order 5, namely

$$
\begin{aligned}
H_5 &= \frac{J_5}{J_3^2} - \frac{5J_4^2}{4J_3^3} = \frac{K_5}{K_3^2} - \frac{5K_4^2}{4K_3^3} \\
&= \frac{2}{(2y'''y' - 3y''^2)^3}\bigg(2y'^3(2y'y''' - 3y''^2)y^{(5)} + 20y'^3y''y'''y^{(4)} - 5y'^4(y^{(4)})^2 \\
&\quad - 16y'^3y'''^3 + 12y'^2y''^2y'''^2 - 18y'y''^4y''' + 9y''^6\bigg)
\end{aligned}
\tag{23}
$$

4. Difference invariants and their continuous limits for the groups $\mathrm{SL}_x(2)$, $\mathrm{SL}_y(2)$ and $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$

4.1. General comments. The cross-ratios

The Lie group actions that we are considering in this article are the standard projective action of $\mathrm{SL}(2)$ on a real or complex line (the action of the Möbius group). The fundamental invariants of this action are well-known (and can easily be reobtained using the prescription (9)). The lowest order invariants involve 4 points and are the cross-ratios (anharmonic ratios):

$$R_{k+3} = \frac{(y_{k+3} - y_{k+1})(y_{k+2} - y_k)}{(y_{k+3} - y_{k+2})(y_{k+1} - y_k)} \tag{24}$$

$$S_{k+3} = \frac{x_{k+3} - x_{k+1})(x_{k+2} - x_k)}{(x_{k+3} - x_{k+2})(x_{k+1} - x_k)} \tag{25}$$

for $\mathrm{SL}_y(2)$ and $\mathrm{SL}_x(2)$ respectively.

For $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$ these are the only four-points invariants and all higher order invariants can be formed by shifting the four points to the right, forming e.g. the cross-ratios $R_{k+4}, R_{k+5} \ldots$, and taking linear combinations of R_{k+3}, $R_{k+4} \ldots$, etc. The problem is to form the linear combinations that will have the chosen differential invariants of Section 3 as a continuous limit.

For $\mathrm{SL}_x(2)$ and $\mathrm{SL}_y(2)$ we have further difference invariants, namely the dependent variables y_n and the independent variables x_n for $\mathrm{SL}_x(2)$ and $\mathrm{SL}_y(2)$, respectively

The cross-ratios S_j will be used to write invariant lattices, e.g. $S_{k+3} = A$ or $S_{k+3} = AS_{k+4}$, where A is a constant.

The cross-ratios R_j will be expanded into power series, and using equations (10) and (11) we relate them to differential invariants.

Let us consider $\mathrm{SL}_y(2)$, $\mathrm{SL}_x(2)$ and $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$ separately.

4.2. The group $\mathrm{SL}_y(2)$

Taking 4 adjacent points, x_0, x_1, x_2, x_3 (we simplify the notation writing $x_{n+k} \equiv x_k$) and expanding around x_0 we obtain:

$$
\begin{aligned}
R_3 = S_3 \Bigg[1 - \frac{1}{6} h_2(h_1 + h_{k+2} + h_{k+3}) \Bigg(J_3 + \frac{1}{4}(3h_1 + 2h_{k+2} + h_3)J_4 \\
+ \frac{1}{60} \Big(3(6h_1^2 + 3h_2^2 + h_3^2 + 8h_1h_2 + 4h_1h_3 + 3h_2h_3)J_5 \\
- 10(7h_1^2 + 4h_2^2 + h_3^2 + 10h_1h_2 + 5h_1h_3 + 4h_2h_3)J_3^2 \Big) \Bigg) \Bigg] \\
+ O(h^5)
\end{aligned}
\tag{26}
$$

We can make a similar expansion for two more sets of four points, x_1, x_2, x_3, x_4 and x_2, x_3, x_4, x_5 (always expanding around x_0). We see that to lowest order in h the difference invariant that has the correct continuous limit is

$$L_{k+3}^{(3)} = \frac{6}{(x_{k+2} - x_{k+1})(x_{k+3} - x_k)} \left(1 - \frac{R_{k+3}}{S_{k+3}} \right); \tag{27}$$

indeed for all values of k (we will only need $k = 0, 1, 2$) we have

$$\lim_{h_j \to 0} L_{k+3}^{(3)} = J_3. \tag{28}$$

In view of (26) we can define a new set of difference invariants:

$$L_{k+4}^{(4)} = \frac{4}{x_{k+4} - x_k} \left(L_{k+4}^{(3)} - L_{k+3}^{(3)} \right) \tag{29}$$

which can be expanded in h as:

$$L_4^{(4)} = J_4 + \frac{1}{15(h_1 + h_2 + h_3 + h_4)} \Bigg[3 \bigg(4h_1^2 + 3h_2^2 + 2h_3^2 + h_4^2$$

$$+ 7h_1 h_2 + 6h_1 h_3 + 5h_1 h_4 + 5h_2 h_3 + 4h_2 h_4 + 3h_3 h_4 \bigg) J_5$$

$$- 2 \left(h_1^2 - 3h_2^2 + 3h_3^2 - h_4^2 - 2h_1 h_2 - h_1 h_3 + h_2 h_4 + 2h_3 h_4 \right) J_3^2 \Bigg]$$

$$+ O(h^2) \tag{30}$$

and a similar expression for $L_5^{(4)}$. In the continuous limit, we have, for all k (in particular, for the only two values we need, $k = 0, 1$):

$$\lim_{h_j \to 0} L_{k+4}^{(4)} = J_4 \tag{31}$$

Similarly, to obtain the fifth order differential invariant in the continuous limit we form another set of difference invariants, namely

$$L_5^{(5)} = \frac{5}{x_5 - x_0} \left(L_5^{(4)} - L_4^{(4)} \right) \tag{32}$$

and

$$L_5^{(5)} = J_5 + \frac{10}{3} \Bigg[\frac{1}{5} - \frac{h_4}{h_1 + h_2 + h_3 + h_4 + h_5}$$

$$- \frac{(h_1 + h_2)(h_2 + h_3)}{(h_1 + h_2 + h_3 + h_4 + h_5)(h_1 + h_2 + h_3 + h_4)}$$

$$+ \frac{(h_2 + h_3)(h_3 + h_4)}{(h_1 + h_2 + h_3 + h_4 + h_5)(h_2 + h_3 + h_4 + h_5)} \Bigg] J_3^2 + O(h). \tag{33}$$

From (26) (e.g. for $k = 0$, 1 and 2) we see that the limit of $L_5^{(5)}$ is not exactly J_5 of (14) but rather a combination of J_5 and J_3^2

$$\lim_{h_j \to 0} L_5^{(5)} = J_5 + W_0 J_3^2 \tag{34}$$

with

$$W_0 = \lim_{h_j \to 0} W \tag{35}$$

and

$$W = \frac{10}{3} \left(\frac{1}{5} - \frac{x_4 - x_3}{x_5 - x_0} - \frac{(x_3 - x_1)(x_2 - x_0)}{(x_5 - x_0)(x_4 - x_0)} + \frac{(x_4 - x_2)(x_3 - x_1)}{(x_5 - x_0)(x_5 - x_1)} \right) \tag{36}$$

The coefficient W depends on the specific form of the lattice, and W_0 is a finite number (not necessarily zero).

For instance, let us consider an $SL_x(2) \times SL_y(2)$ invariant lattice given by

$$S_j = K, \quad \forall j \tag{37}$$

where K is a constant. This equation (37) was solved in [11] for arbitrary values of K. The result is particularly simple for $K = 4$. A particular solution, not contained in the general one is

$$x_m = Am + B \tag{38}$$

and the general one is

$$x_m = \frac{1}{Am + B} + C \tag{39}$$

(A, B, C are constants). Solution (38) corresponds to a uniform lattice with $h_m = A$ for all m. For (38) and (39) we have

$$W_0 = 0, \quad W_0 = \frac{2A^2(8A^2 - 5AB - B^2)}{(A + B)(2A + B)(3A + B)(4A + B)} \tag{40}$$

respectively.

Thus, W_0 is a definite number (and can be set equal to zero in the case (39) by choosing $B = \frac{1}{2}(A(-5 \pm \sqrt{57}))$.

We see that it is not difficult to construct $\mathrm{SL}_y(2)$ difference invariants of arbitrary orders. The challenge is to find an appropriate basis for these invariants that in the continuous limit reproduces the chosen basis of difference invariant.

The connection between the difference and differential $\mathrm{SL}_y(2)$ invariants of order 3, 4 and 5 is given by equations (27), (29) and (32), respectively.

4.3. The group $\mathrm{SL}_x(2)$

The approach used for $\mathrm{SL}_y(2)$ must be modified since the differences h_k are not invariant under $\mathrm{SL}_x(2)$. Instead of (27) we expand the $\mathrm{SL}_x(2)$ invariants:

$$M_{k+3}^{(3)} = \frac{6}{(y_{k+3} - y_k)(y_{k+2} - y_{k+1})} \left(1 - \frac{R_{k+3}}{S_{k+3}}\right) \tag{41}$$

and obtain

$$M_3^{(3)} = K_3 + \frac{1}{4}(3h_1 + 2h_2 + h_3)y'K_4$$

$$+ \frac{1}{120}\bigg(6(6h_1^2 + 3h_2^2 + h_3^2 + 8h_1h_2 + 4h_1h_{k+3} + 3h_2h_3)y'^2K_5$$

$$- 4(19h_1^2 + 12h_2^2 + 4h_3^2 + 27h_1h_2 + 11h_1h_3 + 12h_2h_3)y'^2K_3^2$$

$$+ 15(3h_1^2 + 2h_2^2 + h_3^2 + 4h_1h_2 + 2h_1h_3 + 2h_2h_3)y''K_4\bigg) + O(h^3) \tag{42}$$

and similar expressions for $k = 1, 2$. Thus we have

$$\lim_{h_j \to 0} M_{k+3}^{(3)} = K_3, \quad \forall k \tag{43}$$

In analogy, we define

$$M_{k+4}^{(4)} = \frac{4}{y_{k+4} - y_k}\left(M_{k+4}^{(3)} - M_{k+3}^{(3)}\right), \quad \forall k \tag{44}$$

satisfying

$$\lim_{h_j \to 0} M_{k+4}^{(4)} = K_4, \quad \forall k \tag{45}$$

Finally, to obtain a fifth order differential invariant in the continuous limit we define

$$M_5^{(5)} = \frac{5}{y_5 - y_0} \left(M_5^{(4)} - M_4^{(4)} \right) \tag{46}$$

satisfying

$$\lim_{h_j \to 0} M_5^{(5)} = K_5 - W_{x,0} K_3^2, \quad W_{x,0} = \lim_{h_j \to 0} W_x \tag{47}$$

where

$$W_x = \frac{20}{6} \left(\frac{4}{5} - \frac{h_3}{h_1 + h_2 + h_3 + h_4 + h_5} \right) \tag{48}$$

As in the case of $\mathrm{SL}_x(2)$ we do not obtain K_5 in the limit but have an additional term involving a scalar multiple, $W_{x,0}$, of a lower order invariant K_3. The number $W_{x,0}$ can be evaluated on any lattice and is

$$W_{x,0} = 2 \tag{49}$$

on a uniform lattice.

4.4. The group $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$

For the group $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$ all invariants must be constructed out of the cross-ratios R_{k+3}, S_{k+3}. To obtain the lowest order differential invariant H_5 given by (23) we need three values of k, e.g. $k = 0, 1, 2$. As always, the difficulty is to identify the combination or combinations of them that go to H_5 in the continuous limit. One way to do this is to expand $Q_{k+3} = 1 - \frac{R_{k+3}}{S_{k+3}}$ for $k = 0, 1, 2$ into power series, and eliminate terms of order h^2 and h^3 using S_{k-3} whenever possible. Another way is to inspire oneself by the continuous limit (23) and build up an invariant with the correct limit using discretized versions of J_5, J_4, J_3 and K_5, K_4 K_3. Both methods are quite laborious, even using computer algebra, and lead to the result

$$\mathsf{H}^{(5)} = \frac{(x_5 - x_4)(x_1 - x_0)}{(x_4 - x_3)(x_2 - x_1)} \frac{1}{L_5^{(3)}} \left(\frac{L^{(5)}}{L_3^{(3)}} - \frac{5}{4} \frac{L_4^{(4)} L_5^{(4)}}{L_3^{(3)} L_4^{(3)}} \right) \tag{50}$$

or explicitly in terms of the invariants R_i and S_i

$$\mathsf{H}^{(5)} = \frac{10}{3} \frac{S_4}{(S_3(1 - S_4) + S_4)(S_4(1 - S_5) + S_5)(S_4(1 - S_3)(1 - S_5) - S_3 S_5)}$$
$$\times \left(S_4(1 - S_5)\frac{1}{Q_5} + S_4(1 - S_3)\frac{1}{Q_3} \right.$$
$$- (1 - S_4)(S_4(1 - S_3)(1 - S_5) - S_3 S_5)\frac{1}{Q_4}$$
$$\left. - S_4(1 - S_3)(1 - S_5)\frac{Q_4}{Q_3 Q_5} \right) \tag{51}$$

where

$$Q_i = 1 - \frac{R_i}{S_i}, \quad i = 3, 4, 5 \tag{52}$$

The continuous limit is given by (the quantity W was defined in (36)):

$$\lim_{h_j \to 0} \mathsf{H}^{(5)} = \left(\lim_{h_j \to 0} \frac{h_5 h_1}{h_4 h_2} \right) \left(H_5 + \lim_{h_j \to 0} W \right) \tag{53}$$

For the case of a uniform lattice ($S_j = 4$, $h_j = h_{j+1}$) we have $W = 0$ and

$$\lim_{h \to 0} \mathsf{H}^{(5)} = H_5 = \frac{J_5}{J_3^2} - \frac{5J_4^2}{4J_3^3} \tag{54}$$

with

$$\mathsf{H}^{(5)} = \frac{16R_5 + R_4(3R_4 + R_5 - 32) + R_3(R_4 - 5R_5 + 16)}{2(R_3 - 4)(R_4 - 4)(R_5 - 4)} \tag{55}$$

For other invariant lattices we have $W = \text{constant} \neq 0$.

The overall conclusion from this section is that we have constructed all difference and differential invariants up to order 5 for all groups considered. We have also shown how to proceed to higher orders, using the fundamental invariants, specifically the cross-ratios.

We now proceed to test the invariant OΔSs presented in this section and numerical schemes for specific equations. So far, only the group $\mathrm{SL}_y(2)$ has been used in this manner and only for ODEs of order 2 or 3 [3, 39].

5. Numerical examples

We present in this section some representative examples of differential equations which are invariant under the groups $\mathrm{SL}_y(2)$, $\mathrm{SL}_x(2)$ and $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$. Some particular solutions are numerically computed using the invariant discretizations studied in this work. In some cases the discretized scheme provides the exact solution. In other cases, the roundoff errors will forbid to find the solutions at some points. Increasing the working precision would allow to find more approximate solutions, but we cannot keep the same working precision beyond a certain point. These quantitative aspects of the theory have been studied elsewhere [3] and will not be discussed in detail in this work. Since our purpose is to provide some qualitative remarks on the invariant method, the standard numerical approach has been carried out using the software Mathematica, which allows a high level performance in all the examples we will discuss below, while being very simple to use. In each case, the program chooses the most appropriate method (for instance, an Adams predictor-corrector or an explicit Runge-Kutta method) and, whenever necessary, a variable step, looking for higher accuracy. In both approaches (standard and invariant) some control parameters, like working precision and accuracy, have also been chosen in order to get the best results.

5.1. Example 1: fourth order equation, invariant under $\mathrm{SL}_y(2)$

The differential equation:

$$\frac{y^{(4)}}{y'} - 4\frac{y''y'''}{y'^2} + 3\frac{y''^3}{y'^3} = \cos x \tag{56}$$

is invariant under $\mathrm{SL}_y(2)$, but not under any transformation of $\mathrm{SL}_x(2)$. Note that the equation can be written as:

$$J_3' = \cos x \tag{57}$$

and a first integral is:
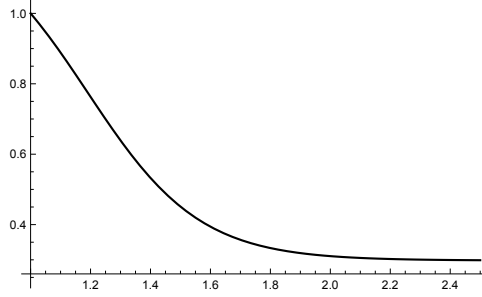
$$J_3 = \sin x + A \tag{58}$$

However, to check the invariant method, we will use the invariant scheme for fourth order differential equations (31).

We choose as initial data:

$$y(1) = 1.0, \quad y'(1) = -1.0, \quad y''(1) = -2.5, \quad y'''(1) = 5.0 \qquad (59)$$

Using a standard numerical solution in the interval $[1, 2.5]$, we obtain the graphics of Figure 1.

Figure 1. Example 1. Solution of equation (56). Standard numerical method.

We use a uniform lattice with step h (which like any lattice depending only on the variables x_i is invariant under $\mathrm{SL}_y(2)$) on a five-point stencil. The equations are

$$
\begin{aligned}
y_n &= \left(1 - \frac{y_{n-3} - y_{n-2}}{y_{n-3} - y_{n-1}} \left(\frac{(y_{n-1} - y_{n-3})(y_{n-2} - y_{n-4})}{(y_{n-1} - y_{n-2})(y_{n-3} - y_{n-4})} - 2h^3 \cos(h(n-2) + x_0) \right) \right)^{-1} \\
&\times \left(y_{n-2} - \frac{y_{n-1}(y_{n-3} - y_{n-2})}{y_{n-3} - y_{n-1}} \left(\frac{(y_{n-1} - y_{n-3})(y_{n-2} - y_{n-4})}{(y_{n-1} - y_{n-2})(y_{n-3} - y_{n-4})} \right. \right. \\
&\qquad\qquad \left. \left. - 2h^3 \cos(x_0 + h(n-2)) \right) \right) \\
x_n &= x_0 + nh
\end{aligned}
\qquad (60)
$$

with initial conditions taken from the numerical solution computed with a standard procedure. We compute the cosine function in the middle point of the stencil and consider three cases, with three different steps h.

Table 1 compares the values at several points using standard numerical methods (explicit Runge-Kutta) and the invariant approach (Inv) with different steps. The accuracy of the method improves when h goes to 0, (although the increasing number of steps will correspondingly increase the roundoff errors. We have used the same working precision in all cases).

x	Standard	Inv $h = 0.1$	Inv $h = 0.01$	Inv $h = 0.001$
1.5	0.451089	0.451095	0.451084	0.451089
2.0	0.310434	0.310466	0.310435	0.310434
2.5	0.298849	0.298885	0.298850	0.298849

Table 1. Equation (56). Values of the solution at points 1.5, 2.0 and 2.5. The columns are the values obtained with a standard numerical procedure and with the invariant one with different steps, respectively.

To test the precision of the results we can use the distance, as mean square averages, between the numerical solutions computed by the invariant scheme and the

standard method, defining a global estimator χ:

$$\chi = \sqrt{\frac{\sum_n (y_n^{\text{Inv}} - y_n)^2}{\sum_n y_n^2}} \tag{61}$$

and get the results in Table 2 at different values of the step, in the interval $[0, 2.5]$. The table shows the deviation of the invariant solution from the standard one. The precision of the method is improved when h diminishes, as it was expected. In other cases when we know the exact solution, we could compare y_n^{Inv} with the exact solution y_n^{Exact} in (61).

h	0.1	0.01	0.001
χ	$4.6729. \times 10^{-5}$	1.7866×10^{-6}	7.2291×10^{-8}

Table 2. Equation (56). Values of χ. Invariant approach with different steps versus standard numerical approach

In Figure 2, the dots are the values of the numerical solutions computed with an invariant approach, for $h = 0.001$. The solid curve is the standard numerical approach.
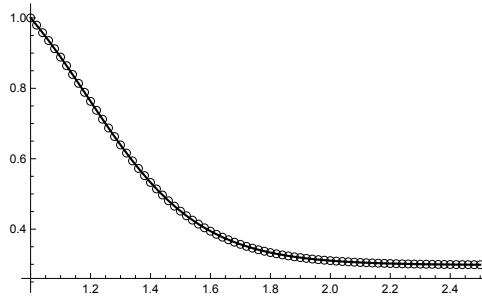
Figure 2. Equation (56). Invariant approach versus standard approach for step $h = 0.001$

5.2. Example 2: third order equation invariant under $\text{SL}_x(2)$

Let us consider the differential equation:

$$\frac{1}{y'^2}\left(\frac{y'''}{y'} - \frac{3y''^2}{2y'^2}\right) = c, \quad c \neq 0 \tag{62}$$

where c is a constant. This equation is invariant under $\text{SL}_x(2)$ (and under translations in y). The order can be easily reduced by one, and the general solution in terms of elementary functions can be written as

$$y(x) = c_3 + \sqrt{\frac{2}{c}}\,\text{arctanh}\,(c_1 x + c_2) \tag{63}$$

A 2-parameter family of particular solutions not contained in (62) is

$$y = a + \frac{1}{\sqrt{2c}}\log(x - b) \tag{64}$$

When $c = 1/2$, a particular solution is:

$$y(x) = \log |x| \tag{65}$$

which has a singularity at $x = 0$. The initial values for the solution in the region $x < 0$ can be chosen as

$$y(-1) = 0, \quad y'(-1) = -1, \quad y''(-1) = -1 \tag{66}$$

and the solution can be numerically computed by any standard method which will obviously stop at $t = 0$.

We shall now apply the method of invariant discretization to calculate solutions of the type (63) and (65) numerically. Let us start with solution (65). The invariant approach is obtained with the difference and x-lattice equations (we choose a uniform lattice, yielding $S = 4$)

$$\frac{6}{(y_{n-1} - y_{n-2})(y_n - y_{n-3})} \left(1 - \frac{(y_{n-3} - y_{n-1})(y_{n-2} - y_n)}{4(y_{n-3} - y_{n-2})(y_{n-1} - y_n)} \right) = \frac{1}{2}, \tag{67}$$

$$x_n = x_0 + nh$$

The initial conditions are taken from the exact solution:

$$x_0 = -1, \; h = 0.0001, \; y_0 = 0., \; y_1 = \log(0.9999), \; y_2 = \log(0.9998)$$

However, in contrast with Example 1, the difference equation is not linear in y_n, but quadratic. Apart from technical difficulties, one obtains two possible solutions. One of them provides the approximate solution. Note that the difference equation could be, in principle, computed beyond the stop point $x = 0$ but y_n becomes complex if x_n is greater than 0. See Figure 3 for a graphics of the approximate solution (dots) versus the exact one (solid line).
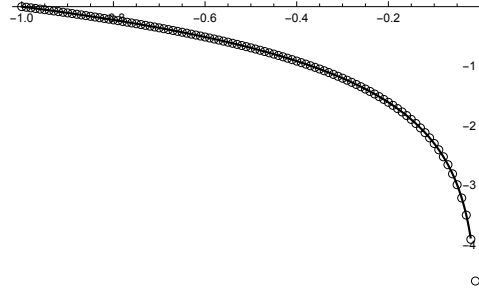
Figure 3. Differential equation (62) $x < 0$. Invariant approach, $h = 0.0001$, (dots) versus exact solution (solid line).

We can also compute the solution in the region $x > 0$. Starting at $x_0 = 1$ and using a negative step $h = -0.0001$, we get for the invariant numerical solution the graphics in Figure 4. As in the previous case, the y_n values become complex when x_n becomes negative. Then, we have reproduce in these graphics, 3 and 4, the two regions of the logarithmic solution.

Let us now choose a different solution of (62) (with $c = 2$), namely

$$y(x) = \operatorname{arctanh} x \tag{68}$$

which exists only in the interval $(-1, 1)$. The initial values for this solution are:
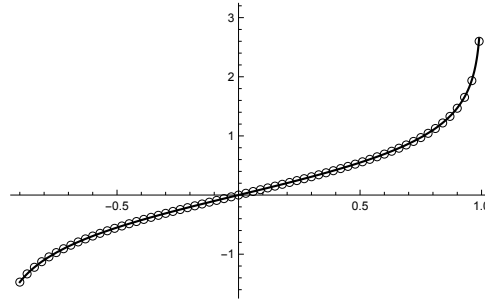
$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0 \tag{69}$$
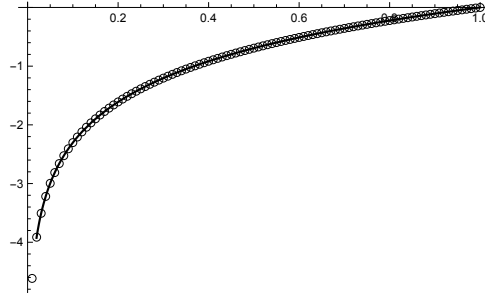
Figure 4. Differential equation (62), $x > 0$. Invariant approach, $h = -0.0001$, (dots) versus exact solution (solid line).

and can be numerically computed by any standard method which will stop at $x = 1$. The invariant approach is obtained with the same difference equation as above (67), but the initial conditions (which are again taken from the exact solution) are:

$x_0 = -0.9$, $h = 0.01$, $y_0 = \arctan(-0.9)$, $y_1 = \arctan(-0.89)$, $y_2 = \arctan(-0.88)$

As above, one of the two solutions for y_n provides the approximate solution. See Figure 5 for a plot of the invariant numerical solution (dots) versus the exact one (solid line). The χ estimator in the interval $[-0.9, 0.9]$ can be computed against the

Figure 5. Differential equation (62). Invariant approach, $h = 0.001$, (dots) versus exact solution (solid line).

exact solution, for different values of the step, see Table 3.

h	0.1	0.01	0.001
χ	0.145901	0.007028	0.001131

Table 3. Equation (62). χ for different values of the step h.

5.3. Example 3: third order equation invariant under $\mathrm{SL}_x(2)$

The differential equation:

$$\frac{1}{y'^2}\left(\frac{y'''}{y'} - \frac{3y''^2}{2y'^2}\right) = y \tag{70}$$

is invariant under $\mathrm{SL}_x(2)$. Apparently, the general solution cannot be constructed in terms of elementary functions.

With the initial values

$$y(0) = 10, \quad y'(0) = -1, \quad y''(0) = -10 \tag{71}$$

the standard numerical methods stop at $x \approx 0.14$. A singularity (or a point with infinite derivative) is expected.

The invariant approach is obtained with the difference and x-lattice equations (we choose a uniform lattice, yielding $S = 4$). The difference equation is a polynomial of third degree in y_n and the search for real solutions becomes rather involved. However, it can be done and the results appear in the tables and graphics we present. The role of the other branches, which can contain complex values, is not well understood.

If the initial conditions are taken from the approximate solution:

$$x_0 = 0., \quad h = 0.001, \quad y_0 = 10., \quad y_1 = 9.9989, \quad y_2 = 9.9979 \tag{72}$$

the graphics of the invariant approach is given in Figure 6. Although the difference equation could be solved beyond the point $x = 1.4$, y_n becomes complex (for the chosen branch).
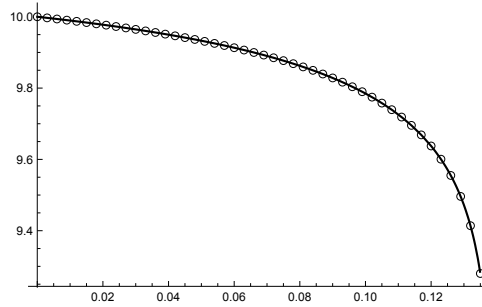
Figure 6. Invariant approach (dots) versus standard approach (solid line) for equation (70).

5.4. Example 4: invariant equation under $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$; a discrete exact solution

We will consider differential equations invariant under the direct product group $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$.

The equations are of the form

$$H_5 = c, \quad y^{(5)} = \frac{1}{y'^3 \left(2y'y''' - 3y''^2\right)} \left(\frac{5}{2} y'^4 (y^{(4)})^2 - 10 y'^3 y'' y''' y^{(4)} + 2(c+4) y'^3 y'''^3 \right.$$
$$\left. - \frac{9}{4}\left(c + \frac{2}{3}\right) \left(4 y'^2 y''^2 y'''^2 - 6 y' y''^4 y''' + 3 y''^6\right)\right) \tag{73}$$

where c is a constant. The equation can be easily solved, although the general solution can adopt several equivalent forms. We will consider the particular solution:

$$y(x) = \frac{1}{1 - e^x}, \quad c = 0 \tag{74}$$

The values corresponding to this solution can be easily obtained by a standard numerical method with initial conditions:

$$y^{(k)}(x) = \frac{\mathrm{d}}{\mathrm{d}x^k} \frac{1}{1 - \mathrm{e}^x}\Bigg|_{x=-1}, \quad k = 0, 1, 2, 3, 4 \tag{75}$$

until $x = 0$ where the function has a singularity (see Figure 7, black curve).
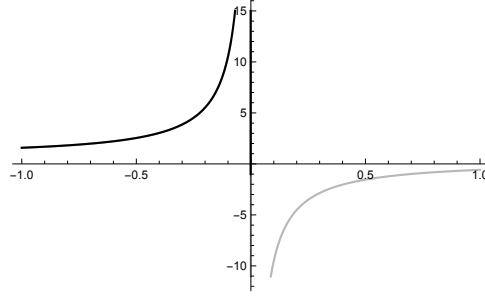
Figure 7. The solution (74) of equation (73) obtained by a standard numerical method. Grey line, the exact solution.

The invariant approach is obtained with the difference and x-lattice equations (we choose a uniform lattice, yielding $S = 4$).

$$3R_4^2 + (R_5 - 32)R_4 + 16R_5 + R_3(R_4 - 5R_5 + 16) = 0, \quad x_n = x_0 + nh \tag{76}$$

where

$$R_3 = \frac{(y_{n-3} - y_{n-5})(y_{n-2} - y_{n-4})}{(y_{n-4} - y_{n-5})(y_{n-2} - y_{n-3})}, \quad R_4 = \frac{(y_{n-2} - y_{n-4})(y_{n-1} - y_{n-3})}{(y_{n-3} - y_{n-4})(y_{n-1} - y_{n-2})},$$

$$R_5 = \frac{(y_{n-1} - y_{n-3})(y_n - y_{n-2})}{(y_{n-2} - y_{n-3})(y_n - y_{n-1})} \tag{77}$$

The difference equation is linear in y_n. The initial conditions are taken from the exact solution:

$$x_0 = 0, \quad h = 0.1, \quad y_k = \frac{1}{1 - \mathrm{e}^{x_k}}, \quad k = 0, 1, 2, 3, 4 \tag{78}$$

and the invariant solution is represented in Figure 8.
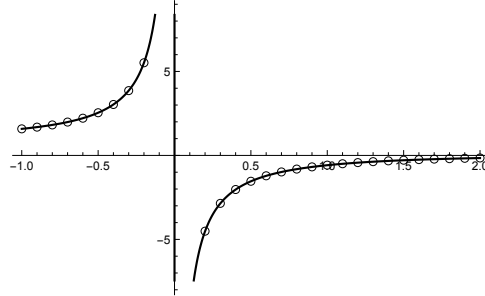
Figure 8. The invariant approach (dots) versus the exact solution (74) of equation (73)

It turns out that the solution provided by the invariant approach is an exact solution:

$$y_n = \frac{1}{1 - e^{x_n}}, \quad x_n = -1 + nh \tag{79}$$

This is easy to check. Compute R_i, for any four consecutive points, for instance R_3, when $y_n = \frac{1}{1-e^{x_n}}$. We get (for a uniform lattice, $x_n = -1 + hn$)

$$R_3 = 2 + e^h + e^{-h} \tag{80}$$

and the same expression for R_4 and R_5. Substituting in H_5 we get zero. This can be also observed from another point of view. The expression for y_n is a solution of the equation:

$$R_k = \alpha \tag{81}$$

and this provides a solution of the difference equation we are considering.

This observation allows to study the problem in the opposite direction. Since $R_k = \alpha$ is a solution of $H_5 = 0$, the equation $R_i = \alpha$ should provide solutions of the difference equation and approximation of solutions of the differential equation. These exact solutions have been computed in [11] (for the case under study, see equation (5.24) of this reference).

5.5. Example 5: invariant equation under $\mathrm{SL}_x(2) \times \mathrm{SL}_y(2)$

Our final example is a discussion of the equation

$$H_5 = 0 \tag{82}$$

and the particular solution:

$$y(x) = \tan \frac{1}{x} \tag{83}$$

The numerical values of this solution can be easily obtained by a standard numerical method, although it cannot be prolonged beyond the singularities (the first one greater than $x = 0.1$ is located at $x = 2/(5\pi)$). We take an initial condition at $x_0 = 0.1$ with the values of the function and its derivatives computed using the exact solution (see Figure 9):

$$y(x_0) = \tan 10, \quad y^{(k)}(x_0) = \left(\tan \frac{1}{x} \right)^{(k)}_{x=0.1}, \quad k = 1, \ldots, 4 \tag{84}$$

The invariant approach is obtained with the difference and x-lattice equations (we choose a uniform lattice, yielding $S = 4$):

$$3R_4^2 + (R_5 - 32)R_4 + 16R_5 + R_3(R_4 - 5R_5 + 16) = 0, \quad x_n = x_0 + nh \tag{85}$$

and is linear in y_n. The initial conditions are also taken from the exact solution:

$$x_0 = 0.1, \quad y_k = \tan \left(\frac{1}{0.1 + kh} \right), \quad k = 0, 1, 2, 3, 4 \tag{86}$$

The scheme is very sensitive to the step size and the fixed working precision. However, it is possible, using the invariant approach, to go beyond the singularities of the solution. We will just present a qualitative summary of results.

The graphics in Figures 10, 11 and 12 represent the solution and the invariant discretization for $h = 0.01$, $h = 0.005$, and $h = 0.001$, respectively.

Smaller steps provide a better approximation, although the roundoff errors prevent us (even at the cost of greater working precisions) to go beyond a certain point.
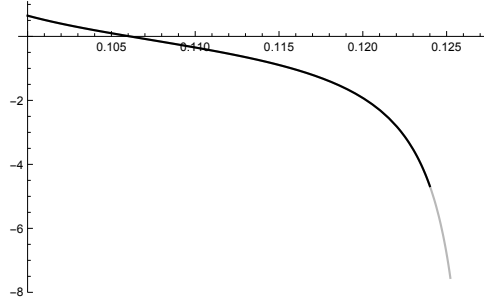
Figure 9. Standard numerical approach versus the exact solution (83) of equation (82). The grey line corresponds to the exact solution.
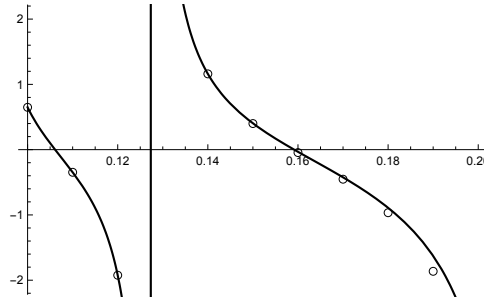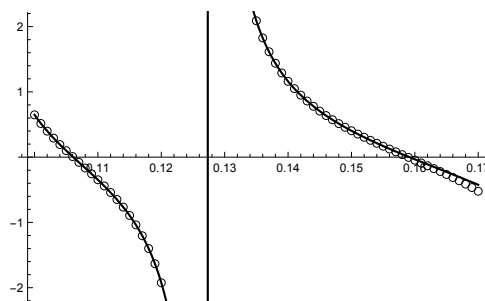

Figure 10. Invariant approximation for solution (82) of equation (83). Step $h = 0.01$
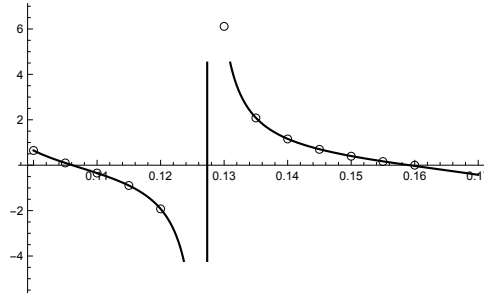
Figure 11. Invariant approximation for solution (82) of equation (83). Step $h = 0.005$

6. Conclusions

The main theoretical results of this paper are contained in Section 4. We have shown that starting from the four-point difference invariants R_i (24) and S_i (25) of $\mathrm{SL}_y(2)$ and $\mathrm{SL}_x(2)$ we can construct difference invariants of arbitrary order, for the 3 groups considered in this article. In the continuous limit they approach the corresponding differential invariants. Explicitly we go up to order $N = 5$ and this provides invariant schemes for solving invariant ODEs of order up to five numerically.

Figure 12. Invariant approximation for solution (82) of equation (83). Step $h = 0.001$

The numerical results are presented in Section 5. We consider several ODEs of order $N = 3, 4$ and 5. The main features that emerge are the following:

(i) Invariant numerical methods and standard methods provide very similar results for smooth solutions.

(ii) For solutions with singularities invariant methods provide significantly better results, specially close to singularities and beyond them.

(iii) In some cases the "invariant numerical" solutions are exact (see example 4 for a fifth order ODE. This is always true for first order ODEs [40] and was already observed for some second order ones [12, 13].

Since symmetries are an essential part of any physical problem preserving them in a discretization is important in itself. This is true independently of whether invariant discretization improves numerical results.

An open question which merits further study is that of identifying equations and initial or boundary conditions for which invariant methods provide exact solutions. Work in this direction is currently in progress. Another line of research is related to the study of the several solutions arising from nonlinear discrete schemes (implicit schemes, where the highest point y_n is not defined as a unique function of the previous points in the stencil).

Acknowledgments

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