The quasiclassical Keldysh Green function
Lectures delivered at the Università di Camerino

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Introductory remarks

- The Keldysh technique is the way to a quantum kinetic equation, but the resulting equations are in general very complicated and difficult to solve.

- The quasiclassical approximation, which works well for degenerate Fermi systems, is a concrete example, where, starting from a quantum formulation, one makes a link to the phenomenological Boltzmann equation for quasiparticles.

- The cases which I include in these lectures show that it is not only possible to give a microscopic justification to the Boltzmann equation, but also to derive systematically correction terms.

- The quasiclassical approximation, furthermore, even though it does not have the elegance of the functional integral methods, provides very often a much clearer physical picture.

- Even at equilibrium, when diagrammatic methods are sufficient, the quasiclassical approximation has the advantage of a more compact derivation.

I hope to convince you! Let us get started.
References

   Where all began

   I find this relatively short account of the basics of the Keldysh technique one of
   the best. You just learn exactly what is necessary to know in pure Landau style.

   This is a classic reference full of historical and technical information. It gives the
   rules of the game

   This deals mainly with quantum corrections.

   A modern and self-contained account based on functional integrals. The authors
   really know the subject.
Program of the lectures

1. The Keldysh technique
2. The quasiclassical approximation
3. Weak localization
4. Quantum interaction correction
5. Thermal transport and Coulomb Blockade
6. Superconductivity and Andreev scattering
1. The Keldysh time path
2. The example of the Fermi gas
3. Dyson equation
4. Perturbation theory
5. Disorder and self-consistent Born approximation
Let us recall the standard perturbation theory

- Split the Hamiltonian into free and interacting parts

\[ H = H_0 + H_I \]

- Go to the interaction picture, which in terms of Schrödinger and Heisenberg pictures reads

\[ \psi_i(t) = e^{iH_0 t} \psi_S(t) \]

\[ \psi_i(t) = e^{iH_0 t} e^{-iH t} \psi_H(t) \]

Note that at \( t = 0 \) all pictures coincide

- The time evolution in the interaction picture is given by the S-matrix

\[ \psi_i(t_1) = S(t_1, t_2) \psi_i(t_2) \]

- The S-matrix is naturally time ordered

\[ S(t_1, t_2) = T \exp \left( -i \int_{t_2}^{t_1} dt \ H_{II}(t) \right) \]

- The problem is how to connect the time evolution of the S-matrix in the interaction picture with the relation between the free and interacting states of the Heisenberg picture
Trick of the adiabatic turning on

• In the far past and future there is no interaction $H_f \rightarrow e^{-\epsilon |t|} H_f$

• Then the interaction picture state vector in the far past and future and at $t = 0$ is related to the free and interacting Heisenberg state vector

$$\psi_i(\pm \infty) = \psi_H$$

$$\psi_i(0) = \psi_H$$

• To have a connection at all times one needs the S-matrix

$$\psi_H = S(0, -\infty) \psi_H_0$$

$$\psi_i(t) = S(t, 0) \psi_H$$

• From state vectors to operators

$$A_H(t) = S^{-1}(t, 0) A_i(t) S(t, 0)$$

• By introducing $S \equiv S(\infty, -\infty)$ the key formula is

$$\langle \psi_H | T(A_H(t_1)B_H(t_2)) | \psi_H \rangle = \langle \psi_H_0 | S^{-1} T(A_i(t_1)B_i(t_2)S) | \psi_H_0 \rangle$$

i.e. we have transformed average over the interacting state in an average over the free one
The Keldysh Time Contour

- The problem with the previous formula is that $S$ is time ordered, while $S^{-1}$ is anti-time ordered.
- The problem is solved by invoking the adiabatic theorem. The state at $t = \infty$, adiabatically evolved from the state at $t = -\infty$ acquires just a divergent phase factor (Gell-Mann and Low (1951))

$$\tilde{\psi}_H = \exp(i\alpha)\psi_H \quad S = \exp(i\alpha)$$

- The unpleasant $S^{-1}$ factor can be moved into the denominator

$$\langle \psi_H | T(A_H(t_1)B_H(t_2)) | \psi_H \rangle = \frac{\langle \psi_H | T(A_i(t_1)B_i(t_2)S) | \psi_H \rangle}{\langle \psi_H | S | \psi_H \rangle}$$

- There is an alternative strategy: the Keldysh contour $C_K = (-\infty, \infty) \cup (\infty, -\infty)$, which includes both time-ordered and antitime-ordered factors via the ordering operator $T_K$. The S-matrix becomes

$$S_K = T_K \exp \left(-i \int_{C_K} dt \, H_i(t) \right)$$

- The key formula for perturbation theory becomes

$$\langle \psi_H | T(A_H(t_1)B_H(t_2)) | \psi_H \rangle = \langle \psi_H | T_K(A_i(t_1)B_i(t_2)S_K) | \psi_H \rangle$$

- For a stationary state both strategies can be used. For a non-equilibrium state one must use the Keldysh strategy.
The beauty of the Keldysh time path is that the definition of the Green function is like the equilibrium case

\[ G(x_1, x_2) = -i \langle T_K \psi(x_1) \psi^\dagger(x_2) \rangle, \quad x_i = (x_i, t_i) \]

The price to pay is that there are actually several Green functions to calculate.

It is convenient to introduce a matrix structure in the so-called Keldysh space to indicate which Keldysh-contour branch the time arguments belong to

\[ G(x_1, x_2) \rightarrow \tilde{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \]

Time - ordered \[ G_{11}(x_1, x_2) = -i \langle T \psi(x_1) \psi^\dagger(x_2) \rangle \]

Lesser (Density matrix) \[ G_{12}(x_1, x_2) = +i \langle \psi^\dagger(x_2) \psi(x_1) \rangle \]

Greater \[ G_{21}(x_1, x_2) = -i \langle \psi(x_1) \psi^\dagger(x_2) \rangle \]

Antitime - ordered \[ G_{22}(x_1, x_2) = -i \langle \tilde{T} \psi(x_1) \psi^\dagger(x_2) \rangle \]

The Green functions are not all independent and obey the causality condition

\[ G_{11} + G_{22} = G_{12} + G_{21} \]
The Keldysh rotation

- Due to the fact that the components of $\tilde{G}$ are not independent, it is convenient to transform the matrix $\tilde{G}$ as

$$\tilde{G} \rightarrow \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{G} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- In this new representation the Green's function is

$$\tilde{G} = \begin{pmatrix} G^R & G^K \\ G^Z & G^A \end{pmatrix} = \frac{1}{2} \begin{pmatrix} G_{11} - G_{22} - G_{12} + G_{21} & G_{11} + G_{22} + G_{12} + G_{21} \\ G_{11} + G_{22} - G_{12} - G_{21} & G_{11} - G_{22} + G_{12} - G_{21} \end{pmatrix}$$

- The rotated components are

$$G^R(x_1, x_2) = -i\Theta(t_1 - t_2)\langle \Psi(x_1)\psi^\dagger(x_2) + \psi^\dagger(x_2)\Psi(x_1) \rangle$$
$$G^A(x_1, x_2) = +i\Theta(t_2 - t_1)\langle \Psi(x_1)\psi^\dagger(x_2) + \psi^\dagger(x_2)\Psi(x_1) \rangle$$
$$G^K(x_1, x_2) = -i\langle \psi(x_1)\psi^\dagger(x_2) - \psi^\dagger(x_2)\psi(x_1) \rangle$$
$$G^Z(x_1, x_2) = 0$$

- The vanishing of $G^Z$ is not automatic if the evolution on the two time paths is not the same. This may happen in the presence of quantum fluctuations. We will come back to this when treating interactions.
The meaning of the rotation

• The rotation defines classical and quantum fields

\[ \psi^{cl} = \frac{1}{\sqrt{2}} (\psi^1 + \psi^2) \]
\[ \psi^{qu} = -\frac{1}{\sqrt{2}} (\psi^1 - \psi^2) \]

• The connection with the various Green functions

\[ G^K \rightarrow \langle \psi^{cl} \psi^{cl} \rangle \]
\[ G^R \rightarrow \langle \psi^{cl} \psi^{qu} \rangle \]
\[ G^A \rightarrow \langle \psi^{qu} \psi^{cl} \rangle \]
\[ G^Z \rightarrow \langle \psi^{qu} \psi^{qu} \rangle \]
A first example: the Fermi gas at equilibrium

• One can compute explicitly all the Green functions starting from

\[ \psi (\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} e^{-i \xi_p t} \quad \xi_p = \frac{p^2}{2m} - \mu \]

• The retarded and advanced Green function provide information about the energy spectrum

\[ G^{R,A}(\mathbf{p}, \epsilon) = [\epsilon - \xi_p \pm i0^+]^{-1}, \]

• The Keldysh Green function tells about the distribution function

\[ G^K(\mathbf{p}, \epsilon) = \left( G^R(\mathbf{p}, \epsilon) F(\epsilon) - F(\epsilon) G^A(\mathbf{p}, \epsilon) \right), \quad F(\epsilon) = \tanh \left( \frac{\epsilon}{2T} \right) \]

This is basically the fluctuation-dissipation theorem

• Observables are given in terms of the lesser Green function

\[ G_{12} = \frac{1}{2} (G^K - G^R + G^A) = -f(\epsilon)(G^R - G^A) \]

where \( f(\epsilon) \) is the Fermi function
Dyson equations

• From the definition, the derivation of the Dyson equation is like in the equilibrium case. For future use, we show both left- and right-side equation of motion

\[
(\tilde{G}_0^{-1} - \tilde{\Sigma}) \tilde{G} = \delta(x_1 - x_2) \quad \tilde{G}(\tilde{G}_0^{-1} - \tilde{\Sigma}) = \delta(x_1 - x_2)
\]

• The differential operator is a matrix in Keldysh space

\[
\tilde{G}_0^{-1}(x_1, x_2) = \left[ i \partial_t + \frac{1}{2m} (eA(x_1)) + \mu \right] \delta(x_1 - x_2)
\]

• While the electromagnetic potentials \( \phi(x), A(x) \) are shown explicitly (with \( e > 0 \)), disorder scattering and electron-electron interactions are contained in the self-energy

• The self-energy has a triangular structure as the Green function

\[
\tilde{\Sigma} = \begin{pmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{pmatrix}
\]

• The Keldysh Green function can always be expressed in terms of the retarded and advanced Green functions and Keldysh self-energy

\[
\left( (G_0^{R,A})^{-1} - \Sigma^{R,A} \right) G^{R,A} = 1 \quad G^K = G^R \Sigma^K G^A
\]
Perturbation expansion in an external potential

• Let $U$ be one-body external potential. The zero and first order terms of the perturbation expansion are

$$
\tilde{G} = \begin{pmatrix} G^K_0 & G^K_A \\ 0 & G^K_A \end{pmatrix} + \begin{pmatrix} G^K_0 & G^K_A \\ 0 & G^K_A \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} G^K_0 & G^K_A \\ 0 & G^K_A \end{pmatrix} + \ldots
$$

• Replace $G^K_0$ with the equilibrium form

$$G^K_0 = G^K_0 R F_0 - F_0 G^K_A$$

$$G^K = (G^K_0 + G^K_0 U G^K_0) F_0 - G^K_0 (UF_0 - F_0 U) G^K_A - F_0 (G^K_A + G^K_A U G^K_0) + \ldots$$

• By observing the partial resummations

$$G^K = G^K_0 + G^K_0 U G^K_0 + \ldots, \quad G^K = G^K_0 + G^K_0 U G^K_0 + \ldots, \quad F = F_0 - [U, F_0] + \ldots$$

One obtains that the equilibrium relation of the Fermi gas is valid in general

$$G^K = G^K R F - FG^K_A$$
The standard model of disorder

- We consider in the Hamiltonian a term
  \[ \int d\mathbf{x} \psi^\dagger(\mathbf{x}) U(\mathbf{x}) \psi(\mathbf{x}) = \sum_{\mathbf{p}, \mathbf{p'}} c_\mathbf{p}^\dagger U(\mathbf{p} - \mathbf{p'}) c_\mathbf{p} \]

- The lowest order Born approximation corresponds to the self-energy

- The average over disorder realizations is indicated by a dashed line and defines the type of disorder model

- The white noise model
  \[ \overline{U(\mathbf{x}) U(\mathbf{x}')} = u_0^2 \delta(\mathbf{x} - \mathbf{x}') \]
  where \( u_0 \) is the scattering amplitude
Self-consistent Born approximation

• Let us evaluate the lowest order self-energy

\[
\Sigma^R = \delta(x - x') u_0^2 \sum_p \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \frac{e^{-i\epsilon(t-t')}}{\epsilon - \xi_p + i0^+}
\]

\[
\approx -i\delta(x - x') 2\pi u_0^2 N_0 \int_{-\infty}^{\infty} d\xi_p e^{-i(\xi_p - i0^+)(t-t')} \mu \sim \infty
\]

\[
= -i \frac{\delta(x - x')}{2\tau} \delta(t - t') \quad \tau^{-1} = 2\pi u_0^2 N_0
\]

• Actually a self-consistent solution is obtained since what matters is the position of the pole of the Green function in the complex plane, but not its distance from the real axis

\[
\Sigma^R = -i \frac{\delta(x - x')}{2\pi N_0 \tau} \sum_p e^{-i(\xi_p - i/2\tau)(t-t')}
\]

• The important physical assumption is that the distance of the pole from the real axis is small compared to the Fermi energy

\[
\epsilon_F \tau \gg 1 \quad \text{or} \quad p_F l \gg 1
\]
The form of the self-energy

- The retarded Green function has the form
  \[ G^R = \left( \epsilon - \xi_p + i/2\tau \right)^{-1} \quad G^A = (G^R)^* \]

- The self-consistent self-energy can be generalized in the Keldysh space
  \[ \tilde{\Sigma} = \frac{\delta(\mathbf{x} - \mathbf{x}')}{2\pi N_0 \tau} \tilde{G}(\mathbf{x}, t; \mathbf{x}, t') \]

- The above self-energy can be inserted into the Dyson equation. This form treats the scattering from different impurities incoherently. We will come back to this.
Summary of the first lecture

- In this lecture we have seen how the perturbation theory based on the Keldysh technique works.
- In particular, we have studied the important case of the perturbation expansion for an external potential.
- We have seen how the impurity technique leads to a functional form for the self-energy.
- In the next lecture, we will introduce the so-called quasiclassical approximation, which will allow us to derive effective kinetic equations.
2. *The Quasiclassical approximation*

1. The Eilenberger equation
2. Meaning of $\xi$-integration
3. Observables
4. Gauge invariance and Drude formula
The quasiclassical approximation

- The idea is that in order to describe the response to an external disturbance, it is necessary to keep all the information at the microscopic level.
- The basic assumption is that external disturbances have length scales much larger than the Fermi wave length.
- The arguments of the Green function can be divided in center-of-mass and relative coordinates:
  \[
  x = \frac{x_1 + x_2}{2}, \quad r = x_1 - x_2, \quad t = \frac{t_1 + t_2}{2}, \quad \eta = t_1 - t_2
  \]
- The Wigner form corresponds to Fourier transform with respect to the relative coordinates:
  \[
  G(p, \eta; x, t) = \int \text{d}r e^{-i p \cdot r} G \left( x + \frac{r}{2}, t + \frac{\eta}{2}; x - \frac{r}{2}, t - \frac{\eta}{2} \right)
  \]
The gradient expansion

• Suppose to consider a convolution \((1 \equiv x_1 \text{ and so for } 2 \text{ and } 3)\)

\[
(A \cdot B)(1, 2) = \int d3 \, A(1, 3)B(3, 2)
\]

\[
\equiv \int d3 \, A\left(\frac{1+3}{2}, 1-3\right)B\left(\frac{2+3}{2}, 3-2\right)
\]

\[
= \int d3 \, A\left(\frac{1+2}{2} + \frac{3-2}{2}, 1-3\right)B\left(\frac{1+2}{2} + \frac{3-1}{2}, 3-2\right)
\]

\[
\approx \int d3 \, A\left(\frac{1+2}{2}, 1-3\right)B\left(\frac{1+2}{2}, 3-2\right) + \ldots
\]

• By making the Fourier transform with respect to the relative coordinate of the free arguments \(1 - 2\)

\[
(A \cdot B)_p = A(x, p)B(x, p) + \ldots
\]

• All other terms can be obtained by a gradient expansion

\[
(AB)(p, x) = e^{-i(\partial^A_p \partial^B_x - \partial^A_x \partial^B_p)/2} A(p, x)B(p, x)
\]
Quasiclassical equation of motion (G. Eilenberger, Z. Phys. 214, 195 (1968))

- Subtract LH and RH Dyson equations from one another so that the delta functions cancel. Then Fourier transform and keep the lowest order term in the gradient expansion

\[
\left[ \imath \partial_t + \frac{1}{m} \mathbf{p} \cdot \partial_x \right] \tilde{G}(\mathbf{p}, \mathbf{x}) = \left[ \tilde{\Sigma}(\mathbf{p}, \mathbf{x}), \tilde{G}(\mathbf{p}, \mathbf{x}) \right]
\]

- The electromagnetic potential enters through the covariant derivatives

\[
\tilde{\partial}_t \tilde{G} = [\partial_t - \imath e \phi(\mathbf{x}, t + \eta/2) + \imath e \phi(\mathbf{x}, t - \eta/2)] \tilde{G}(\mathbf{p}, \eta; \mathbf{x}, t)
\]

\[
\tilde{\partial}_x \tilde{G} = [\partial_x + \imath e \mathbf{A}(\mathbf{x}, t + \eta/2) - \imath e \mathbf{A}(\mathbf{x}, t - \eta/2)] \tilde{G}(\mathbf{p}, \eta; \mathbf{x}, t)
\]

- Define the quasiclassical Green function

\[
\tilde{g}(\hat{\mathbf{p}}, \eta; \mathbf{x}, t) = \frac{\imath}{\pi} \int d\xi \tilde{G}(\mathbf{p}, \eta; \mathbf{x}, t), \quad \xi = \frac{\mathbf{p}^2}{2m} - \mu
\]

The Eilenberger equation

\[
\left[ \tilde{\partial}_t + \mathbf{v}_F \hat{\mathbf{p}} \cdot \tilde{\partial}_x \right] \tilde{g}(\hat{\mathbf{p}}, \eta, \mathbf{x}, t) = -\imath \left[ \tilde{\Sigma}(\hat{\mathbf{p}}, \mathbf{x}), \tilde{g}(\hat{\mathbf{p}}, \eta, \mathbf{x}, t) \right], \quad \tilde{g} \tilde{g} = \hat{1}
\]

The normalization condition is necessary since the equation is homogeneous
Eilenberger decomposition

\[ \int_{-\infty}^{+\infty} d\xi \ldots = \frac{1}{2} \int_{C_{\text{low}}} d\xi \ldots + \frac{1}{2} \int_{C_{\text{high}}} d\xi \ldots \]
Meaning of the $\xi$-integration

The momentum integration can be divided into an energy integration, which corresponds to the $\xi$-integration and an integration over the Fermi surface

$$G^R(x_1, x_2) = \sum_p \frac{e^{ip \cdot (x_1 - x_2)}}{\epsilon - \xi + i0^+}, \quad r = x_1 - x_2$$

$$= \int \frac{dp \cdot dp}{(2\pi)^2} \frac{e^{ip \cdot r}}{\epsilon - \nu_F(p \cdot p_F) - \frac{p^2}{2m} + i0^+}$$

$$= -i e^{i(pF + \nu_F)r} \nu_F \int \frac{dp}{2\pi} e^{-ip^2 r / 2pF}$$

$$= -\sqrt{2\pi i \frac{m}{\nu_F r}} e^{i(pF + \nu_F)r}$$

$$= G^R_0(r, \epsilon = 0) g^R(x_1, x_2)$$

At large $r = |x_1 - x_2|$ integral dominated by the extrema in the exponential and the Green function is factorized in rapidly and slowly varying terms

- The definition of the quasiclassical Green function appears to be defined by a non-convergent integral

\[ g^R(x_1, x_2) = \frac{i}{2\pi} \int d\xi \frac{e^{i\xi r/v_F}}{\epsilon - \xi + i0^+} = e^{i\epsilon r/v_F} \]

- The slowly varying part is, however, well defined due to the exponential

\[ g^R(x_1, x_2) = \frac{i}{2\pi} \int d\xi \, e^{i\xi r/v_F} G^R(p, x), \quad p = p\hat{r} \]

- According to Shelankov, in general, we may write

\[ g^R(x_1, x_2) = \frac{i}{2\pi} \int d\xi \, e^{i\xi r/v_F} G^R(p, x), \quad p = p\hat{r} \]

- This leads to a well-defined quasiclassical Green function

\[ g^R(\hat{p}, \eta; x, t) = \lim_{r \to 0} \frac{i}{\pi} \int d\xi \cos \left( \frac{\xi r}{v_F} \right) G^R(p, \eta; x, t) \]

- In the Fermi gas, for instance,

\[ g^R(\hat{p}, \eta; x, t) = \delta(\eta) = -g^A(\hat{p}, \eta; x, t) \]
**Connection to observables and quasiclassical Green function**

- The observables are obtained by evaluating the Green function at equal-time and for doing so it is important to integrate energy first and momentum after

$$\rho(x, t) = (-e)(-i) \sum_p \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} G_{12}(p, \epsilon)$$

- The momentum integration is divided into $\xi$-integration and angle integration

$$\sum_p \cdots = \langle \int d\xi \cdots \rangle, \quad \langle \cdots \rangle = \int d\hat{p} \cdots$$

- The quasiclassical Green function at equal-time interchanges the correct order of integration

$$g_{12}(\hat{p}, \eta = 0; x, t) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \langle g_{12}(\hat{p}, \epsilon; x, t) \rangle$$

- Already, at equilibrium, for the Fermi gas case, this leads to problems since

$$g_{12}(\hat{p}, \epsilon; x, t) = -2f(\epsilon), \quad f \quad \text{Fermi function}$$

which is not integrable
Continuity equation

- At equilibrium the Keldysh component is well defined since
  \[ g^K(\hat{p}, \epsilon; x, t) = g^R(\hat{p}, \epsilon; x, t)F_0(\epsilon) - F_0(\epsilon)g^R(\hat{p}, \epsilon; x, t) = 2(1 - 2f(\epsilon)) \]
  is integrable due to antisymmetry of the hyperbolic tangent and
  \[ g^K(\hat{p}, \eta = 0; x, t) = 0 \]

- The Eilenberger equation for \( g^K \) with \( \phi = 0, A = 0 \) and \( \Sigma = 0 \) reads
  \[ (\partial_t + v_F \hat{p} \cdot \partial_x)g^K = 0 \]

- Then by averaging over \( \hat{p} \) and integrating over \( \epsilon \)
  \[ \partial_t \frac{eN_0}{2} \int_{-\infty}^{\infty} d\epsilon \langle g^K \rangle + \partial_x \frac{eN_0}{2} \int_{-\infty}^{\infty} d\epsilon \langle v_F \hat{p} g^K \rangle = 0 \]
  which yields the continuity equation for charge and current
  \[ \delta \rho(x, t) = \frac{eN_0}{2} \int_{-\infty}^{\infty} d\epsilon \langle g^K(\hat{p}, \epsilon; x, t) \rangle \quad j = \frac{eN_0}{2} \int_{-\infty}^{\infty} d\epsilon \langle v_F \hat{p} g^K \rangle \]
Continuity equation in the presence of potentials

Let us insert the potentials $\phi$ and $A$

$$(\partial_t - e\partial_t\phi(x, t)\partial_\epsilon + v_F \hat{p} \cdot \partial_x + ev_F \hat{p} \cdot \partial_t A(x, t)\partial_\epsilon)g^K = 0$$

Again by averaging over $\hat{p}$ and integrating over $\epsilon$

$$\partial_t \delta\rho(x, t) + \partial_x \cdot j(x, t) = 0$$

with

$$\delta\rho(x, t) = \frac{eN_0}{2} \int_{-\infty}^{\infty} d\epsilon \langle g^K(\hat{p}, \epsilon; x, t) \rangle - 2e^2N_0\phi(x, t)$$

$$j(x, t) = \frac{eN_0}{2} \int_{-\infty}^{\infty} d\epsilon \langle v_F \hat{p} g^K \rangle$$

since

$$\int_{-\infty}^{\infty} d\epsilon \partial_\epsilon g^K(\hat{p}, \epsilon; x, t) = g^K(\hat{p}, \epsilon; x, t)|_{-\infty}^{\infty} = g^K_{eq}(\hat{p}, \epsilon; x, t)|_{-\infty}^{\infty} = 4$$

The expression for the density shows clearly how the quasiclassical Green function captures the low energy part.
Disorder in the Born approximation just shifts the pole in the complex plane for the retarded and advanced Green functions, which then do not change \( g^R = -g^A = 1 \)

The RHS of the Keldysh component of the Eilenberger equation yields

\[
\begin{align*}
\left[ \tilde{\Sigma}, \tilde{g} \right]^K &= \Sigma^R g^K + \Sigma^K g^A - g^R \Sigma^K - g^K \Sigma^A \\
&= -\frac{i}{\tau} g^K - 2\Sigma^K \\
&= -\frac{i}{\tau} g^K - 2 \frac{1}{2\pi N_0 \tau} \sum_p G^K(p, \epsilon; x, t) \\
&= -\frac{i}{\tau} (g^K - \langle g^K \rangle)
\end{align*}
\]

which has the form of the collision integral with the scattering-\textit{in} and scattering-\textit{out} terms well-known in the Boltzmann’s equation within the relaxation-time approximation.
\[ \mathbf{A} = -t \mathbf{E} \]

\[
(\partial_t + v_F \hat{\mathbf{p}} \cdot \partial_x - ev_F \hat{\mathbf{p}} \cdot \mathbf{E} \partial_\epsilon) g^K = \frac{1}{\tau} (\langle g^K \rangle - g^K)
\]

- By looking for a uniform solution, we take the \( p \)-wave component of both sides
  \[
  \langle \hat{\mathbf{p}} g^K \rangle = \frac{e v_F \tau}{d} \mathbf{E} \partial_\epsilon g^K
  \]

- To linear order in the electric field
  \[
  \langle \hat{\mathbf{p}} g^K \rangle = \frac{e v_F \tau}{d} \mathbf{E} \partial_\epsilon g_{eq}^K
  \]

- By integration over the energy \( \epsilon \)
  \[
  \mathbf{j} = \frac{e N_0 v_F}{2} \langle \hat{\mathbf{p}} g^K \rangle = \frac{e N_0 v_F}{2} \frac{4 e v_F \tau}{2} \mathbf{E} = \sigma_D \mathbf{E}
  \]
Drude conductivity and gauge invariance: scalar gauge

- \( \phi(x) = -E \cdot x \) Apparently the scalar potential drops out

\[
(\partial_t + v_F \hat{p} \cdot \partial_x)g^K = \frac{1}{\tau} (\langle g^K \rangle - g^K)
\]

- However

\[
\langle \hat{p} g^K \rangle = -\frac{v_F \tau}{2} \partial_x \langle g^K \rangle
\]

- We must use the condition of charge uniformity

\[
\partial_x \delta \rho(x, t) = \partial_x \left( \frac{e N_0}{2} \int_{-\infty}^{\infty} d\epsilon \langle g^K(\hat{p}, \epsilon; x, t) \rangle - 2e^2 N_0 \phi(x, t) \right) = 0
\]

The *bottom line* is the minimal substitution also in this case

\[
\partial_x \rightarrow \partial_x - eE \partial_\epsilon
\]
Summary of the second lecture

- In this lecture we have derived the Eilenberger equation for the quasiclassical Green function.
- We have applied this equation to derive the semiclassical theory of electrical transport.
- In the next lectures we are going to apply the Eilenberger equation to different situations to obtain corrections to the semiclassical level.
3. Weak Localization

1. Crossed diagrams
2. Backscattering
3. Corrections to conductivity
Some historical remarks

- Anderson localization was invented in 1958, half a century ago and perhaps would deserve an entirely dedicated lecture course.
- Here, I concentrate on the phenomenon of weak localization, which is a perturbative effect on the metallic behavior.
- My emphasis will be on the advantage of the quasiclassical method in order to capture the physical origin of the phenomenon.
Quantum interference and crossed diagrams

The Born approximation corresponds to the *Rainbow* diagrams. Interference between scattering events at different sites is not allowed.

The maximally crossed diagrams describe the interference between *time-reversed paths* (G. Bergmann, *Phys. Rep.*, 107, 1 (1984)).
Corrections to the self-energy

Let us consider a term for the Keldysh Green function

\[ G^R U G^R \ldots G^R U G^K U G^A \ldots G^A U G^A \]

Note that an equal number of \( G^R \) and \( G^A \) precede and follow \( G^K \)

We take the disorder average selecting the diagrams with \textit{maximum} crossing

\[ \delta \Sigma^K = \]

Disorder average yields momentum conservation at each impurity line

\[ Q = p + p' \]

\[ p' = Q - p \]
\[ p'_1 = Q - p_1 \]
\[ p'_2 = Q - p_2 \]
\[ p'_3 = Q - p_3 \]
The so-called Cooperon is the sum of all diagrams with an arbitrary number of crossed impurity lines (J. S. Langer and T. Neal, Phys. Rev. Lett. 16, 984 (1966).)

\[ C(Q, \omega) = \epsilon - \frac{\omega}{2} \epsilon + \frac{\omega}{2} \]

The correction to the self-energy is a convolution with respect to the relative momentum

\[ \delta \Sigma^K(p, \epsilon; q, \omega) = \sum_Q C(Q, \omega) G^K(Q - p, \epsilon; q, \omega) \]

The momentum \( Q \) describes the effect of multiple scattering
**Diffusion pole and backscattering**

By using the **time-reversal invariance** one line can be reversed turning the crossed diagrams into a geometric series of ladder diagrams. The ladder is a particle-particle scattering channel as in the Cooper mechanism for superconductivity.

The building block

$$\eta^{RA} = \sum_p G^R(p, \epsilon + \frac{\omega}{2}) G^A(Q - p, \epsilon - \frac{\omega}{2}) = 2\pi N_0 \tau \left[ 1 + i\omega \tau - DQ^2 \tau + \ldots \right]$$

$$C(Q, \omega) = \frac{1}{2\pi N_0 \tau} \eta^{RA} \frac{1}{2\pi N_0 \tau} + \ldots = \frac{1}{2\pi N_0 \tau^2} \left[ \frac{1}{DQ^2 - i\omega - 1} \right]$$

The diffusion pole selects processes with $p' \approx -p$
\[
\sum_p G^R(p, \epsilon)G^A(p, \epsilon) = \int_{-\mu}^{\infty} d\xi \frac{N(\xi)}{(\epsilon - \xi + i/2\tau)(\epsilon - \xi - i/2\tau)} \\
\approx N_0 \int_{-\infty}^{\infty} d\xi \frac{1}{(\epsilon - \xi + i/2\tau)(\epsilon - \xi - i/2\tau)} \\
= 2\pi N_0 \tau
\]

All integrals involving a number of retarded and advanced Green function can be made in the same way by using the residue Cauchy theorem. It is important to note that if all the Green functions are retarded or advanced the integral vanishes.
Quasiclassical procedure

The $\xi$-integration yields the final form to be used in the Eilenberger equation

$$\frac{1}{\pi} \int d\xi \left[ \delta \tilde{\Sigma}, \tilde{G} \right]^K = \frac{1}{\pi} \int d\xi \left( \delta \Sigma^R G^K + \delta \Sigma^K G^A - G^R \delta \Sigma^K - G^K \delta \Sigma^A \right)$$

By using

$$G^K = G^R F - FG^A = \frac{1}{2} \left( G^R g^K - g^K G^A \right)$$

Correction to the scattering-in term of the kinetic equation

$$\delta \sigma^k = -\frac{1}{\pi} \int d\xi \left[ G^R \left( -p, \epsilon + \frac{\omega}{2} \right) - G^A \left( -p, \epsilon - \frac{\omega}{2} \right) \right] \frac{1}{2} g^K \left( -\hat{p}, \epsilon, q, \omega \right) \sum Q C(Q, \omega)$$

(S. Hershfield and V. Ambegaokar, Phys. Rev. B 34, 2147 (1986).)

But the story is not complete yet, since we don't know about diagrams when the numbers of $G^R$ and $G^K$ before and after $G^K$ is not the same.
The arising of the Hikami box

In addition to this diagram, we must consider that obtained by the interchange of $R$ and $A$

$$p' = Q - p + q/2$$

$$p'_1 = Q - p_1 + q/2$$

$$p'_2 = Q - p_2$$

$$p'_3 = Q - p_3$$

Correction to the scattering-out term of the kinetic equation

$$\delta \Sigma^K = \frac{1}{2\pi N_0 \tau} \sum_{p_1} G^R \left( \mathbf{q} - \frac{\mathbf{p}_1}{2}, \epsilon + \frac{\omega}{2} \right) G^R \left( \mathbf{q} - \frac{\mathbf{p}}{2}, \epsilon + \frac{\omega}{2} \right) \times G^K (p_1, q, \epsilon, \omega) \sum_{\mathbf{q}} C(Q, \omega)$$
Kinetic equation with quantum corrections

\[
(\partial_t + v_F \hat{p} \cdot \partial_x) g^K(\hat{p}, \eta, \mathbf{x}, t) = -\frac{1}{\tau} \left[ g^K(\hat{p}, \eta, \mathbf{x}, t) - \langle g^K(\hat{p}, \eta, \mathbf{x}, t) \rangle \right] \\
+ \frac{1}{\tau} \int_{-\infty}^{\infty} dt' \alpha(t - t') \\
\times \left[ g^K(-\hat{p}, \eta, \mathbf{x}, t') - \langle g^K(\hat{p}, \eta, \mathbf{x}, t') \rangle \right]
\]

\[
\alpha(t - t') = 2\tau^2 \sum_Q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} C(Q, \omega)
\]

- The correction to the in- and out- terms maintain charge conservation
- The non local-in-time response in the collision integral corresponds to the memory effect of a self-retracing trajectory
Weak localization correction

\[ j = \left[ 1 - \int_{-\infty}^{\infty} dt' \alpha(t - t') \right] \sigma_D E \]

\[ \alpha_{2D}(t - t') = \frac{\Theta(t - t')}{(2\pi)^2 N_0 D} \frac{1}{t - t'} \]

\[ \int_{-\infty}^{\infty} dt' \alpha(t - t') \Rightarrow \int_{-\tau}^{t - \tau} dt' \frac{1}{(2\pi)^2 N_0 D} \frac{1}{t - t'} = \frac{1}{(2\pi)^2 N_0 D} \ln \frac{\tau_{\phi}}{\tau} \]

The controlling parameter is the conductance measured in units of the quantum of conductance

\[ \sigma = \sigma_D - \frac{2e^2}{h} \frac{1}{2\pi} \ln \frac{\tau_{\phi}}{\tau} \]
Scaling theory and MIT

- $\tau$ time above which diffusive motion set in
- $\tau_{\phi}$ maximum time over which phase memory lasts
- in general $\tau_{\phi} \sim T^{-p}$
- in finite system at low temperatures $\tau_{\phi} \to L^2/D$

Dimensionless conductance

$$g = \frac{\sigma_D L^{d-2}}{2e^2/h}$$

Effective perturbation theory

$$\sigma = \sigma_D \left(1 - \frac{1}{\pi} \frac{1}{g} \ln \frac{L}{l}\right)$$

Summary of the third lecture

• In this lecture we have applied the quasiclassical Green function approach to weak localization

• We have derived corrections term to the kinetic equation associated to the backscattering mechanism originating from the interference of self-retracing trajectories

• In the next lecture we are going to illustrate a different quantum correction originating from the electron-electron interaction
4. Quantum Interaction Corrections

1. Electron-electron interaction
2. Density of states
3. Electrical conductivity
Diffusive approximation in general

- Let us recall the Eilenberger equation for the quasiclassical Green function

\[
[\partial_t + v_F \hat{p} \cdot \partial_x] \hat{g} = -\frac{1}{2\tau} [\langle \hat{g} \rangle, \hat{g}], \quad \hat{g} \hat{g} = \hat{1}
\]

- Diffusive motion makes the Green function almost isotropic and allows expansion in s-wave and p-wave

\[
\hat{g} = g_s + \hat{p} \cdot \hat{g}_p + \ldots, \quad g_s \hat{g}_s = \hat{1}, \quad \{ \hat{g}_s, \hat{g}_p \} = 0
\]

One obtains the Usadel diffusive equation (K. D. Usadel, Phys. Rev. Lett. 25, 507 (1970).)

\[
\partial_t \hat{g}_s - D \partial_x \cdot \hat{g}_s \partial_x \hat{g}_s = 0
\]

The generalized current is related to the gradient of the s-wave component

\[
\hat{g}_p = -v_F \tau \hat{g}_s \partial_x \hat{g}_s
\]
**How to include electron-electron interaction**

Hubbard-Stratonovich transformation

$$D(x_1, x_2) = -i \langle T_K \phi(x_1) \phi(x_2) \rangle$$

**Boson on the Keldysh contour**

$$D^{11}(x_1, x_2) = -i \langle T \phi(x_1) \phi(x_2) \rangle$$
$$D^{12}(x_1, x_2) = -i \langle \phi(x_2) \phi(x_1) \rangle$$
$$D^{21}(x_1, x_2) = -i \langle \phi(x_1) \phi(x_2) \rangle$$
$$D^{22}(x_1, x_2) = -i \langle \tilde{T} \phi(x_1) \phi(x_2) \rangle$$

As in the case of Fermions, also Boson Green functions obey the condition

$$D^{11} + D^{22} = D^{12} + D^{21}$$

**Boson-fermion interaction matrix structure**

In Keldysh space

$$\psi^\dagger \tilde{T} \phi \psi \equiv \psi^\dagger \begin{pmatrix} \phi^1 & 0 \\ 0 & -\phi^2 \end{pmatrix} \psi \quad \phi^1(x) \quad t \in (-\infty, \infty)$$
$$\phi^2(x) \quad t \in (\infty, -\infty)$$
By making the Keldysh rotation as for fermions

\[ D^K(x_1, x_2) = -i\langle \{ \phi(x_1), \phi(x_2) \} \rangle \]
\[ D^R(x_1, x_2) = -i\Theta(t_1 - t_2)\langle [\phi(x_1), \phi(x_2)] \rangle \]
\[ D^A(x_1, x_2) = -i\Theta(t_2 - t_1)\langle [\phi(x_1), \phi(x_2)] \rangle \]

Notice the anticommutator in the Keldysh component and the commutator in the retarded and advanced components.

By means of the Keldysh rotation \( \psi^\dagger \hat{\phi} \psi \Rightarrow \psi^\dagger \hat{\phi}' \psi' \)

\[ \hat{\phi} = \begin{pmatrix} \phi^1 & 0 \\ 0 & -\phi^2 \end{pmatrix} \Rightarrow \hat{\phi}' = \frac{1}{2} \begin{pmatrix} \phi^1 + \phi^2 & \phi^1 - \phi^2 \\ \phi^1 - \phi^2 & \phi^1 + \phi^2 \end{pmatrix} \equiv \phi^1' \sigma^0 + \phi^2' \sigma^1 \]

- \( \phi^1' \) classical component
- \( \phi^2' \) quantum component

\[ \langle T_K \hat{\phi}'(x_1) \hat{\phi}'(x_2) \rangle = \frac{i}{2} \begin{pmatrix} D^K(x_1, x_2) & D^R(x_1, x_2) \\ D^A(x_1, x_2) & 0 \end{pmatrix} \]
**Interaction propagator in a diffusive (Fermi gas) metal**

As for fermions, the Keldysh component is expressed in terms of the distribution function, which at equilibrium has the standard form of the Bose statistics

\[ \mathcal{D}^K(q, \omega) = \left[ \mathcal{D}^R(q, \omega) - \mathcal{D}^A(q, \omega) \right] \coth \left( \frac{\omega}{2T} \right) \]

The retarded and advanced part are given by the RPA resummation for standard screening

\[ \mathcal{D}^R(q, \omega) = \frac{V(q)}{1 - V(q) \chi(q, \omega)} \quad \chi(q, \omega) = -2N_0 \frac{Dq^2}{Dq^2 - i\omega} \]

<table>
<thead>
<tr>
<th>Regime</th>
<th>Condition</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$Dq^2 &gt; \omega$</td>
<td>$\mathcal{D}^R(q, \omega) \approx 2\pi e^2 \frac{\kappa^2}{\kappa^2_D}$</td>
</tr>
<tr>
<td>II</td>
<td>$Dq^2 &lt; \omega &lt; Dq\kappa^2_D$</td>
<td>$\mathcal{D}^R(q, \omega) \approx -2\pi e^2 \frac{i\omega}{Dq^2}$</td>
</tr>
<tr>
<td>III</td>
<td>$Dq\kappa^2_D &lt; \omega$</td>
<td>$\mathcal{D}^R(q, \omega) \approx \frac{2\pi e^2}{q}$</td>
</tr>
</tbody>
</table>

Various regimes of transferred momentum and frequency

- I. Physics of diffusive modes and interference effects
- II. Physics of coupling to modes of the e.m. environment
Eilenberger equation with interaction: three steps

- Expand in powers of $\phi$

  $\ddot{g} = \ddot{g}^{(0)} + \ddot{g}^{(1)} + \ddot{g}^{(2)} + \ldots,$

  $\ddot{g}^{(0)} = \begin{pmatrix} 1 & 2F \\ 0 & 1 \end{pmatrix}$

- Solve up to quadratic order in $\phi$

  $\partial_t \dot{g}_s - D \partial_x \cdot \dot{g}_s \partial_x \dot{g}_s = i \left[ \hat{\phi}, \dot{g} \right]$

  $\hat{\phi} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}$

- Average over the fluctuations of $\phi$

Notice that for an external (classical) field $\phi^2 = 0$ due to causality. Since for a quantum field $\phi^2 \neq 0$, the Green function has an extra component $g^Z$, which, however, must vanish after averaging.
How it actually works in more detail

The normalization condition $\bar{g}^2 = 1$ makes only two components independent

$$g^R = 1 - \frac{1}{2}(g^{(1)Z} + g^{(1)K} g^{(1)Z}) + \ldots$$

$$g^A = -1 + \frac{1}{2}(g^{(1)K} + g^{(1)Z} g^{(1)K}) + \ldots$$

Up to second order in $\phi$, $g^{(1)Z}$ and $g^{(1)K}$ are enough, because the current operator

$$\langle (\bar{g} \partial_x \bar{g})^K \rangle_\phi = \langle g^R \partial_x g^K + g^K \partial_x g^A \rangle_\phi$$

has a correction of the form

$$\langle \delta(\bar{g} \partial_x \bar{g})^K \rangle_\phi = \partial_x \langle g^{[2]K} \rangle_\phi + \frac{1}{2} \langle [g^{(1)K} (\partial_x g^{(1)Z}) 2F + 2F (\partial_x g^{(1)Z}) g^{(1)K}] \rangle_\phi$$

After averaging, the Keldysh component must be related to retarded and advanced components via the distribution function

$$\langle g^{(2)K} \rangle_\phi = \langle g^{(2)R} \rangle_\phi F - F \langle g^{(2)A} \rangle_\phi$$
First order equations

One obtains non homegenous linear differential equations

\[(\partial_t - D\partial_x^2)g^{(1)K}(\eta, x, t) = 2i \left[ \phi^1(t + \eta/2, x) - \phi^1(t - \eta/2, x) \right] F(\eta, t)\]

\[(\partial_t + D\partial_x^2)g^{(1)Z}(\eta, x, t) = 2i\delta(\eta)\phi^2(t, x)\]

The solution can be written in terms of the appropriate Green function for the diffusion operator

\[g^{(1)K}(\eta, x, t) = 2i \int dt't' x' L_{tt'}(x, x') \left[ \phi^1(t' + \eta/2, x') - \phi^1(t' - \eta/2, x') \right] F(\eta, t')\]

\[g^{(1)Z}(\eta, x, t) = -2i\delta(\eta) \int dt't' x' L_{tt'}(x, x')\phi^2(t', x')\]

The Green function for the diffusion operator corresponds to the ladder resummation of impurity lines in the diagrammatic approach

\[(\partial_t - D\partial_x^2)\mathcal{L}_{tt'}(x, x') = \delta(t - t')\delta(x - x')\]
Density of states

\[
\frac{\delta N}{N_0} = \langle g^{(2)R}(\epsilon, x, t)\rangle_\phi = -i \sum_q \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{D^R(q, \omega)}{(Dq^2 - i\omega)^2} \tanh \left( \frac{\epsilon - \omega}{2T} \right)
\]

- The diffusion poles come from \( g^{(1)Z} \) and \( g^{(1)K} \)
- The distribution function comes from \( g^{(1)K} \)
- The interaction comes from \( \langle \phi^1 \phi^2 \rangle \)

Degree of singularity of the corrections and physical origin

- In 2D for short range interactions the correction is logarithmic

\[
\frac{\delta N(\epsilon)}{N_0} = N_0 V^R(0, 0) t \ln |\epsilon \tau|
\]

and controlled by the parameter \( t = 1/(2\pi)^2 N_0 D \propto 1/g \) as in the weak localization

- This is the famous zero-bias anomaly in DOS

- However for long range interaction the correction is log-square!

\[
\delta N(\epsilon) = -\frac{t}{4} \ln (|\epsilon| \tau) \ln \left( \frac{|\epsilon|}{\tau D^2 \kappa_{2D}^4} \right)
\]

This is due to the fact that the integral is dominated by regime II for transferred momentum and frequency

Regime II corresponds to long distances where interference effects are no longer effective. We will come back to this point.
\[ j = -\frac{en_0D}{2} \int d\epsilon (\tilde{g}_s \partial_x \tilde{g}_s)^K = \delta j_{\text{Born}} + \delta j_{\text{QC}} \]

- Diagrams (c), (d) and (e) correspond to \( \delta j_{\text{Born}} \) and eventually cancel
- (a) and (b) correspond to \( \delta j_{\text{QC}} \) and yield the interaction corrections

Structure of the corrections

Note that $\delta j_{\text{Born}}$

- has the same structure as the density of states
- gives a term proportional to the gradient of the density
- and vanishes at uniform density as in a linear response calculation

$$\delta j_{\text{Born}} = -D \partial_x \frac{eN_0}{2} \int_{-\infty}^{\infty} d\epsilon \delta \langle g^{(2)K}(\epsilon, x, t) \rangle_{\phi} = -D \partial_x \delta \rho(x, t)$$

$\delta j_{\text{QC}}$

- has three ladders due to the internal derivative
- the factor $Dq^2$ and $F \partial_x F$ make relevant regime I for the Coulomb interaction
- yields the quantum interaction correction to the electrical conductivity

$$\delta j_{\text{QC}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\epsilon \sum_q (F(\epsilon, x) \partial_x F(\epsilon - \omega, x)) \frac{D^2 q^2 D^R(q, \omega)}{(Dq^2 - i\omega)^3}$$

$$\delta \sigma = -\frac{e^2}{2\pi^2 \hbar} \ln \left( \frac{1}{T_T} \right)$$
Interference of scattering from impurities and self-consistent potential

An electron moves along path A or B
The phase difference between paths A and B is due to the extra closed loop of path B

A background electron goes around the same closed loop along path C and interacting with the other electron cancels the extra phase of path B

Interference between paths A and B

Phenomena in regime I

\[ l < L < L_T, \quad \omega > T \]
Effect of electromagnetic environment

Transport across a junction

Environmental impedance \( Z(\omega) \)

Getting a charge across the Junction has an energy cost

\[ E_C = \frac{e^2}{2C} \]

Coulomb blockade

But electrons get energy from the environment

Quantum fluctuating e.m. modes of the environment weaken the blocking

Phenomena in regime II

\[ L_T < L < \frac{D\kappa}{\omega}^{2d} , \quad \omega < T \]
Summary of the fourth lecture

- We have set the formalism for e-e interaction in disordered systems
- Corrections to density of states and conductivity
- Identification of two important regimes of effects
5. Thermal transport and Coulomb blockade

1. Definition of heat current
2. Quantum corrections and W-F law
3. Transport through a tunnel junction
4. Coulomb Blockade
Some history

Pre-history
- Wiedemann-Franz
- Boltzmann-Drude-Sommerfeld

Classical period
- Chester and Tellung (1960)
- Langer (1962)

Modern period
- Livamov, Reizer, and Sergeev (1991)
- Niven and Smith (2003)

\[ \frac{\kappa}{\sigma T} = L_0 = \frac{\pi^2 k_B^2}{3e^2} \]

General validity based on:
- Independent quasiparticles
- Fermion statistics
- Elastic scattering

With quantum corrections
Heat current at semiclassical level

• Consider the energy density in terms of the quasiclassical Green function

\[ \rho_Q = \pi N_0 (-i \partial_\eta) g_s^K (\eta, x, t)|_{\eta=0} = -N_0 \int_{-\infty}^{\infty} d\epsilon \, \epsilon g_s^K (\epsilon, x, t) \]

• Check that it gives the correct value for the specific heat of the electron gas by using

\[ g_s^K (\eta, x, t) = -\frac{2i}{\pi} \mathcal{P} \frac{1}{\eta} + 2i\pi \frac{T^2}{6} \eta + \ldots \]

so that

\[ \rho_Q = \frac{2\pi^2 N_0}{6} T^2 \]

• Derive each term of the Eilenberger equation with respect to \( \eta \), the relative time, set \( \eta = 0 \) and take the angle average

\[ \partial_t (-i \partial_\eta) g_s^K (\eta, x, t)|_{\eta=0} + \partial_x \frac{v_F}{d} (-i \partial_\eta) g_p^K (\eta, x, t)|_{\eta=0} = 0 \]

• One gets a continuity equation for the energy and an expression for the energy current

\[ j_Q = \frac{\pi N_0 v_F}{d} (-i \partial_\eta) g_p^K (\eta, x, t)|_{\eta=0} \]
A fourier transform

\[ a \frac{\epsilon + i \pi}{2} \epsilon \left( \frac{\epsilon}{2\pi} \right) \Rightarrow \int_0^\infty \frac{d\epsilon}{2\pi} e^{-i \epsilon (\eta + i0^+)} f_n \left( \frac{\epsilon}{\pi T} \right) \]

\[ + \int_0^\infty \frac{d\epsilon}{2\pi} e^{-i \epsilon (\eta - i0^+)} f_n \left( \frac{\epsilon}{2\pi T} \right) \]

\[ \kappa_n = (2n+1) \pi \frac{T}{\pi} \]

\[ - \frac{2\pi I}{2\pi} + \sum_{n=0}^\infty \left[ \frac{-(2n+1)\pi \pi T (\eta + i0^+)}{e} - \frac{(2n+1)\pi \pi T (\eta - i0^+)}{e} \right] \]

\[ = - \frac{iT}{2} \left[ \frac{\Lambda}{\sinh(\pi T (\eta + i0^+))} + \frac{\Lambda}{\sinh(\pi \pi T (\eta - i0^+))} \right] \]
The semiclassical result

- The p-wave is related to the s-wave by the usual relation
  \[ g^p_K(\eta, x, t) = -l \partial_x g^s_K(\eta, x, t) \]

- Space dependence via local equilibrium \( T(x) \)
  \[ -l \partial_x g^K_s(\eta, x, t) = -2i\pi l \frac{T(x)}{3} \eta \partial_x T(x) \]

- By taking the derivative with respect to \( \eta \)
  \[ -i \partial_{\eta} \left( -2i\pi l \frac{T(x)}{3} \eta \partial_x T(x) \right) = (-\partial_x T(x)) 2\pi l \frac{T(x)}{3} \]

- The thermal current becomes
  \[ j_Q = (-\partial_x T(x)) \frac{\pi^2}{3} \frac{2N_0 v_F l}{d} T(x) = (-\partial_x T(x)) \frac{\pi^2}{3e^2 \sigma_D} T \]
Quantum Corrections to Heat Current

The strategy is as for electrical current:

- There are two terms

  \[ \delta j_Q = \frac{N_0 D}{2} \int d\epsilon \epsilon \delta(\hat{g}_s \partial_x \hat{g}_s)^K = \delta j^a_Q + \delta j^b_Q \]

- Instead of the charge vertex \( e \), there appears an energy vertex \( \epsilon \)

- The first term has the same structure as the density of states

  \[ \delta j^a_Q = DN_0 \nabla T \int d\epsilon \int \frac{d\omega}{2\pi} \partial_T(F_{\epsilon-\omega}(\mathbf{x})F_{\epsilon}(\mathbf{x})) \times \text{Im} \sum_q \frac{D^R(\omega, \mathbf{q})}{(-i\omega + D\mathbf{q}^2)^2} \]

- The second term instead is like the quantum corrections to the electrical conductivity

  \[ \delta j^b_Q = DN_0 \nabla T \int d\epsilon \int \frac{d\omega}{2\pi} F_{\epsilon}(\mathbf{x})\partial_T F_{\epsilon-\omega}(\mathbf{x}) \times \frac{4}{d} \text{Im} \sum_q \frac{D\mathbf{q}^2 D^R(\omega, \mathbf{q})}{(-i\omega + D\mathbf{q}^2)^3} \]
Conclusions about the Wiedemann-Franz law

- Previous literature shows some controversy
  - Livanov, Reizer, Sergev (1991) finds extra terms that violate W-F
  - Niven and Smith (2003) also find extra terms
  - Apparently no explanation for the different results
  - Our analysis clarifies the issue

- The final result shows an extra term which violates the W-F law
  \[
  \kappa = \frac{\pi^2}{3} \frac{k_B^2 T}{e^2} \left( \sigma_D + \delta \sigma + \frac{1}{2} \frac{e^2}{\pi \hbar} \ln\left( \frac{\hbar D \kappa_2}{k_B T} \right) \right)
  \]

- The term that is in agreement with W-F comes from integration of diffusive modes corresponding to energies \( T < \omega < \tau^{-1} \) which is regime I

- The term violating W-F is due to energies \( \omega < T \) in regime II and is beyond the reach of RG analysis since the singularity is purely infrared
The Coulomb blockade theory

The physics associated to the violation of the W-F is also clearly seen when considering the Coulomb interaction affecting transport through a tunnel junction

In the Eilenberger equation we neglect space dependence but for which side the quasiclassical Green function belongs to

\[ \partial_t \tilde{g}_i(\eta, t) = i\hbar \hat{\phi}_i, \tilde{g}_i, \hat{g}_i, i = L, R \]

The solution can be expressed as a gauge transformation, since the fluctuating Hubbard-Stratonovich field depends on time only

\[ \tilde{g}_i(\eta, t) = e^{i\hat{\phi}_i(t+\eta/2)} \tilde{g}_{0,i}(\eta, t) e^{-i\hat{\phi}_i(t-\eta/2)} \]

\[ \hat{\phi}_i(t) = \int dt \hat{\phi}_i(t) \]

If only one electrode were present the field \( \phi \) could be eliminated by a gauge transformation. This explains why the singularities in the density of states drop out from physical quantities.
We need boundary conditions connecting the quasiclassical Green functions on the two sides of the junction.

In general, for arbitrary tunnel transmittance this is a difficult problem.

We confine to the tunneling limit within the Bardeen model with tunneling amplitude $T_{LR}$:

$$\check{\Sigma}_L = T_{LR} \check{G}_R(0, 0) T_{LR}^* \quad \check{\Sigma}_R = T_{RL} \check{G}_L(0, 0) T_{RL}^*$$

The tunneling current is then the Keldysh component of the commutator:

$$j = \frac{G_T}{8e} \int_{-\infty}^{\infty} d\epsilon \langle [\check{g}_L(\epsilon, t), \check{g}_R(\epsilon, t)]^K \rangle_\phi$$

$G_T \propto T_{LR} T_{LR}^*$ tunneling conductance.

In the absence of interaction standard tunneling theory.
How interaction affects tunneling by changing the density of the states

- Gauge transformation: eliminate the interaction from one lead, say the left one
- The average over the fluctuating field factorizes

\[ j = \frac{G_T}{8e} \int_{-\infty}^{\infty} d\epsilon (g^R_L - g^A_L)(g^R_R - g^A_R)(F_R - F_L) \quad F_{L,R} = \tanh \left( \frac{\epsilon + eV_{L,R}}{2T} \right) \]

What is left is the average over the \( \phi \) field of the retarded Green function

\[ \langle g^R_R(t_1, t_2) \rangle_{\phi} = \langle \left( e^{i\hat{\phi}(t_1)} \right)_{1i} \left( \begin{array}{cc} \delta(t_1 - t_2) & 2F(t_1, t_2) \\ 0 & -\delta(t_1 - t_2) \end{array} \right)_{ij} \left( -e^{i\hat{\phi}(t_1)} \right)_{j1} \rangle_{\phi} \]

The matrix structure is associated to the quantum component \( \phi^2 \)

\[ e^{\pm i\hat{\phi}(t)} = e^{\pm i\varphi^1(t)} \left( \begin{array}{cc} \cos \varphi^2(t) & \pm i \sin \varphi^2(t) \\ \pm i \sin \varphi^2(t) & \cos \varphi^2(t) \end{array} \right) \langle e^{i\phi} \rangle_{\phi} = e^{-(1/2)\langle \phi^2 \rangle} \]
The effect of the interaction is condensed into a single function

\[ J(t) = e^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \text{Im} D^R(\omega) \left[ \frac{1 - \cos(\omega t)}{\omega^2} - i \frac{\sin(\omega t)}{\omega} \right] \]

and in the Fourier transform of its exponential

\[ \mathcal{P}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} e^{J(t)} \]

The correction to the density of states depends on the spectral density of the interaction

\[ \delta \langle g^R_R(\epsilon) \rangle_\phi = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{2} \left( F_R(\epsilon + \omega) - F_R(\epsilon - \omega) \right) \mathcal{P}(\omega) \]

When an electron tunnels through the junction, a charge is added to the (right) lead.

The charge creates an extra potential.

After some time, the charge relaxes and the lead goes back to a charge neutrality condition.

The process may be represented in terms of an equivalent circuit with a capacitor and a resistor in parallel.
Resistor-Capacitor Model: continued

The solution of the circuit equations allows to obtain the effective interaction

\[ Q(t) = qe^{-t/\tau} \]
\[ U(t) = \frac{q}{C} e^{-t/\tau} \]

where \( \tau = RC \)

\[ \partial_t Q(t) = q\delta(t) \]
\[ U(t) = \int_{-\infty}^\infty dt' \mathcal{D}^R(t - t')Q(t') \]

which defines the screened interaction

By setting \( q = e \) and defining the charging energy \( E_c = e^2/(2C) \)

\[ J(t) = E_c \int_{-\infty}^\infty \frac{d\omega}{\pi \tau} \frac{1}{\omega^2 + \tau^{-2}} \left[ \frac{1 - \cos(\omega t)}{\omega^2} - \frac{i \sin(\omega t)}{\omega} \right] \]

In the limit of large \( \tau \)

\[ J(t) \approx iE_c t - E_c Tt^2 \]

and for \( T \to 0 \)

\[ \mathcal{P}(\omega) = 2\pi \delta(\omega + E_c) \]

- Suppression of the density of states and blocking of tunneling
- Resummation of the zero-bias anomaly in DOS

\[ \langle g_R^R(\epsilon) \rangle_\phi = \Theta(\vert \epsilon \vert - E_c) \]
Summary of the fifth lecture

- Thermal transport: continuity equation and heat current
- Two types of corrections to the heat current
- Coulomb blockade
6. Superconductivity

1. General equation
2. Landau-Ginzburg limit
3. Andreev scattering and ZBA
Quasiclassical formulation

Eilenberger's eq.
\[ \partial_t \{ \hat{\sigma}_z, \hat{g} \} - i \epsilon [\hat{\sigma}_z, \hat{g}] - D \partial_x \hat{g} \partial_x \hat{g} = i [\hat{\Delta}, \hat{g}] \quad \hat{g} \hat{g} = \mathbf{1} \]

We consider the diffusive limit, but the clean limit can be also analyzed

Keldysh space
\[ \hat{\sigma}_z = \begin{pmatrix} \hat{\sigma}_z & \hat{0} \\ \hat{0} & \hat{\sigma}_z \end{pmatrix}, \quad \hat{g} = \begin{pmatrix} \hat{g}^R & \hat{g}^K \\ \hat{0} & \hat{g}^A \end{pmatrix}, \quad \hat{\Delta} = \begin{pmatrix} \hat{\Delta} & \hat{0} \\ \hat{0} & \hat{\Delta} \end{pmatrix} \]

The phase of the order parameter is fixed

Nambu space
\[ \hat{g}^{R,A} = G^{R,A} \hat{\sigma}_z + i F^{R,A} \hat{\sigma}_y, \quad \hat{g}^K = \hat{g}^R \hat{i} - \hat{i} \hat{g}^A, \quad \hat{f} = f_0 \hat{\sigma}_0 + f_z \hat{\sigma}_z \]

The distribution function \( f_0 \pm f_z \) refer to particles and holes, respectively

Self-consistency
\[ \hat{\Delta} = \frac{gN_0 \pi}{2} \hat{g}^K (\eta = 0, x, t) \quad \hat{\Delta} = i \Delta \hat{\sigma}_y \]

Larkin A I and Ovchinikov Y N in *Non equilibrium superconductivity* 1986 eds.
Langeberg D N and Larkin A I (NorthHolland)
Uniform equilibrium case

Distribution functions

\[ f_0 = \tanh \left( \frac{\epsilon}{2T} \right) \quad f_z = 0 \]

Normalization condition

\[ (G^R)^2 - (F^R)^2 = 1 \]

Explicit solution

\[ G^{R,A}(\epsilon) = \pm \frac{\epsilon}{\sqrt{(\epsilon + i0^+)^2 - \Delta^2}} \]
\[ F^{R,A}(\epsilon) = \pm \frac{\text{sign}(\epsilon)\Delta}{\sqrt{(\epsilon + i0^+)^2 - \Delta^2}}, \]

\[ \Delta = \frac{gN_0\pi}{2} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) (F^R(\epsilon) - F^A(\epsilon)) \]
Direct evaluation from BCS theory

The same expression can of course be obtained by the $\xi$-integration of the BCS Green function.

In BCS we have

$$u_p^2 = \frac{1}{2} \left( 1 \pm \frac{\xi_p}{E_p} \right) \quad \nu_p^2 = 1 - u_p^2 \quad E_p^2 = \xi_p^2 + \Delta^2$$

$$G^R = \frac{i}{\pi} \int_{-\infty}^{\infty} d\xi_p \left( \frac{u_p^2}{\epsilon - E_p + i0^+} + \frac{\nu_p^2}{\epsilon + E_p + i0^+} \right)$$

$$= \int_{-\infty}^{\infty} d\xi_p \left( u_p^2 \delta(\epsilon - E_p) + \nu_p^2 \delta(\epsilon + E_p) \right)$$

$$= \int_{0}^{\infty} d\xi_p \delta(\epsilon - E_p)$$

$$= \frac{\epsilon}{\sqrt{(\epsilon + i0^+)^2 - \Delta^2}}$$
Landau-Ginzburg Equation for the order parameter

- Consider the Eilenberger equation for the anomalous part $F$ and express $G$ using the normalization condition

\[-2i\epsilon \mp D\partial_x^2) F^{R,A}(\epsilon, x) = \mp 2i\Delta(x)(1 + (1/2)(F^{R,A}(\epsilon, x))^2)\]

- At first limit to linear term in $\Delta$ and $F$

\[F^{R,A}(\epsilon, x) = \pm \int d' x' P^{R,A}_\epsilon(x - x')\Delta(x')\]

- The Green function $P$ plays the role of propagator of the Cooper pair

\[-2i\epsilon \mp D\partial_x^2) P^{R,A}_\epsilon(x - x') = -2i\delta(x - x')\]

- To solve the integral equation, expand the order parameter

\[\Delta(x') = \Delta(x' + x - x) = \Delta(x) + (x' - x) \cdot \partial_x \Delta(x') + \frac{1}{2} (x' - x)^i (x' - x)^j \partial_{ij}^2 \Delta(x) + \ldots\]

- Plug $F$ into the self-consistency equation

\[gN_0 \left[ \ln \frac{T}{T_c} + \frac{7\zeta_R(3)}{8\pi^2 T^2} \Delta^2(x) - \frac{\pi D}{8T_c} \partial_x^2 \right] \Delta(x) = 0\]
A technical detail: the coefficient of the linear term

\[
\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) (F^R(\epsilon, \mathbf{x}) - F^A(\epsilon, \mathbf{x}))
\]

\[= \Delta(\mathbf{x}) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) \left( \int d\mathbf{x}' \mathcal{P}^R_\epsilon(\mathbf{x} - \mathbf{x}') + \int d\mathbf{x}' \mathcal{P}^A_\epsilon(\mathbf{x} - \mathbf{x}') \right) \]

\[= \Delta(\mathbf{x}) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) \left( \frac{1}{\epsilon + i0^+} + \frac{1}{\epsilon - i0^+} \right) \]

\[\approx 4\Delta(\mathbf{x}) \ln \frac{\omega_D}{T} \]
A technical detail: the coefficient of the derivative term

\[
\frac{1}{6} \partial_x^2 \Delta(x) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) \left( \int d' x' \mathcal{P}_\epsilon^{R}(x') x'^2 + \int d' x' \mathcal{P}_\epsilon^{A}(x') x'^2 \right)
\]

\[
= \frac{i D}{2} \partial_x^2 \Delta(x) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) \left( \frac{1}{(\epsilon + i0^+)^2} - \frac{1}{(\epsilon - i0^+)^2} \right)
\]

\[
= -\frac{i D}{2} \partial_x^2 \Delta(x) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) \partial_\epsilon \left( \frac{1}{(\epsilon + i0^+)} - \frac{1}{(\epsilon - i0^+)} \right)
\]

\[
= \frac{\pi D}{2T}
\]
A technical detail: the coefficient of the cubic term

Solve the cubic term by using the first order result

\[ F^{R,A(3)}(\epsilon, x) = \pm \frac{1}{2} \left( \int d\mathbf{x'} P^{R,A}_\epsilon(\mathbf{x'}) \right)^3 \Delta^3(x) = \pm \frac{1}{2} \frac{\Delta^3(x)}{(\epsilon \pm i0^+)^3} \]

To plug it into the self-consistency condition, integrate over energy

\[
\frac{1}{8} \Delta^3(x) \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tanh \left( \frac{\epsilon}{2T} \right) \left( \frac{1}{(\epsilon + i0^+)^3} + \frac{1}{(\epsilon - i0^-)^3} \right)
\]

\[
= - \frac{1}{\pi^2 T^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}
\]

\[
= - \frac{7 \zeta_R(3)}{8\pi^2 T^2}
\]
**Tunneling current**

- The boundary condition in the tunneling limit as in the case of Coulomb blockade (M. Y. Kuprianov and V. F. Lukichev, Sov. Phys. JETP 64, 139(1988).)

\[ \dot{I} = \frac{\sigma}{e} \dot{g}_R \partial_x \dot{g}_R = \frac{G_T}{2e} [\dot{g}_R, \dot{g}_L], \]

- We allow for the phase of the order parameter

\[ \hat{g}_i^{R(A)} = (iF_i^{R(A)} \sin(\phi_i), iF_i^{R(A)} \cos(\phi_i), G_i^{R(A)}) \]

- The current has two components

\[ j = \frac{G_T}{8e} \int_{-\infty}^{\infty} d\epsilon (I_J + I_PI) \]

- The first term \( I_J \) is the Josephson current which is different from zero only the order parameter on the two sides of the junction has a different phase

\[ I_J = i \sin(\phi_1 - \phi_2) \left[ f_{0R}(F_R^R - F_R^A)(F_L^R + F_L^A) + f_{0L}(F_R^R + F_R^A)(F_L^R - F_L^A) \right] \]

- In the Josephson only the symmetrix part of the distribution function enters

\[ f_{0L,R} = \frac{1}{2} (tanh((\epsilon + eV_{L,R})/2T) + tanh((\epsilon - eV_{L,R})/2T)) \]

Clearly the Josephson current exists even at zero voltage drop

- The second term is called quasiparticle and interference current

\[ I_{PI} = \left[ (G_L^R - G_L^A)(G_R^R - G_R^A) + \cos(\phi_1 - \phi_2)(F_R^R + F_R^A)(F_L^R + F_L^A) \right] (f_{zL} - f_{zR}). \]

- It can be non-zero only when a bias is applied

\[ f_{zL,R} = \frac{1}{2} (\tanh((\epsilon + eV_{L,R})/2T) - \tanh((\epsilon - eV_{L,R})/2T)) \]

- The terms with \( G_R \) and \( G_L \) yield the standard quasiparticle current
- The terms with \( F_L \) and \( F_R \) yield Andreev scattering

Andreev scattering and proximity effect

These two phenomena are two aspects of the same thing: at the S-N interface one electron is scattered back as a hole and this makes the Gorkov’s function \( F \) leaking in the normal side, i.e., yielding the proximity effect.
Andreev scattering in pictures

Normal scattering  Andreev Scattering

\[ G_{NS} = 2 \left( \frac{2e^2}{h} \right) \sqrt{\frac{T_N^2}{(E - T_N)^2}} \]

For \( T_N < 1 \)
\[ G_N \sim T_N \]
\[ G_{NS} \sim T_N^2 \ll G_N \]

Blonder, Tinkham, Klapwijk (1982), Shelankov (1980)
Experiments show an enhancement at low voltage (Kastalskii et al. (1991)) that cannot be explained by the simple BTK theory.

\[ j = \frac{G_T}{8e} \int_{-\infty}^{\infty} d\epsilon \left[ (F_R^R + F_A^R)(F_L^R + F_A^L) \right] (f_{zL} - f_{zR}). \]

\[ F_L^R \neq 0 \quad F_R^R(\epsilon, x), f_z(\epsilon, x) \]

Superconductor at equilibrium

To be determined by solving the kinetic equations
Kinetics equations (Zaitsev (1990), Volkov and Klapwijk (1992))

- First solve the spectral problem
  \[ \partial_x \hat{g}^{R(A)} \partial_x \hat{g}^{R(A)} = i\epsilon \left[ \hat{\sigma}_z, \hat{g}^{R(A)} \right] \]

- Plug the result into the equation for the Kedysh component
  \[ \partial_x (\hat{g}^R \partial_x \hat{g}^K + \hat{g}^K \partial_x \hat{g}^A) = 0 \]

- Express \( \hat{g}^K = \hat{g}^{R\hat{f}} - \hat{f} \hat{g}^A \)
  \[ \partial_x \left[ \partial_x \hat{f} - \hat{g}^R (\partial_x \hat{f}) \hat{g}^A \right] - (\hat{g}^R \partial_x \hat{g}^R) (\partial_x \hat{f}) - (\partial_x \hat{f}) (\hat{g}^A \partial_x \hat{g}^A) = 0 \]

- Finally obtain a diffusion-like equation with an energy and position-dependent coefficient
  \[ \partial_x \left[ (1 - G^R G^A - F^R F^A) \partial_x f_z \right] = 0 \]

\[ f_z(x) = m(x) \frac{f_z(L) - f_z(0)}{m(L)} + f_z(0) \quad m(x) = \int_0^x \frac{dx'}{1 - G^R(x') G^A(x') - F^R(x') F^A(x')} \]
Zero bias anomaly (ZBA)

- On the normal side the diffusion equation with boundary condition (for \( L \to \infty \)) \( F(\epsilon, \infty) = 0 \) yields

\[
F^R(\epsilon, x) = F^F(\epsilon, 0)e^{i\sqrt{2i|\epsilon|/Dx}} = F^R_R e^{i\sqrt{2i|\epsilon|/Dx}}
\]

- To connect the Gorkov’s function on the two sides we use the standard boundary conditions (current conservation) which are valid for a weak proximity effect

\[
\frac{\sigma S}{e} \hat{g}_R^R \partial_R \hat{g}_R^R = \frac{G_T}{2e} [\hat{g}_R^R, \hat{g}_L^L] \quad \Rightarrow \quad iF^R_R = \sqrt{\frac{D}{2i|\epsilon|}} \frac{G_T}{2\sigma S} F^L_L
\]

One sees that the Gorkov function on the normal side is smaller by a factor \( G_T^2 \) with respect to that on the superconducting side but there is square-root of the energy at the denominator

- The smallness of the factor \( G_T^2 \) is compensated by the an effective conductance of the normal side (at low \( T, L_T \gg L \))

\[
G_{NIS} \approx \frac{G_T^2}{G_N^{\text{eff}}} \quad G_N^{\text{eff}} = \frac{\sigma S}{L_T} \quad L_T = \sqrt{\frac{D}{T}}
\]
Summary of all the lectures

• With superconductivity we have ended our tour through the applications of the quasiclassical Green function approach
• My aim has been that of showing the flexibility of the method and its vast range of applicability
• I think that in most cases is more powerful of the conventional diagrammatics, but its application is not straightforward. It requires some physical understanding of the physical phenomenon under study
• A current field of application of the method is to systems with spin-orbit interaction and spin-splitted Fermi surfaces

Thank you for listening!