Lie-point symmetries of the discrete Liouville equation

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Abstract

The Liouville equation is well known to be linearizable by a point transformation. It has an infinite dimensional Lie point symmetry algebra isomorphic to a direct sum of two Virasoro algebras. We show that it is not possible to discretize the equation keeping the entire symmetry algebra as point symmetries. We do however construct a difference system approximating the Liouville equation that is invariant under the maximal finite subalgebra $SL_x(2,\mathbb{R}) \otimes SL_y(2,\mathbb{R})$. The invariant scheme is an explicit one and provides a much better approximation of exact solutions than comparable standard (non invariant) schemes.

1 Introduction

The purpose of this article is to investigate the possibility of discretizing the Liouville equation

$$z_{xy} = e^z, \tag{1.1}$$

or its algebraic version

$$u\,u_{xy} - u_x\,u_y = u^3, \qquad u = e^z, \tag{1.2}$$

while preserving all of its Lie point symmetries. This is quite a challenge, since the Lie point symmetry group of these equations is infinite dimensional. We shall call (1.2) the *algebraic Liouville equation*.

The article is part of a general program on the study of continuous symmetries of discrete equations [2–6, 10–15, 17, 23–27]. This program has several aspects each possibly requiring different approaches. They are:

1. In relativistic and nonrelativistic quantum mechanics or field theory on a discrete space–time, a problem is to discretize the continuous theory while preserving continuous symmetries such as rotational, Lorentz, Galilei or conformal invariance. One possible way of doing this is the way explored in the present article, namely to not use a preconceived constant lattice. Instead one can construct an invariant set of equations defining both the

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lattice and system of difference equations. The lattice thus appears as part of a solution of a set of discrete equations and the symmetry group acts on the solutions of the equation and on the lattice.

2. The study of symmetries of genuinely discrete phenomena, such as molecular or atomic chains, where the discrete lattice is given a priori.

3. The third aspect of this program fits into the general field of geometrical integration [7,9,19, 20]. The basic idea is to improve numerical methods of solving specific ordinary and partial differential equations, by incorporating important qualitative features of these equations into their discretization. Such features may be integrability, linearizability, Lagrangian or Hamiltonian formulation, or some other features.

We concentrate on the preservation of Lie point symmetries. In our case the idea is to take an ordinary or partial differential equation (ODE or PDE) with a known Lie point symmetry algebra $\mathcal{L}$ realized by vector-fields. The differential equation is then approximated by a difference system with the same symmetry algebra. The difference system consists of a set of difference equations, describing both the approximation of the ODE (PDE) and the lattice. The difference system is constructed out of the invariants of the Lie point symmetry group $\mathcal{G}$ of the original ODE (PDE). The Lie algebra $\mathcal{L}$ of $\mathcal{G}$ is realized by the same vector fields as for the continuous equation, however its action is prolonged to all points of the lattice, rather than to derivatives.

In Section 2 we present the Lie point symmetry algebra of the continuous algebraic Liouville equation and the corresponding vector fields depending on two arbitrary functions of one variable each. The symmetry algebra is isomorphic to the direct sum of two Virasoro algebras (with no central extension). We also give the two second order invariants of the maximal finite-dimensional subgroup $SL_x(2,\mathbb{R}) \otimes SL_y(2,\mathbb{R})$ of the corresponding infinite dimensional symmetry group. Section 3 is devoted to a brief exposition of the method of discretizing differential equations while preserving their point symmetries. In Section 4 we discretize the Liouville equation on a four-point stencil. The discretization is invariant under the maximal finite dimensional subgroup, not however under the entire infinite -dimensional group. Section 5 is devoted to numerical experiments. We choose 3 different exact solutions of the continuous Liouville equation and then formulate a boundary value problem that leads to these solutions. The boundary value problem is then solved numerically, using a standard discretization and our invariant one. The results are compared to the exact solutions. In all three cases the invariant discretization is shown to perform considerably better than the standard one. An alternative symmetry preserving discretization of the Liouville equation due to Rebelo and Valiquette [23] is discussed in Section 6. They have succeeded in preserving the entire symmetry group but as generalized symmetries rather than point ones (only translations and dilations remain as point symmetries). Finally, in Section 7 we discuss a linearizable discretization due to Adler and Startsev [1] and show that it has no continuous Lie point symmetries at all. The last Section 8 is devoted to conclusions.

2 Lie point symmetries of the continuous Liouville equation

The Liouville system (1.1) is a remarkable equation that has already been thoroughly investigated. It was shown by Liouville himself [18] that it is linearized into the linear wave equation by the transformation

$$z = \ln\left[2\frac{\phi_x\,\phi_y}{\phi^2}\right], \qquad \phi_{x\,y} = 0. \tag{2.1}$$

Putting $\phi(x,y) = \phi_1(x) + \phi_2(y)$, where $\phi_i,\ i = 1,2$ are arbitrary functions, we get a very general class of solutions of (1.1) (and (1.2)), namely

$$z = \ln\left[2\frac{\phi_{1,x}\,\phi_{2,y}}{\left(\phi_1+\phi_2\right)^2}\right]. \tag{2.2}$$

In view of (2.1) the Liouville equation is linearizable and it is not surprising that its symmetry algebra is infinite dimensional, as was already known in 1898 [21]. The symmetry algebra of the algebraic Liouville equation (1.2) is given by the vector fields

$$X(f(x)) = f(x)\,\partial_x - f_x(x)\,u\,\partial_u, \qquad Y(g(y)) = g(y)\,\partial_y - g_y(y)\,u\,\partial_u, \tag{2.3}$$

where $f = f(x)$ and $g = g(y)$ are arbitrary smooth functions. The nonzero commutation relations of the vector fields (2.3) are

$$\left[X(f), X(\tilde f)\right] = X\left(f\tilde f_x - \tilde f f_x\right), \quad \left[Y(g), Y(\tilde g)\right] = Y\left(g\,\tilde g_y - \tilde g\,g_y\right), \quad \left[X(f), Y(g)\right] = 0. \tag{2.4}$$

The algebra (2.3)-(2.4) is isomorphic to the direct sum of two Virasoro algebras. We denote it $L = vir_x \oplus vir_y$. Its maximal finite dimensional subalgebra is $sl_x(2,\mathbb{R}) \bigoplus sl_y(2,\mathbb{R})$, obtained by restricting $f(x)$ and $g(y)$ to be second order polynomials. Limiting ourselves to a neighborhood of the origin, the above vector fields can be expanded in the basis $\{X(x^n)\}_{n\in\mathbb{N}}$ and $\{Y(y^n)\}_{n\in\mathbb{N}}$, which leads to the commutation relations

$$\begin{aligned}\left[X(x^m), X(x^n)\right] &= (n-m)\,X\left(x^{m+n-1}\right), \quad \left[Y(y^m), Y(y^n)\right] = (n-m)\,Y\left(y^{m+n-1}\right),\\ \left[X(x^m), Y(y^n)\right] &= 0.\end{aligned} \tag{2.5}$$

As said above, the maximal finite subalgebra corresponds to the basis elements with $m,n = 0,1,2$.

Let us find the most general second order expression of the form $I(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy})$ invariant under the group corresponding to the algebra (2.3). The second order prolongation of $X(f)$ is

$$\begin{aligned}\mathrm{pr}^{(2)}X(f) &= f\partial_x - f'\left[u\,\partial_u + 2u_x\,\partial_{u_x} + u_y\,\partial_{u_y} + 2u_{xy}\,\partial_{u_{xy}} + 3u_{xx}\,\partial_{u_{xx}} + u_{yy}\,\partial_{u_{yy}}\right]\\ &\quad - f''\left[u\,\partial_{u_x} + u_y\partial_{u_{xy}} + 3u_x\partial_{u_{xx}}\right] - f'''u\partial_{u_{xx}}\end{aligned} \tag{2.6}$$

and similarly for $Y(g)$. We see that the last term in (2.6) is absent in the subalgebra.

The group $SL_x(2,\mathbb{R}) \otimes SL_y(2,\mathbb{R})$ allows two functionally independent "strong" invariants, namely

$$I_1 = \frac{uu_{xy} - u_x\,u_y}{u^3}, \qquad I_2 = \frac{\left(2uu_{xx} - 3u_x^2\right)\,\left(2uu_{yy} - 3u_y^2\right)}{u^6}. \tag{2.7}$$

We have

$$\mathrm{pr}^{(2)}X(f)\ I_1 = \mathrm{pr}^{(2)}Y(g)\ I_1 = 0 \tag{2.8}$$

for arbitrary f and g, but

$$\mathrm{pr}^{(2)}X(f)\ I_2 = \frac{2f_{xxx}\left(3u_y^2 - 2uu_{yy}\right)}{u^4}, \quad \mathrm{pr}^{(2)}Y(g)\ I_2 = \frac{2g_{yyy}\left(3u_x^2 - 2uu_{xx}\right)}{u^4}. \tag{2.9}$$

Thus, I_1 is invariant under the direct product the two Virasoro groups $VIR(x) \otimes VIR(y)$. The PDE $I_1 = A$, for any real constant A, is invariant under this group. For $A \neq 0$ we scale to $A = 1$ and obtain the equation (1.2). For $A = 0$ we obtain an equation equivalent to the linear wave equation $z_{xy} = 0$, namely

$$uu_{xy} - u_xu_y = 0. \tag{2.10}$$

On the other hand I_2 is invariant only for $f_{xxx} = g_{yyy} = 0$, i.e. it is only invariant under $SL_x(2,\mathbb{R}) \otimes SL_y(2,\mathbb{R})$. Even the equation $I_2 = 0$ is only invariant on the manifold satisfying the system

$$2uu_{xx} - 3u_x^2 = 0, \qquad 2uu_{yy} - 3u_y^2 = 0, \tag{2.11}$$

i.e. on a very restricted class of solutions, namely

$$u = (a\,x\,y + b\,x + c\,y + d)^{-2}\,, \tag{2.12}$$

for arbitrary constants $a, \ldots, d$.

3 Symmetry preserving discretization of partial difference equations.

The basic idea of the invariant discretization of a PDE is to replace it by a system of difference equations, formed out of invariants of the action of the symmetry group of the PDE. This difference system (ΔS) describes both the original PDE and a lattice [4,5,15,26,27].

To be specific, let us restrict to the case of one scalar PDE involving two independent variables (x, y) and one dependent one $u(x, y)$. The PDE is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \cdots) = 0 \tag{3.1}$$

and its Lie point symmetry group $\mathcal{G}$ is assumed to be known, together with its symmetry algebra $\mathcal{L}$. The ΔS describing (3.1) will have the form

$$E_\alpha(x_{m+i,n+j}, y_{m+i,n+j}, u_{m+i,n+j}) = 0, \tag{3.2}$$
$$\alpha = 1, \ldots, N, \quad i_{min} \le i \le i_{max}, \quad j_{min} \le j \le j_{max}.$$

On Fig.1 we depict a general lattice, a priori extending indefinitely in all directions. An orthogonal lattice (not necessarily uniform) is obtained by setting $\epsilon_{ik} = 0$, $\delta_{ik} = 0$. The difference system (3.2) is written on a *stencil*: a finite number N of adjacent points, sufficient to reproduce, in the continuous limit, all derivatives figuring in the differential equation (3.1). For instance, for a first order PDE the minimal number of points on a stencil is three: (m, n) $(m + 1, n)$ $(m, n + 1)$. Since the system (3.2) is *autonomous*, i.e. the labels (m, n) do not figure in the ΔS (3.2) explicitly, we can shift the stencil around on the lattice arbitrarily. For convenience we will choose the reference point to be $(m, n) = (0, 0)$ and build the stencil around it. Thus, in (3.2) we start with $m = n = 0$ and then shift as needed.

For a first order initial value problem

$$F(x, y, u, u_x, u_y) = 0, \quad u(x, 0) = \phi(x) \tag{3.3}$$

it would be sufficient to choose $N = 3$ in (3.2) and give as initial data $x_{m,0}$, $y_{m,0}$, $u_{m,0}$ for all m.

On the first stencil we know $x_{0,0}$, $x_{1,0}$, $y_{0,0}$, $y_{1,0}$, $u_{0,0}$, $u_{1,0}$ and calculate $x_{0,1}$, $y_{0,1}$, $u_{0,1}$ from (3.2). Then we shift the stencil one step in any direction and calculate further values till we fill the entire lattice.

To facilitate the calculations of the continuous limit we perform a transformation of variables on the stencil, introducing differences between coordinates and discrete partial derivatives [11, 12,15,16]. The new coordinates are $\{x_{0,0},\, y_{0,0},\, u_{0,0},\, h_{1,0},\, \epsilon_{0,1},\, k_{0,1},\, \delta_{1,0},\, u_x^d,\, u_y^d\}$, with

$$\begin{aligned}
h_{1,0} &= x_{1,0} - x_{0,0},\ k_{0,1} = y_{0,1} - y_{0,0},\ \delta_{1,0} = y_{1,0} - y_{0,0},\ \epsilon_{0,1} = x_{0,1} - x_{0,0}, && (3.4)\\
u_x^d &= \frac{1}{\mathcal{D}}[(y_{1,0} - y_{0,0}(u_{0,1} - u_{0,0}) - (y_{0,1} - y_{0,0})(u_{1,0} - u_{0,0})], && (3.5)\\
u_y^d &= \frac{1}{\mathcal{D}}[(x_{0,1} - x_{0,0}(u_{1,0} - u_{0,0}) - (x_{1,0} - x_{0,0})(u_{0,1} - u_{0,0})],\\
\mathcal{D} &= \epsilon_{0,1}\delta_{1,0} - h_{1,0}k_{0,1} \ne 0.
\end{aligned}$$

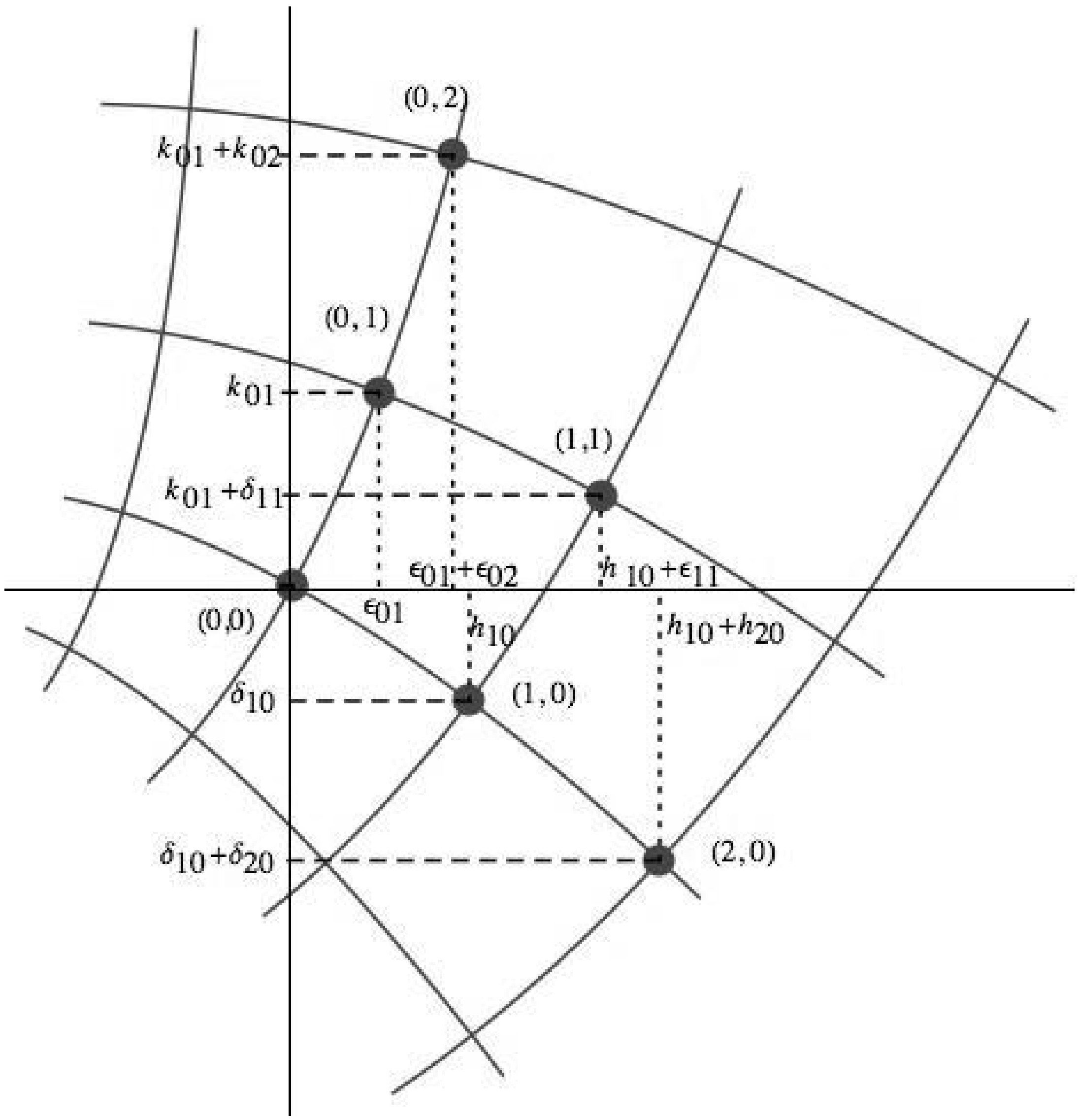

Figure 1: Points on a general lattice, e.g. $x_{0,0} = x$, $x_{1,0} = x + h_{1,0}$, $x_{0,1} = x + \epsilon_{0,1}$, $x_{1,1} = x + h_{1,0} + \epsilon_{1,1}$, $x_{2,0} = x + h_{1,0} + h_{2,0}$, $x_{0,2} = x + \epsilon_{0,1} + \epsilon_{0,2}$, $y_{0,0} = y$, $y_{0,1} = y + k_{0,1}$, $y_{1,0} = y + \delta_{1,0}$, $y_{1,1} = y + k_{0,1} + \delta_{1,1}$, $y_{0,2} = y + k_{0,1} + k_{0,2}$, $y_{2,0} = y + \delta_{1,0} + \delta_{2,0}$.

To describe an arbitrary second order PDE we need a stencil consisting of at least six points. A possible choice is to take points $\{(0,0),(1,0),(0,1),(1,1),(2,0),(0,2)\}$. For PDEs of the type

$$u_{xy} = F(x, y, u, u_x, u_y), \tag{3.6}$$

i.e. not involving u_{xx}, u_{yy}, it might be sufficient to take four points: $\{(0,0),(1,0),(0,1),(1,1)\}$.

An element of the symmetry algebra $\mathcal{L}$ of the PDE (3.1) will have the form

$$\hat{Z} = \xi(x,y,u)\partial_x + \eta(x,y,u)\partial_y + \phi(x,y,u)\partial_u \tag{3.7}$$

where the smooth functions ξ, η and ϕ are known (obtained by a standard algorithm for PDEs [22]).

In order to obtain an *invariant* ΔS (3.2) we must construct it out of difference invariants of the group $\mathcal{G}$, the Lie point symmetry group of the PDE (3.1). To calculate these invariants we consider the action of the vector field $\hat{Z}$ at some reference point $\{x_{0,0},\, y_{0,0},\, u_{0,0}\}$ and prolong it to all points figuring on a chosen stencil. This amounts to a prolongation to the discrete jet space:

$$\mathrm{pr}\hat{Z} = \sum_{i,j}(\xi_{i,j}\partial_{x_{i,j}} + \eta_{i,j}\partial_{y_{i,j}} + \phi_{i,j}\partial_{u_{i,j}}). \tag{3.8}$$

As in the continuous case, we can use both *strong* and *weak* invariants. The strong and weak invariants satisfy

$$\mathrm{pr}\hat{Z}I_s = 0, \tag{3.9}$$

$$\mathrm{pr}\hat{Z}I_w\,|_{I_w=0} = 0, \tag{3.10}$$

respectively. To determine both types of invariants we choose a basis $\left\{\hat{Z}_1,\ \cdots,\ \hat{Z}_A\right\}$ $(A = \mathrm{dim}\mathcal{L})$ for the Lie algebra $\mathcal{L}$ and solve the set of equations

$$\mathrm{pr}\hat{Z}_a I(x_{i,j}, y_{i,j}, u_{i,j}) = 0, \quad a = 1, \cdots, A. \tag{3.11}$$

For strong invariants the rank r of the matrix of coefficients in (3.11) is maximal and the same for all points $(m+j, n+k)$. Invariants exist if we have $r = A < N$. Weak invariants are only invariant on some manifold in the space of points, obtained by requiring that the rank of coefficients in (3.11) be less than maximal. Thus, there may be more weak invariants than strong ones (strong invariants satisfy both (3.9) and (3.10)). The number of strong invariants is n = N-A.

4 Invariant discretization of the algebraic Liouville equation on a four-point stencil

We choose the four points $\mathfrak{s}_4^0 \equiv \{(0,0)\,,(0,1)\,,(1,0)\,,(1,1)\}$ on Fig.1 and can translate them to any stencil $\mathfrak{s}_4^{m,n} = \{(m,n)\,(m+1,n)\,(m,n+1)\,(m+1,n+1)\}$ on the (x,y) plane. The vector fields (2.3) of the symmetry algebra $\mathcal{L}$ can be discretized and prolonged to all points of the stencil:

$$\begin{aligned} X^D(f) = \mathrm{pr}\,X(f) &= \sum_{(m,n)\in\mathfrak{s}_4^{m,n}} \left[f(x_{m,n})\,\partial_{x_{mn}} - f'(x_{m,n})\,u_{mn}\,\partial_{u_{mn}}\right], \\ Y^D(g) = \mathrm{pr}\,Y(g) &= \sum_{(m,n)\in\mathfrak{s}_4^{m,n}} \left[g(y_{m,n})\,\partial_{y_{mn}} - \dot{g}(y_{m,n})\,u_{mn}\,\partial_{u_{mn}}\right]. \end{aligned} \tag{4.1}$$

The prime and the dot denote (continuous) derivatives with respect to x and y, respectively.

Let us first restrict to the maximal finite-dimensional subalgebra $sl_x(2,\mathbb{R}) \bigoplus sl_y(2,\mathbb{R})$. The corresponding group acts transitively on the space of the continuous variables $(x,y,u) \in \mathbb{R}^3$, and sweeps out an orbit of codimension 6 on the 12-dimensional direct product $\mathbb{R}^3 \bigotimes \mathfrak{f}_4$. Hence we obtain 6 functionally independent invariants. A simple basis for these invariants is given by

$$\begin{aligned} \xi_1 &= \tfrac{(x_{0,1}-x_{0,0})(x_{1,1}-x_{1,0})}{(x_{0,0}-x_{1,0})(x_{0,1}-x_{1,1})} = \tfrac{\epsilon_{0,1}\epsilon_{1,1}}{h_{1,0}(h_{1,0}+\epsilon_{1,1}-\epsilon_{0,1})}, \\ \eta_1 &= \tfrac{(y_{0,0}-y_{1,0})(y_{0,1}-y_{1,1})}{(y_{0,1}-y_{0,0})(y_{1,1}-y_{1,0})} = \tfrac{\delta_{1,0}\delta_{1,1}}{k_{0,1}(k_{0,1}+\delta_{1,1}-\delta_{1,0})} \end{aligned} \tag{4.2}$$

$$\begin{aligned} H_1 &= u_{0,0}u_{0,1}\epsilon_{0,1}^2 k_{0,1}^2 \\ H_2 &= u_{1,0}u_{1,1}\epsilon_{1,1}^2(k_{0,1}+\delta_{1,1}-\delta_{1,0})^2 \\ H_3 &= \tfrac{u_{1,0}(h_{1,0}-\epsilon_{0,1})^2(k_{0,1}-\delta_{1,0})^2}{u_{0,0}\,\epsilon_{0,1}^2\,k_{0,1}^2} \\ H_4 &= \tfrac{u_{1,1}\epsilon_{1,1}^2(k_{0,1}+\delta_{1,1}-\delta_{1,0})^2}{u_{0,0}\,h_{1,0}^2\,\delta_{1,0}^2} \end{aligned} \tag{4.3}$$

The quantities $h_{1,0}$, $k_{0,1}$, $\epsilon_{0,1}$, $\epsilon_{1,1}$, $\delta_{1,0}$ and $\delta_{1,1}$ are defined on Fig. 1. The invariants ξ_1 and η_1 can be conveniently used to define an invariant lattice, e.g. by putting $\xi_1 = A$, $\eta_1 = B$, where A and B are constants. We choose the simplest possibility, namely

$$\xi_1 = 0, \qquad \eta_1 = 0. \tag{4.4}$$

This implies that e.g. $x_{0,1} - x_{0,0} = \epsilon_{0,1} = 0$ and also as a consequence $x_{1,1} - x_{1,0} = \epsilon_{1,1} = 0$. Similarly $\delta_{1,0} = \delta_{1,1} = 0$. Thus we have

$$x_{m,n} = x_m, \qquad y_{m,n} = y_n, \tag{4.5}$$

i.e. $x_{m,n}$ depends only on the first index, $y_{m,n}$ only on the second one. We thus obtain an orthogonal lattice (in an invariant manner). The quantities ξ_1 and η_1 are only invariant under $SL_x(2) \otimes SL_y(2)$, however we have

$$\begin{aligned} &\hat{X}^D(x^3)\xi_1 = (x_{1,1}-x_{0,0})(x_{1,0}-x_{0,1})\xi_1\,|_{\xi_1=0} = 0 \\ &\hat{X}^D(x^3)\eta_1 = 0. \end{aligned} \tag{4.6}$$

It follows from the commutation relations (2.4) that a quantity annihilated by $\hat{X}^D(x^3)$ is also annihilated by $\hat{X}^D(x^n)$ for any n. Thus the lattice condition (4.4) is invariant under $VIR(x) \otimes VIR(y)$. On the other hand the equations $\xi_1 = A$, $\eta_1 = B$, where A and B are nonzero constants are not Virasoro invariant. We conclude that an orthogonal lattice is obligatory if we define it in terms of ξ_1 and η_1 alone. Conditions (4.4) and (4.5) are compatible with choosing a uniform orthogonal lattice

$$x_m = am + x_0, \qquad y_n = bn + y_0, \tag{4.7}$$

where $a > 0$, $b > 0$, x_0, y_0 are constants, but this choice is not obligatory.

The invariants $H_1, \cdots, H_4$ of (4.3) are not suitable on the lattice (4.4) since they all vanish or become infinite on the lattice. Before specifying the lattice we must choose new invariants (functions of those in (4.2) and (4.3)) which remain finite and nonzero for $\epsilon_{i,j} = \delta_{i,j} = 0$. Only two such $SL_x(2) \otimes SL_y(2)$ invariants exist, namely:

$$J_1 = H_1 H_3 = u_{0,1}u_{1,0}h_{1,0}^2 k_{0,1}^2, \tag{4.8}$$

$$J_2 = \frac{1}{\xi_1^2}\frac{H_2}{H_3} = u_{0,0}u_{1,1}h_{1,0}^2 k_{0,1}^2. \tag{4.9}$$

Neither of them is strongly invariant under the Virasoro group, since we have

$$\hat{X}^D(x^3)J_1 = -h_{1,0}^2 J_1, \qquad \hat{X}^D(x^3)J_2 = -h_{1,0}^2 J_2. \tag{4.10}$$

The equation $J_2 - J_1 = 0$ is Virasoro invariant (on its solution set) and this equation is a discretization of $uu_{xy} - u_x u_y = 0$ (equivalent to the wave equation $z_{xy} = 0$).

Putting $u_{0,0} = u(x,y)$, $u_{1,0} = u(x+h_{1,0}, y)$, $u_{0,1} = u(x, y+k_{0,1})$ and $u_{1,1} = u(x+h_{1,0}, y+k_{0,1})$, expanding in a Taylor series and keeping only the lowest order terms, we find

$$J_2 - J_1 = h_{1,0}^3 k_{0,1}^3 (uu_{xy} - u_x u_y). \tag{4.11}$$

The Liouville equation is approximated by the difference scheme

$$\begin{gathered} J_2 - J_1 = a|\,J_1\,|^{3/2} + bJ_1|\,J_2\,|^{1/2} + c|\,J_1\,|^{1/2} J_2 + d|\,J_2\,|^{3/2}, \\ \xi_1 = 0, \quad \eta_1 = 0, \qquad a+b+c+d = 1. \end{gathered} \tag{4.12}$$

Indeed the Taylor expansion yields

$$\begin{aligned} & J_2 - J_1 - \left[aJ_1^{3/2} + bJ_1 I_2^{1/2} + cJ_1^{1/2} J_2 + dJ_2^{3/2}\right] = \\ & = h_{1,0}^3 k_{0,1}^3 \left[uu_{xy} - u_x u_y - u^3\right] + h_{1,0}^4 k_{0,1}^3 \left[\frac{1}{2} u_y u_{xx}(u-1) - \frac{3}{2} u^2 u_x\right] + \\ & + h_{1,0}^3 k_{0,1}^4 \left[\frac{1}{2} u_x u_{yy}(u-1) - \frac{3}{2} u^2 u_y\right] + \mathcal{O}(h_{1,0}^4 k_{0,1}^4), \end{aligned} \tag{4.13}$$

where the constant $a,\ b,\ c,\ d$ appear in the $O(h_{1,0}^4 k_{0,1}^4)$ terms. The ΔS (4.12) is $SL_x(2) \otimes SL_y(2)$ invariant, not however Virasoro invariant. The scheme is suited for solving a boundary value problem. Give (x,y) in the points $(m,0)$, $(0,n)$ then start from $(0,0)$, $(1,0)$, $(0,1)$ and calculate $(x_{1,1}, y_{1,1}, u_{1,1})$. Then move the stencil up or to the right and cover the entire first quadrant in the computational space (m,n).

5 Numerical results and analysis

In order to to test the efficiency of the numerical algorithms based on the invariant difference scheme (4.12), we will solve a set of boundary value problems for the Liouville equation on a uniform lattice $h_{m,n} = h,\ k_{m,n} = k$. Then, we will compare the results with the analytic solutions and with the corresponding ones obtained by the standard finite difference approximation

$$u_{1,1} u_{0,0} - u_{0,1} u_{1,0} = hk\ u_{0,0}^3. \tag{5.1}$$

Both the equations (4.12) and (5.1) relate the values at the corner of a rectangle of meshes of length h and k, respectively. Then a natural class of boundary value problems consists in giving the value of u on two sets of points of the form $(m,0)$ and $(0,n)$ for $m,n \in \mathbb{N}$ in the computational basis. Thus, one can proceed in calculating the fourth value of u from three given values on each rectangle, as depicted on Fig.2, starting from the left bottom one at the corner. The problem with the formula (4.12) is that it involves algebraic functions. However, a possibility is to make a special choice for the parameters, namely set $b = d = 0$, which leads to a linear equation for u_{11} and hence to an explicit scheme. More precisely, we have a 1-parameter family of recursion formulae

$$u_{1,1} = \frac{u_{0,1} u_{1,0} \left(ahk\sqrt{u_{0,1} u_{1,0}} + 1\right)}{u_{0,0} \left((a-1)hk\sqrt{u_{0,1} u_{1,0}} + 1\right)} \qquad (a \neq 0, 1)\,, \tag{5.2}$$

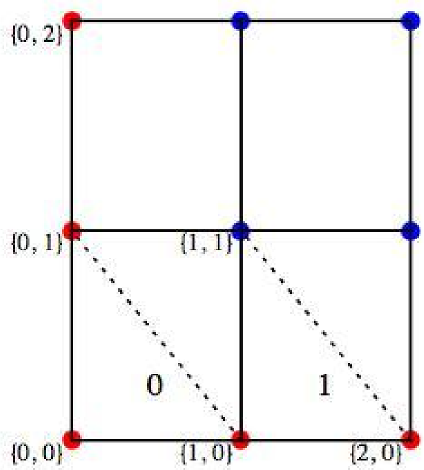

Figure 2: In the 4 point scheme, adopted both in the standard discretization of the Liouville equation (5.1) and in the invariant discretization (4.12), the value of u at the right top point in each rectangle is obtained using the values in the three other vertices. In the considered boundary value problem, the values of u are given in the points (m,0) and (0,n). Then, starting from the rectangle at the left bottom corner, denoted by 0, one gets the value u_{11} from the data connected by the dotted diagonal. This can be used to evaluate the right top point of the rectangle denoted by 1 together with the data in $(1, 0)$ and $(2, 0)$. Proceed further, till the first row of rectangles is completed, then repeat the same procedure for the second row, involving also the data in $(0, 2)$. In the figure are indicated the pair of points involved in the computation of the invariants in each rectangle.

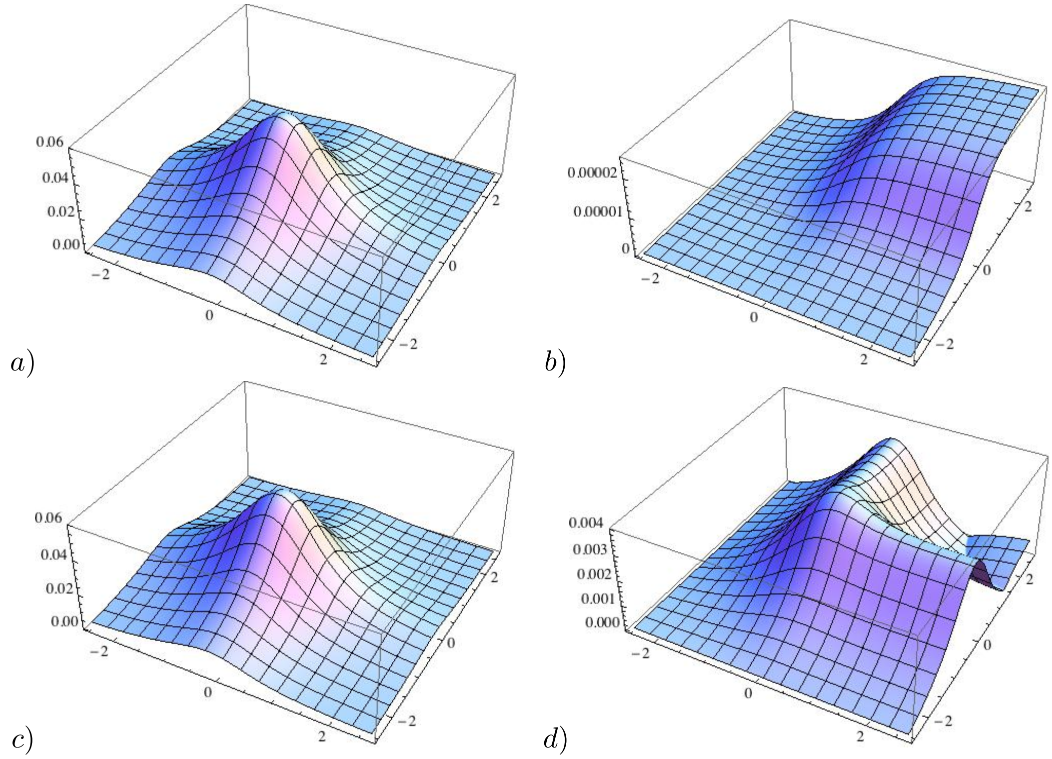

Figure 3: The solution s_1 with the choice of parameters $\alpha = 6$, $\beta = 1$, $\gamma = 1$, $\delta = 1$ is numerically computed giving a boundary value problem on a lattice with corner point $(x_{00}, y_{00}) = (-2.5, -2.5)$ and steps of equal length $h = k = 0.02$ for a lattice of 260×260 points. Numerical results using the invariant formula (5.2) are shown in a), and the relative error with respect to the analytic solution in b). Analogously, numerical results obtained by the standard formula (5.1) are reported in c) and the corresponding relative error in d). Despite the generic similarities of the two results, the difference of two orders of magnitude in the relative errors is remarkable.

for arbitrary real a (with $c = 1 - a$). Furthermore, to simplify calculations we require that the unknown function is strictly positive. In the actual calculations we used the symmetric case $a = c = \frac{1}{2}$.

We used different exact solutions of the Liouville equations, among them for instance

$$s_1 = \frac{2\beta\gamma\delta}{(\beta^2x^2+1)(\delta^2y^2+1)\left(\tan^{-1}(\beta x)+\gamma\tan^{-1}(dy)+\alpha\right)^2}, \tag{5.3}$$

$$s_2 = \frac{2As^2e^{s(x+y)}}{(Ae^{sy}+e^{sx})^2}, \tag{5.4}$$

$$s_3 = \frac{8\left(1-4\left(x+\frac{1}{2}\right)\right)(1-4y)\exp\left(-4\left(x+\frac{1}{2}\right)^2+2\left(x+\frac{1}{2}\right)-4y^2+2y\right)}{\left(e^{2\left(x+\frac{1}{2}\right)-4\left(x+\frac{1}{2}\right)^2}+e^{2y-4y^2}+1\right)^2}, \tag{5.5}$$

for certain values of the constants $A, s, \alpha, \beta, \gamma, \delta$. Once the values for the lattice constants h and k and the corner point $(0,0)$ are fixed the values of the analytic solution on the points of the boundary are computed and used as initial data for the numerical calculations. For some of the functions defined above, both the invariant formula (5.2) and the standard formula (5.1) are used to compute the solutions and compare them with the known analytically computed values at the lattice points. As an illustrative example, in Figure 3, we report the calculations made for the solution s_1.

Supplementary material of the same kind is provided in Fig.4 and Fig. 5 for s_2 and s_3. In all cases the agreement with the exact formulas is much better for the invariant schemes than for the standard ones.

In order to provide a rough evaluation of how correctly the numerical calculations reproduce the analytical solutions below we give a table, where the distances, as mean square averages (or the normalized $L^2_{\mathbb{R}^2}$ metric), between the numerical solutions computed by the invariant scheme and the standard method, respectively, w.r.t. the analytic ones are compared:

	χ_{Inv}	χ_{stand}
s_1	6.4×10^{-16}	7.2×10^{-5}
s_2	1.6×10^{-7}	7.0×10^{-1}
s_3	1.7×10^{-2}	6.0×10^{-1}

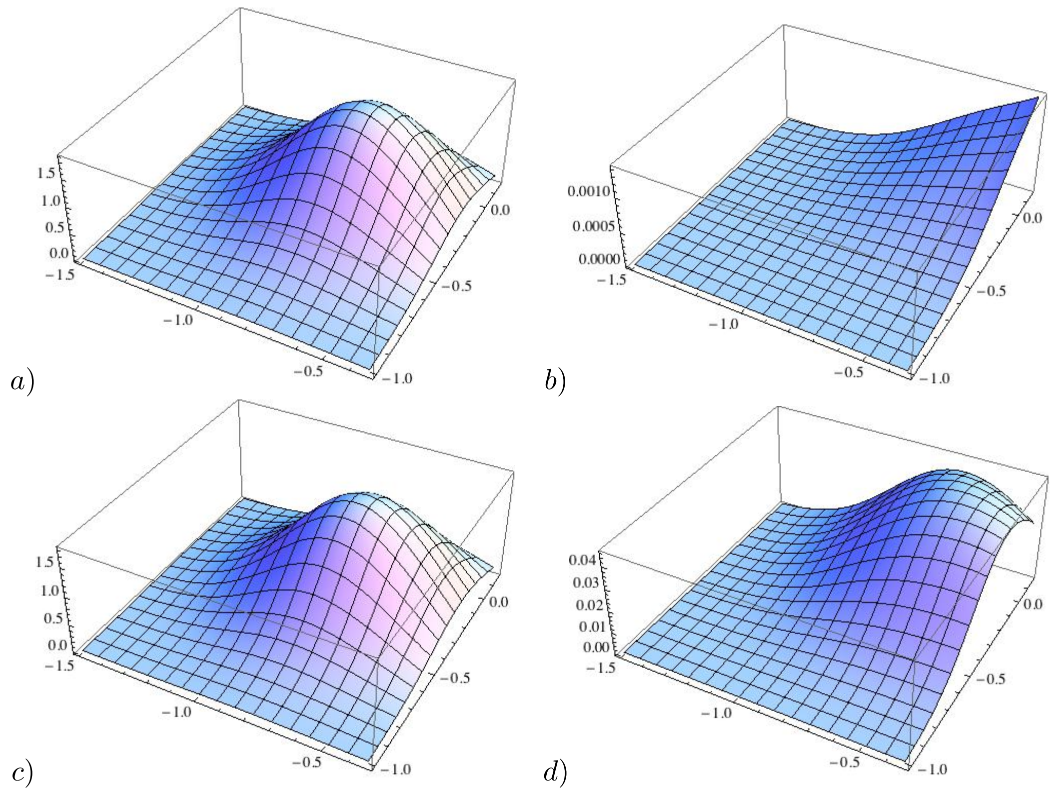

Figure 4: The same analysis as above for the solution s_2 on a lattice with corner point $(x_{00}, y_{00}) = (-1.5, -1.0)$ and steps of equal length $h = k = 0.02$ for a lattice of 60×60 points.

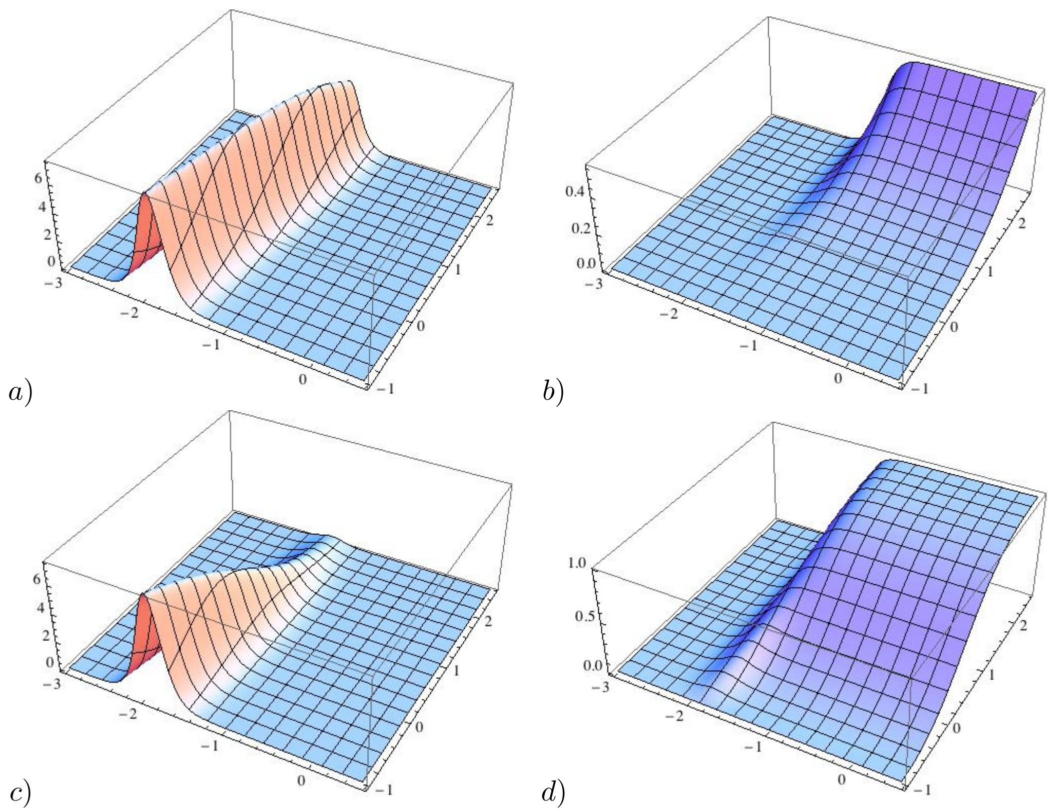

Figure 5: The same analysis as above for the solution s_3 for the choice of parameters $A = 12.8397, s = 3.86233$ on a lattice with corner point $(x_{00}, y_{00}) = (-3, -1)$ and steps of equal length $h = k = 0.02$ for a lattice of 180×180 points.

6 Symmetries of Rebelo-Valiquette Liouville discretized equation

In [24] Rebelo and Valiquette considered a symmetry preserving discretization of the Liouville equation (1.2) namely:

$$L^D_{RV} = u_{11}u_{00} - u_{10}v_{01} - u_{00}u_{01}u_{10}(x_{10}-x_{00})(y_{01}-y_{00}) = 0, \tag{6.1}$$
$$x_{01} = x_{00}, \quad y_{10} = y_{00}.$$

The equation for the lattice clearly states that $x_{ij} = x_i$ and $y_{ij} = y_j$, so the lattice coincides with the one we used above. They constructed (6.1) from the invariance with respect to the pseudo–group

$$\tilde{x}_i = F(x_i), \quad \tilde{y}_j = G(y_j), \quad \tilde{u}_{ij} = \frac{u_{ij}}{\frac{F(x_{i+1})-F(x_i)}{x_{i+1}-x_i}\,\frac{G(y_{j+1})-G(y_j)}{y_{j+1}-y_j}} \tag{6.2}$$

for arbitrary regular F and G.

First, let us notice that the equation (6.1) is not invariant with respect the algebra $sl_x(2,\mathbb{R}) \oplus sl_y(2,\mathbb{R})$ considered in the previous sections. In fact it results that

$$X^D\left(x^2\right) L^D_{RV}|_{L^D_{RV}=0} = u_{00}u_{01}u_{10}(x_{10}-x_{00})^2(y_{01}-y_{00}) \tag{6.3}$$

and similarly for $Y^D\left(y^2\right)$.

Thus, let us look here for infinitesimal symmetries of (6.1) of the form

$$\hat{X} = Q^{(1)}_{ij}(x_{ij}, y_{ij}, u_{ij})\partial_{x_{ij}} + Q^{(2)}_{ij}(x_{ij}, y_{ij}, u_{ij})\partial_{y_{ij}} + Q^{(3)}_{ij}(x_{ij}, x_{i+1,j}, y_{ij}, y_{i,j+1}, u_{ij})\partial_{u_{ij}}. \tag{6.4}$$

The determining equations are:

$$Q^{(1)}_{01} = Q^{(1)}_{00}, \tag{6.5}$$
$$Q^{(2)}_{10} = Q^{(2)}_{00}, \tag{6.6}$$
$$\begin{aligned} Q^{(3)}_{11}u_{00} + u_{11}Q^{(3)}_{00} - u_{10}Q^{(3)}_{01} - u_{01}Q^{(3)}_{10} &= Q^{(3)}_{00}u_{01}u_{10}(x_{10}-x_{00})(y_{01}-y_{00}) + \\ &+ Q^{(3)}_{10}u_{01}u_{00}(x_{10}-x_{00})(y_{01}-y_{00}) + Q^{(3)}_{01}u_{00}u_{10}(x_{10}-x_{00})(y_{01}-y_{00}) + \\ &+ u_{00}u_{01}u_{10}(Q^{(1)}_{10} - Q^{(1)}_{00})(y_{01}-y_{00}) + u_{00}u_{01}u_{10}(x_{10}-x_{00})(Q^{(2)}_{01} - Q^{(2)}_{00}). \end{aligned} \tag{6.7}$$

We put $x_{01} = x_{00}$, $y_{10} = y_{00}$ and $u_{11} = u_{01}u_{10}\left[\frac{1}{u_{00}} + (x_{10}-x_{00})(y_{01}-y_{00})\right]$ so that x_{00}, y_{00}, y_{01}, x_{10}, u_{00}, u_{01} and u_{10} are independent variables in the determining equations. From (6.5) we deduce that $Q^{(1)}_{ij} = f(x_i)$ and from (6.6) $Q^{(2)}_{ij} = g(y_j)$ where f and g are arbitrary functions of their arguments. Dividing (6.7) by u_{00} and applying the operator $A = u_{10}\partial_{u_{10}} - u_{01}\partial_{u_{01}}$ (we have $A\phi(u_{11}) = 0$ for any function ϕ) and we get

$$\frac{Q^{(3)}_{01}}{u_{01}} - \frac{\partial Q^{(3)}_{01}}{\partial u_{01}} = \frac{Q^{(3)}_{10}}{u_{10}} - \frac{\partial Q^{(3)}_{10}}{\partial u_{10}}, \tag{6.8}$$

i.e. the quantity $\frac{Q^{(3)}_{ij}}{u_{ij}} - \frac{\partial Q^{(3)}_{ij}}{\partial u_{ij}} = h(i+j)$. So

$$Q^{(3)}_{ij} = u_{ij}\left[h(i+j)\log_e(u_{ij}) + A_{ij}(x_{ij}, x_{i+1,j}, y_{ij}, y_{i,j+1})\right]. \tag{6.9}$$

Introducing this result into (6.7) and taking into account that $\log_e(u_{11}) = \log_e(u_{10}) + \log_e(u_{01}) + \log_e\left[\frac{1}{u_{00}} + (x_{10}-x_{00})(y_{01}-y_{00})\right]$ we find from the coefficient of $\log_e\left[\frac{1}{u_{00}} + (x_{10}-x_{00})(y_{01}-y_{00})\right]$

that $h(i+j)=0$. Thus $Q_{ij}^{(3)} = u_{ij}A_{ij}(x_{ij}, x_{i+1,j}, y_{ij}, y_{i,j+1})$. Introducing this last result into (6.7) we find two equations for $A_{ij}(x_{ij}, x_{i+1,j}, y_{ij}, y_{i,j+1})$

$$A_{00} + A_{11} - A_{01} - A_{10} = 0, \tag{6.10}$$

$$A_{ij} = -\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} - \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}. \tag{6.11}$$

Eq. (6.10) is identically satisfied by the result obtained in (6.11) and as a consequence the symmetry algebra of the Liouville equation presented by Rebelo and Valiquette is indeed the sum of two Virasoro algebras determined by the two functions f and g:

$$\hat{X}\,(f,g) = f(x_i)\partial_{x_i} + g(y_j)\partial_{y_j} - \left[\frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} + \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}\right]\partial_{u_{ij}}. \tag{6.12}$$

The main difference between these generators and those in (4.1) is that in (4.1) the coefficients of $\partial_{u_{ij}}$ are locally dependent on the space points, while two points are involved in (6.12). Thus, the expression (6.4) has to be understood as a summation over all points of the lattice. On the contrary (4.1) contains only finite sums over the stencil points. Thus the Rebelo–Valiquette discretization of the Liouville equation is invariant under $VIR(x) \otimes VIR(y)$, but these are generalized symmetries rather than point ones. These are actually very special generalized symmetries: The Lie algebra (6.12) can be integrated to the finite transformations (6.2). These finite transformations were actually the starting point in the Rebelo-Valiquette approach.

7 Lie point symmetries of a linearizable Liouville equation.

Adler and Startsev [1] have presented a discretization of the algebraic Liouville equation (1.2) on a four-point lattice, namely

$$u_{i+1,j+1}(1 + \frac{1}{u_{i+1,j}})(1 + \frac{1}{u_{i,j+1}})u_{i,j} = 1. \tag{7.1}$$

This equation is linearizable by the substitution

$$u_{i,j} = -\frac{(v_{i+1,j} - v_{i,j})(v_{i,j+1} - v_{i,j})}{v_{i+1,j}v_{i,j+1}}, \tag{7.2}$$

where $v_{i,j}$ satisfies the linear equation

$$v_{i+1,j+1} - v_{i+1,j} - v_{i,j+1} + v_{i,j} = 0. \tag{7.3}$$

Hence the general solution of (7.1) is

$$u_{i,j} = -\frac{(c_{i+1} - c_i)(k_{j+1} - k_j)}{(c_{i+1} + k_j)(c_i + k_{j+1})}, \tag{7.4}$$

where c_i, k_j are arbitrary functions of one index each. We restrict (7.1) to the stencil with $i = j = 1$, i.e.

$$E = u_{11}(u_{10} + 1)(u_{01} + 1)u_{00} - u_{10}u_{01} = 0, \tag{7.5}$$

and calculate the Lie point symmetries of this equation. The equation is autonomous, the lattice is fixed (orthogonal and uniform). Hence the symmetry algebra is generated by vector fields of the form

$$\hat{X}_e = Q_{ij}(u_{ij})\partial_{u_{ij}}, \tag{7.6}$$

satisfying

$$\hat{X}E|_{E=0} = 0. \tag{7.7}$$

We obtain

$$Q_{11}(u_{01}+1)(u_{10}+1)u_{00} + Q_{10}u_{11}(u_{01}+1)u_{00} + Q_{01}u_{11}(u_{10}+1)u_{00} + \\ +Q_{00}u_{11}(u_{01}+1)(u_{10}+1) = Q_{10}u_{01} + Q_{01}u_{10}. \tag{7.8}$$

We eliminate u_{11} from (7.8) using (7.5), then differentiate with respect to u_{00} and obtain

$$\frac{Q_{11}}{u_{11}} - \frac{dQ_{11}}{du_{11}} = \frac{Q_{00}}{u_{00}} - \frac{dQ_{00}}{du_{00}}. \tag{7.9}$$

The general solution of (7.9) is

$$Q_{ij} = u_{ij}\left[g_{ij} + f(i-j)\log_e(u_{ij})\right] \tag{7.10}$$

where g_{ij} and $f(i-j)$ are functions of i and j. Substituting (7.10) into (7.8) we find $g(i,j) = f(i-j) = 0$.

It follows that the linearizable discrete Liouville equation has no continuous point symmetries at all!

Two comments are in order.

1. The equation (7.1) is linearizable and hence must have generalized symmetries.

2. Ref. [1] also contains a linearizable differential - difference Liouville equation:

$$\dot{u}_{i+1}u_i - u_{i+1}\dot{u}_i = u_{i+1}u_i(u_{i+1}+u_i). \tag{7.11}$$

   where the dot denotes the derivative of $u_n(x)$ with respect to the continuous variable x. It can be shown using the formalism presented in [17] that (7.11) does have an infinite dimensional Lie point symmetry algebra, isomorphic to the Virasoro algebra. The algebra is realized by evolutionary vector fields of the form

$$\hat{X}_e = Q_i(x, u_i, \dot{u}_i)\partial_{u_i}, \quad Q_i = f(x)\dot{u}_i + \dot{f}(x)u_i. \tag{7.12}$$

   This corresponds to the standard factor fields

$$\hat{X} = f(x)\partial_x - \dot{f}(x)u\partial_u. \tag{7.13}$$

8 Conclusions

We have shown that at least on a four–point lattice it is not possible to discretize the Liouville equation (1.2) (nor (1.1)) while preserving $VIR(x) \otimes VIR(y)$ as the Lie point symmetry group. That is also impossible on a six–point lattice. On the other hand, Rebelo and Valiquette [24] have introduced a special type of generalized symmetries that leave their discretization of the algebraic Liouville equation invariant. In the continuous case these symmetries reduce to point ones. In the discrete case they are special in that the vector fields can be integrated to group transformations acting on the equation and on the lattice. This is somewhat similar to the case of the symmetries of the Toda hierarchy [8] where some generalized symmetries *contract* to point ones in the continuous limit.

From the point of view of numerical methods it remains to explore which discretization provides better results. A discretization preserving the maximal finite subgroup of an infinite dimensional point symmetry group, or one that transforms point symmetries into generalized ones.

As stated in the Introduction, the main purpose of this article is to investigate how continuous physical theories can be discretized while preserving their continuous Lie point symmetries. For the Liouville equation we have shown that in a complete discretization it is possible to preserve invariance under under the maximal finite subgroup. The infinite dimensional Lie pseudogroup does not survive as a group of point symmetries. Rebelo and Valiquette have shown that the entire Virasoro pseudogroup does survive in a different discretization [24], but as generalized symmetries.

In Section 5 we have tested the quality of our invariant discretization as a numerical method. We have shown that it actually performs very well. We are of course aware that what we here call "standard" methods can be improved in many other ways. The use of point symmetries in numerical solutions of partial differential equations deserves a further detailed analysis.

Another interesting point is that the linearizable discretization of Adler and Startsev preserves no point symmetries. It is thus important to decide which features of a continuous theory one wishes to preserve in a discretization. In this case linearizability is incompatible with the preservation of point symmetries.

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