Introduction to discussion

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3 days on Painlevé equations and their applications
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What this introduction isn't: not a talk, neither a review of most of the results about targeted aspects of Painlevé equations.

What this introduction would like to be: a collection of (few) results, questions and ideas to launch a discussion on open problems about Painlevé equations. Please interrupt!

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I shall focus solely on *continuous* Painlevé equations.

The general questions inspiring this introduction is the following: can we consider the Painlevé equations special functions? What we have for classical special functions?

- Tabulation of values
- Properties: addition and multiplication formulae, poles distributions, critical behaviours (behaviour of solutions close to critical points), asymptotic expansions, representations (integral, Mittag-Leffler ...), total integrals.
- The *connection problem*, i.e. the explicit connection among solutions at different critical points plays a fundamental role for Painlevé equations (see e.g. [Guzzetti])
- The identification of “special cases”, i.e. solutions reducible to simpler known functions.

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The Painlevé Project

Painlevé Project: it proposes an organization and tabulation of the properties (algebraic, analytic, asymptotic, numerical) of the Painlevé functions.
Look at the NIST Digital Library of Mathematical Functions (http://dlmf.nist.gov/)

Some literature

[Bassom et al., ’92], [Clarkson, ’10], [Deift et al., ’95], [Dubrovin et al., ’00], [Fokas et al, ’92], [Gromak et al, ’02], [Guzzetti, ’14], [Hinkannen et al, ’04], [Hone et al, ’13], [Its et al., ’92], [Jimbo et al, ’80], [Joshi et al, ’88], [Kapaev et al, ’89], [Mazzocco, ’01], [Novokshenov, ’14], [Shimomura, ’00], [Steinmetz, ’01], [Umemura, ’98]...

Tabulation of values

This is one of the ending goals. Two main methods:

- Use of expansions (Taylor, Laurent (e.g. Weierstrass functions), Padé approximants (e.g. Riccati equations)) close to regular points or poles, e.g. [Novokshenov, 2014] or [Fornberg and Weideman, 2011].
- To restrict, by means of symmetries, to finite domains (also w.r.t. the parameters) and use the above methods or the properties of the function in the domain. (e.g. Weierstrass $\wp$ function and "Weyl chambers" for P_{IV})

Properties

Bäcklund transformations as difference equations among solutions with different parameters: can be useful to evaluate recursively the values of the Painlevé transcendents? (e.g. Bessel functions) But need a normalization relation.

Painlevé I equation ($u'' = 6u^2 - 6\lambda z - \frac{g_2}{2}$)

$$\tau(z, g_2, \lambda, g_3) = Be^{A(z-p)}\tau(z-p, g_2 + 12\lambda p, \lambda, g_3 + 12\lambda A)$$

(see Weierstrass σ, i.e. $\lambda = 0$). Also

$$u(z) = \frac{1}{z^2} + \sum_{\substack{\text{poles } \Omega \\ \Omega \neq 0}} \left(\frac{1}{(z-\Omega)^2} - \frac{1}{\Omega^2} \right)$$

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Properties continued 1)...

Other interesting representations (for differential equation) The P_{VI} equation

$$y_{xx} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y_x + \\ + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha - \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} - (\delta - \frac{1}{2})\frac{x(x-1)}{(y-x)^2}\right)$$

can be remarkably written in the new variables (z, τ)

$$x = \frac{\vartheta_4^4(\tau)}{\vartheta_3^4(\tau)}, \quad y = \frac{1}{3} + \frac{\vartheta_4^4(\tau)}{3\vartheta_3^4(\tau)} - \frac{4}{\pi^2}\frac{\wp(z|\tau)}{\vartheta_3^4(\tau)}$$

as

$$-\frac{\pi^2}{4}\frac{d^2 z}{d\tau^2} = \alpha\wp'(z|\tau) + \beta\wp'(z-1|\tau) + \gamma\wp'(z-\tau|\tau) + \delta\wp'(z-\tau-1|\tau)$$

Lot of consequences...

In some cases total integral formulae are known, e.g. for the "Ablowitz-Segur" solution of P_{II} [Baik et al., 2011]

$$u(x) \to \mathrm{i}s\mathrm{Ai}(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right) \quad \text{as } x \to -\infty$$
$$\int_{-\infty}^{+\infty} u(y)dy = \frac{1}{2}\log\left(\frac{1+\mathrm{i}s}{1-\mathrm{i}s}\right) \tag{1}$$

Similar formulae hold for other solutions of P_{II} (e.g., the Hastings-McLeod solution of P_{II})

Poles distribution

The poles identify the function (see e.g. the Mittag-Leffler expansion for P_I). See also the critical behaviour of solutions of NLS near the “gradient catastrophe” [Dubrovin et al., 2009].

The asymptotic distribution of poles close to critical points is known for P_{VI} [see e.g. Guzzetti 2014].

The global distribution of poles of P_I, P_{II} and P_{IV} is known from the Nevanlinna value distribution theory for meromorphic functions (see e.g. [Steinmetz, 2002], [Gromak et al. 2002]).

The asymptotic distribution of poles (for large $|z|$) for P_I, P_{II} and P_{IV} is known from the corresponding asymptotics of the solutions in terms of elliptic functions.

The distribution of poles of particular solutions is known (in some cases): e.g. for special solutions or for “truncated solutions” of P_I and P_{II} (Dubrovin conjecture, Novokshenov conjecture...)

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Poles distribution

The exact distribution of poles is known only in few cases.

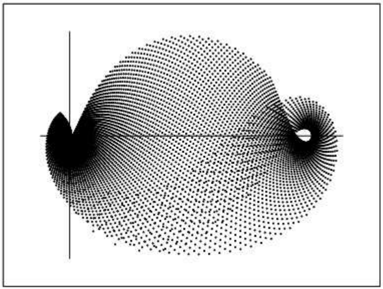

Figure: Picture from [Brezhnev, 2010]

The picture shows the poles of a solution of P_{VI}: the poles are explicitly described by $\Omega_{n,m} = \frac{\vartheta_4^4}{\vartheta_3^4}\left(0, \frac{m-B}{n+A}\right)$, $(n,m) \in \mathbb{Z}$.

Special cases-solutions

Painlevé I equation does not possesses rational solutions. Also it cannot be reduced to Riccati equations.

All rational solutions of P_{II} and P_{IV} are known, also rational solutions of P_{III}-P_V-P_{VI} are known. Also $P_{II} - P_{VI}$ can be reduced, for particular values of the parameters, to Riccati equations (Airy, Bessel, hypergeometric, Whittaker...). Further there are algebraic solutions of $P_{III} - P_V - P_{VI}$ (see e.g. [Dubrovin-Mazzocco] or [Brezhnev, 2010]) for P_{VI}.

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Comments, remarks, observations?