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Painleve equations, elliptic integrals and elementary functions

Abstract. The six Painleve equations can be written in the Hamiltonian forms with time dependent Hamilton functions. We present a rather new approach to this result, leading to rational Hamilton functions. By a natural extension of the phase space one gets corresponding autonomous Hamiltonian systems with two degrees of freedom. We realize the Backlund transformations of the Painleve equations as symplectic birational transformations in C^4 and we interpret the cases with classical solutions as the cases of partial integrability of the extended Hamiltonian systems. We prove that the extended Hamiltonian systems do not have any additional algebraic first integrals besides the known special cases of the third and the fifth Painleve equations. We also show that the original Painleve equations admit the first integrals expressed in terms of the elementary functions only in the special cases mentioned above. In the proofs we use equations in variations with respect to a parameter and Liouville's theory of elementary functions. This is a joint work with G. Filipuk (University of Warsaw).

Painlevé Equations, Elliptic Integrals and Elementary Functions

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Jointly with

Galina Filipuk (University of Warsaw)

Three days on the Painlevé equations and their applications

Rome, December 18-20, 2014

Plan of the Talk:

- The six Painlevé equations are written in the Hamiltonian form with time dependent **rational Hamilton functions**.

- By a natural extension of the phase space one gets corresponding **autonomous Hamiltonian systems with two degrees of freedom**.

- The **Bäcklund transformations** of the Painlevé equations are realized as **symplectic birational transformations in** $\mathbb{C}^4$**.**

- The cases with **classical solutions** are interpreted as the **cases of partial integrability of the extended Hamiltonian systems**.

- It is proved that the extended Hamiltonian systems **do not have any additional algebraic first integral** besides the known special cases of the third and fifth Painlevé equations.

- It is shown that the original Painlevé equations admit the first integrals expressed in terms of the **elementary functions** only in the special cases mentioned above. In the proofs equations in variations with respect to a parameter and Liouville's theory of elementary functions are used.

The Painlevé Equations

$$\ddot{x} = 6x^2 + t \qquad (P_I)$$

$$\ddot{x} = 2x^3 + tx + \alpha \qquad (P_{II})$$

$$\ddot{x} = \frac{\dot{x}^2}{x} - \frac{\dot{x}}{t} + \frac{1}{t}\left(\alpha x^2 + \beta\right) + \gamma x^3 + \frac{\delta}{x} \qquad (P_{III})$$

$$\ddot{x} = \frac{\dot{x}^2}{2x} + \frac{3x^3}{2} + 4tx^2 + 2(t^2 - \alpha)x + \frac{\beta}{x} \qquad (P_{IV})$$

$$\ddot{x} = \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 - \frac{\dot{x}}{t} + \frac{(x-1)^2}{t^2}\left(\alpha x + \frac{\beta}{x}\right) + \frac{\gamma x}{t} + \frac{\delta x(x+1)}{x-1} \qquad (P_V)$$

$$\ddot{x} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\dot{x}^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\dot{x} + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{x^2} + \gamma\frac{t-1}{(x-1)^2} + \delta\frac{t(t-1)}{(x-t)^2}\right), \qquad (P_{VI})$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ are arbitrary complex parameters (and the dot denotes d/dt).

- The Painlevé equations $P_I - P_{VI}$ possess the **Painlevé property**.

- Solutions of $P_I - P_{VI}$ (the Painlevé transcendents) are **meromorphic functions** on the universal cover of $\mathbb{CP}^1 \setminus \{\text{singular points}\}$.

- $P_I - P_{VI}$ are **not integrable** in terms of the known functions.

- The Painlevé equations have **numerous applications** in mathematics and mathematical physics nowadays.

- Equations $P_I - P_{VI}$ can be written in the Hamiltonian form

$$\frac{dx}{dt} = \frac{\partial h}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial h}{\partial x}, \tag{1}$$

  where $h = h(x, y, t)$ is some (time dependent) Hamilton function (papers of K. Okamoto, also J. Malmquist). They have **3/2** degrees of freedom.

Okamoto's (polynomial) Hamiltonians

$$
\begin{aligned}
\tilde{h}_I &= \frac{1}{2}y^2 - 2x^3 - tx = h_I, \\
\tilde{h}_{II} &= \frac{1}{2}y^2 - \left(x^2 + \frac{1}{2}t\right) y - \left(\alpha + \frac{1}{2}\right) x, \\
\tilde{h}_{III} &= \frac{1}{t}\left\{2x^2y^2 + (\theta_0 + \theta_\infty)\eta_\infty \cdot tx - \left[(1 + 2\theta_0)x - 2\eta_0 t + 2\eta_\infty \cdot tx^2\right] y\right\}, \\
\tilde{h}_{IV} &= 2xy^2 - (x^2 + 2tx + 2\kappa_0)y + \theta_\infty x, \\
\tilde{h}_V &= \frac{1}{t}\{x(x-1)^2y^2 - \left[\kappa_0(x-1)^2 + \theta x(x-1) - \eta tx\right] y + \kappa(x-1)\}, \\
\tilde{h}_{VI} &= \frac{1}{t(1-t)}\{x(x-1)(x-t)y^2 + \kappa(x-t) \\
&\quad - \left[\kappa_0(x-1)(x-t) + \kappa_1 x(x-t) + (\theta - 1)x(x-1)\right] y\},
\end{aligned}
$$

where the parameters above are defined explicitly via the parameters $\alpha, \beta, \gamma, \delta$ in P_J.

Extended Hamiltonian Function

$$\frac{dx}{dt} = \frac{\partial h}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial h}{\partial x},$$

After renaming the 'time' t by a new 'coordinate' q, introducing a new 'momentum' p and extending the Hamilton function,

$$H(x, y, q, p) = h(x, y, q) + p, \tag{2}$$

one obtains the **autonomous Hamiltonian system**

$$\dot{x} = H'_y, \quad \dot{y} = -H'_x, \quad \dot{q} = H'_p = 1, \quad \dot{p} = -H'_q. \tag{3}$$

Here the dot denotes differentiation with respect to a new time τ. We shall denote the corresponding vector field by X_H.

E. Horozov and Ts. Stoyanova considered the question of integrability of system (3) in the **Liouville–Arnold** sense (or of its complete integrability). It means that there should exist a function $F(x, y, q, p)$ in involution with H : $\{H, F\} = \dot{F} = 0$. They applied a version of the **Ziglin** method, developed by **J.-P. Ramis and Morales-Ruiz**.

It uses the monodromy group (or the differential Galois group) of the normal variation equation for a particular algebraic solution of the corresponding Hamiltonian system. *In the case of complete integrability with meromorphic first integrals the identity component of this differential Galois group should be abelian.*

Suitable algebraic solutions of the Painlevé equations exist for special values of the parameters. By direct computation of the monodromy group (and, for some equations, of Stokes operators) Horozov and Stoyanova show that **the identity component of the differential Galois group of the normal variation equation is not abelian**. The method works only for **special values of the parameters** (but not discrete).

Our method of proof of the non-integrability is different. By a suitable normalization of the variables we arrive at a perturbation of a completely integrable system with two algebraic first integrals. Then we consider the equation in variations with respect to a parameter (denoted by ε) around a particular solution which is a rather general elliptic curve. Then analysis of few initial terms in powers of ε of a possible first integral of the perturbed system leads to some properties of elliptic integral which cannot be true.

New Hamiltonians

The Painlevé equations are of the **Liénard type**:

$$\ddot{x} = A(x,t)\dot{x}^2 + B(x,t)\dot{x} + C\,(x,t) \tag{4}$$

with rational coefficients (with possible poles at $t = 0$, $t = 1$, $t = \infty$, $x = 0$, $x = 1$, $x = \infty$ and $x = t$).

Let

$$y = \dot{x}/D(x,t).$$

The divergence of the nonautonomous vector field

$$V(x,y,t) = Dy\frac{\partial}{\partial x} + \left\{\left(AD - D'_x\right)y^2 + \left(B - D'_t/D\right)y + C/D\right\}\frac{\partial}{\partial y}$$

equals

$$\mathrm{div}V = \left(2AD - D'_x\right)y + \left(B - D'_t/D\right) = 0,$$

which implies

$$D'_x/D = 2A, \quad D'_t/D = B.$$

Hence, if the condition

$$2A_t' = B_x' \tag{5}$$

is fulfilled, then Eq. (4) takes the **Hamiltonian form in the variables**

$$(x, y) = (x, \dot{x}/D)\,, \tag{6}$$

where

$$D(x,t) = \exp\left(\int^{(x,t)} 2A\mathrm{d}x + B\mathrm{d}t\right). \tag{7}$$

The corresponding Hamilton function is given by

$$h(x,y,t) = D(x,t)\frac{y^2}{2} + h_0(x,t), \tag{8}$$

$$h_0 = -\int^x \frac{C}{D}\mathrm{d}x. \tag{9}$$

Moreover, if the 1−form $2A\mathrm{d}x + B\mathrm{d}t$ has only simple poles with integer residua at them, then the function $D(x,t)$ is rational. If, additionally, the 1−form $\frac{C}{D}\mathrm{d}x$ has vanishing residua at its poles then the Hamilton function (8) is rational.

List of New Hamiltonians

$$
\begin{aligned}
h_I &= \frac{1}{2}y^2 - 2x^3 - tx, \\
h_{II} &= \frac{1}{2}y^2 - \frac{1}{2}x^4 - \frac{1}{2}tx^2 - \alpha x, \\
h_{III} &= \frac{x^2}{t} \cdot \frac{y^2}{2} - \alpha x + \frac{\beta}{x} - \frac{\gamma}{2}x^2 t + \frac{\delta}{2}\frac{t}{x^2}, \\
h_{IV} &= x \cdot \frac{y^2}{2} - \frac{x^3}{2} - 2tx^2 - 2(t^2 - \alpha)x + \frac{\beta}{x}, \\
h_V &= \frac{x(x-1)^2}{t} \cdot \frac{y^2}{2} - \alpha\frac{x}{t} + \frac{\beta}{tx} + \frac{\gamma}{x-1} + \delta\frac{tx}{(x-1)^2}, \\
h_{VI} &= \frac{x(x-1)(x-t)}{t(t-1)} \cdot \frac{y^2}{2} \\
&\quad - \frac{1}{t(t-1)}\left\{\alpha x - \beta\frac{t}{x} - \gamma\frac{t-1}{x-1} - \delta\frac{t(t-1)}{x-t}\right\},
\end{aligned}
$$

where $y = dx/dt$.

Symplectic Bäcklund Transformations

Bäcklund transformations are birational changes of the variables x, t which transform a given equation P_J with given parameters to the same P_J but with different parameters.

In the series of papers of **K. Okamoto** it is proved that these transformations can be extended to the so-called **canonical transformations**

$$(x, y, t, \tilde{h}) \longmapsto (x', y', t', \tilde{h}')$$

which preserve the **canonical form**

$$\widetilde{\Omega} = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}t \wedge \mathrm{d}\tilde{h}.$$

The new Hamiltonian $\tilde{h}' = \tilde{h}'_J$ is from the same list, but with different parameters.

The corresponding changes of the parameters induce the action on the **parameter space**. It turns out that the latter action is equivalent (after a proper choice of coordinates) to an action of some group generated by reflections, **an affine Weyl group** associated with some root system.

The **finite Weyl group** $W(R)$, associated with a root system $R \subset \mathbb{R}^n$, is generated by reflections $s_\alpha : x \longmapsto x - 2\frac{(\alpha,x)}{(\alpha,\alpha)}\alpha,\ \alpha \in R$. They are orthogonal reflections with respect to the hyperplanes $L_\alpha = \{(\alpha,x) = 0\}$.

The **affine Weyl group** $W_a(R)$, associated with the root system R, is generated by the reflections $s_{\alpha,k} : x \longmapsto x - 2\frac{(\alpha,x)-k}{(\alpha,\alpha)}\alpha,\ \alpha \in R,\ k \in \mathbb{Z}$; i.e., by the orthogonal reflections with respect to hyperplanes $L_{\alpha,k} = \{(\alpha,x) = k\}$. Of course, by rescaling the $x \in \mathbb{R}^n$ we can represent the generators of $W_a(R)$ as the above reflections, but with $k \in \mu\mathbb{Z}$ for some $\mu \neq 0$.

$$
\begin{array}{lll}
W_a(A_1) & \text{for} & P_{II}, \\
W_a(B_2) & \text{for} & P_{III}, \\
W_a(A_2) & \text{for} & P_{IV}, \\
W_a(A_3) & \text{for} & P_{V}, \\
W_a(D_4) & \text{for} & P_{VI}.
\end{array}
$$

For new extended Hamilton functions

$$H = H_J(x, y, q, p) = h_J(x, y, q) + p,$$

we want to realize the groups above as the groups of **symplectic transformations in the extended space** with coordinates x, y, q, p and with the **symplectic form**

$$\Omega = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}q \wedge \mathrm{d}p. \tag{10}$$

Equation P_{II}

The **new extended Hamitonian function** is given by

$$H = H^{(\alpha)} = \frac{1}{2}y^2 - \frac{1}{2}x^4 - \frac{1}{2}qx^2 - \alpha x + p.$$

The change

$$(x, y, q, p) = U(x', y', q', p') = \left(x', y' - (x')^2 - \frac{1}{2}q', q', p' - \frac{1}{8}(q')^2 - \frac{1}{2}x'\right)$$

(which is symplectic) transforms the Hamiltonian $H^{(\alpha)}$ to the **extended Okamoto Hamiltonian**

$$U^* H^{(\alpha)} = \tilde{H}^{(\alpha)} = \frac{1}{2}(y')^2 - \left((x')^2 + \frac{1}{2}q'\right)y' - \left(\alpha + \frac{1}{2}\right)x' + p',$$

which equals $\tilde{h}_{II} + p'$.

Define the following **symplectic transformations**

$$(x, y, q, p) \longmapsto \left(x', y', q', p'\right) :$$

$$\begin{aligned}
&S_1 : (x, y, q, p) = S_1(x', y', q', p') = (-x', -y', q', p')\,, \\
&\tilde{S}_2 : (x, y, q, p) = \tilde{S}_2(x', y', q', p') = \left(x' - \tfrac{\alpha+1/2}{y'}, y', q', p'\right), \\
&S_2 = U \tilde{S}_2 U^{-1}.
\end{aligned}$$

We have

$$S_1^* H^{(\alpha)} = H^{(-\alpha)},$$

$$\tilde{S}_2^* \tilde{H}^{(\alpha)} = \tilde{H}^{(-\alpha-1)},$$

$$S_2^* H^{(\alpha)} = H^{(-\alpha-1)}.$$

Therefore, the birational maps S_1 and S_2 are Bäcklund transformations inducing the reflections

$$s_1 : \alpha \longmapsto -\alpha, \quad s_2 : \alpha \longmapsto -\alpha - 1.$$

The latter two maps are reflections generating the affine Weyl group $W_a(A_1)$.

Equation P_{III}, $\gamma = -\delta = 1$

The **new extended Hamitonian function** is given by

$$H^{(\alpha,\beta)} := H_{III}^{(\alpha,\beta,1,-1)} = \frac{x^2y^2}{2q} - \alpha x + \frac{\beta}{x} - \frac{1}{2}qx^2 - \frac{q}{2x^2} + p. \tag{11}$$

The **extended Okamoto Hamiltonian** is given by

$$\tilde{H}^{(\alpha,\beta)} = \frac{x^2}{2q}\left[y^2 + 2\left(\frac{1-\beta}{x} + \frac{q}{x^2} - q\right)y\right] + (\beta - \alpha - 2)\,x + p. \tag{12}$$

It equals $4\tilde{h}_{III}(x, y/4, q) + p$ with $\eta_0 = \frac{1}{2}$, $\eta_\infty = \frac{1}{2}$, $\theta_0 = \frac{1}{2}\beta - 1$, $\theta_\infty = -\frac{1}{2}\alpha$; we also have $\gamma = 4\eta_\infty^2 = 1$, $\delta = -4\eta_0^2 = -1$.

The transformation

$$\begin{aligned} U &= U_\beta : (x, y, q, p) = \left(x', y' + \frac{1-\beta}{x'} + \frac{q'}{(x')^2} - q', q', p' - x' - \frac{1}{x'} + g(q') \right), \\ g(q) &= -\frac{(1-\beta)^2}{2q} + q, \end{aligned}$$

is **symplectic** and has the following property:

$$U^* H^{(\alpha,\beta)} = \tilde{H}^{(\alpha,\beta)}.$$

Define the following symplectic transformations:

$$
\begin{array}{l}
S_1:(x,y,q,p)=(-ix',iy',iq',-ip')\,, \quad i=\sqrt{-1}, \\
S_2:(x,y,q,p)=\left(-1/x',(x')^2y',q',p'\right)\,, \\
\tilde{S}_3:(x,y,q,p)=\left(x'+\frac{\beta-\alpha-2}{y'},y',q',p'+\frac{(\beta-\alpha-2)(\alpha+\beta)}{2q'}\right)\,, \\
S_3=U_\beta\circ\tilde{S}_3\circ U_{\alpha+2}^{-1}.
\end{array} \tag{13}
$$

They imply the following changes in the Hamiltonians:

$$
\begin{array}{rcl}
S_1^*H^{(\alpha,\beta)} & = & -iH^{(\alpha,-\beta)}=-\left(dq'/dq\right)\cdot H^{(\alpha,-\beta)}, \\
S_2^*H^{(\alpha,\beta)} & = & H^{(\beta,\alpha)}, \\
\tilde{S}_3^*\tilde{H}^{(\alpha,\beta)} & = & \tilde{H}^{(\beta-2,\alpha+2)}, \\
S_3^*H^{(\alpha,\beta)} & = & H^{(\beta-2,\alpha+2)}.
\end{array}
$$

In the parameter space we have:

$$
\begin{array}{l}
s_1:(\alpha,\beta)\longmapsto(\alpha,-\beta), \\
s_2:(\alpha,\beta)\longmapsto(\beta,\alpha)\,, \\
s_3:(\alpha,\beta)\longmapsto(\beta-2,\alpha+2)\,.
\end{array} \tag{14}
$$

They are orthogonal reflections with respect to the lines $\{\beta=0\}$, $\{\beta-\alpha=0\}$ and $\{\beta-\alpha-2=0\}$. Such reflections generate the affine Weyl group associated with the root system B_2.

In fact, the changes $\tilde{S}_3$ and S_3 can be generalized to the corresponding changes $\tilde{S}_{\varepsilon,\epsilon}$ and $S_{\varepsilon,\epsilon}$ (where $\varepsilon, \epsilon = \pm 1$ are related with the possible choices of $\eta_0 = \frac{\epsilon}{2}$ and $\eta_\infty = \frac{\varepsilon}{2}$) leading to reflections with respect to the lines $\epsilon\beta - \varepsilon\alpha - 2 = 0$.

Equation P_{IV}

The **new extended Hamitonian function** is given by

$$H_{IV} = H^{(\alpha,\beta)} = \frac{1}{2}xy^2 - \frac{1}{2}x^3 - 2qx^2 - 2q^2x + 2\alpha x + \frac{\beta}{x} + p. \qquad (15)$$

The **extended Okamoto Hamitonian function** is given by

$$\tilde{H}_{\pm}^{(a,b)} = \frac{1}{2}x\left[y^2 + 2\left(\pm(x+2q) - \frac{b}{2x}\right)y\right] \mp ax + p. \qquad (16)$$

Here $\tilde{H}_{-}^{(a,b)}$ equals $4\tilde{h}_{IV}(x, y/4, q) + p$, where $\tilde{h}_{IV}$ is the Okamoto Hamiltonian with $\kappa_0 = \frac{1}{4}b$ and $\theta_\infty = \frac{1}{4}a$, and $\tilde{H}_{+}^{(a,b)}$ is analogously expressed via a modified Okamoto Hamiltonian denoted by $\bar{H}$ in Okamoto's paper. Moreover, we have

$$\alpha = \mp\left(\frac{1}{2}a - \frac{1}{4}b + 1\right), \quad \beta = -\frac{1}{8}b^2. \qquad (17)$$

The Hamiltonians above are related by means of the following symplectic maps:

$$U_{\pm} : (x, y, q, p) = \left(x', y' \pm (x' + 2q') - \frac{b}{2x'}, q', p' \pm bq' \pm 2x'\right),$$

i.e.,

$$U_\pm^* H^{(\alpha,\beta)} = \tilde{H}_\pm^{(a,b)}.$$

Moreover,

$$\begin{aligned} V^* \tilde{H}_-^{(a,b)} &= \tilde{H}_+^{(b-a-4,b)}, \\ V &= V_b = U_-^{-1} \circ U_+. \end{aligned} \tag{18}$$

Below we use $\tilde{H}_-^{(a,b)}$ as a reference Hamiltonian; thus we will get maps in the (a,b) −plane.

Now we introduce the following **symplectic transformations**:

$$\begin{array}{l} S_1 : (x,y,q,p) = \left(x' \frac{x'y'-b}{x'y'-a}, y' \frac{x'y'-a}{x'y'-b}, q', p' \right), \\ S_2 = V_b \circ S_1 \circ V_{b-a-4}^{-1}, \\ S_3 = (x,y,q,p) = \left(x' + \frac{a}{y'}, y', q', p' + 2aq' \right). \end{array} \tag{19}$$

We have

$$\begin{array}{rcl} S_1^* \tilde{H}_{\pm}^{(a,b)} & = & \tilde{H}_{\pm}^{(b,a)}, \\ S_2^* \tilde{H}_{-}^{(a,b)} & = & \tilde{H}_{-}^{(-a-8,b-a-4)}, \\ S_3^* \tilde{H}_{-}^{(a,b)} & = & \tilde{H}_{-}^{(-a,b-a)}. \end{array}$$

Therefore, we get the following changes in the parameter plane:

$$\begin{array}{l} s_1 : (a,b) \longmapsto (b,a), \\ s_2 : (a,b) \longmapsto (-a-8, b-a-4), \\ s_3 : (a,b) \longmapsto (-a, b-a). \end{array} \tag{20}$$

They are involutions. Next, we have $s_1 \circ s_3 : (a,b) \longmapsto (b-a, -a)$, i.e., a linear map equivalent to the rotation of order 3. Therefore, the maps s_1 and s_3 generate the Weyl group $A_2 \simeq S(3)$.

We can realize this group in a subspace of $\mathbb{R}^3$ with zero sum of coordinates:

$$\mathbb{X} = \left\{ v \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \right\}$$

by taking

$$v_1 = \frac{1}{12}(a - 2b), \quad v_2 = \frac{1}{12}(b - 2a), \quad v_3 = \frac{1}{12}(a + b);$$

thus $v \in \mathbb{X}$. Then the reflections (20) take the following form:

$$\begin{array}{l} s_1 : v \longmapsto (v_2, v_1, v_3)\,, \\ s_3 : v \longmapsto (v_1, v_3, v_2), \\ s_2 : v \longmapsto (v_1, v_3 + 1, v_2 - 1)\,. \end{array} \tag{21}$$

We see that s_1 and s_3 are orthogonal reflections in the plane $\mathbb{X}$ with respect to the lines $\{v_1 = v_2\}$ and $\{v_2 = v_3\}$, while s_2 is an orthogonal reflection with respect to the line $\{v_2 = v_3 + 1\}$ (parallel to the first line). These maps are the standard generators of the affine Weyl group $W_a(A_2)$.

Equation P_V, $\delta = -1/2$

The **new extended Hamitonian function** is given by

$$H^{(\alpha,\beta,\gamma)} = H_V|_{\delta=-1/2} = \frac{x(x-1)^2}{2q}y^2 - \alpha\frac{x}{q} + \frac{\beta}{qx} + \frac{\gamma}{x-1} - \frac{qx}{2(x-1)^2} + p.$$

The **extended Okamoto Hamiltonian** is given by

$$\tilde{H}^{(a,b,c)} = \frac{1}{2q}\left\{x(x-1)^2\left[y^2 - \left(\frac{a}{x} + \frac{b}{x-1} + \frac{2q}{(x-1)^2}\right)y\right] + c(x-1)\right\} + p.$$

It equals $2\tilde{h}_V(x, y/2, q) + p$, where $\tilde{h}_V$ is the Okamoto Hamiltonian with $\kappa_0 = a/2$, $\theta = b/2$, $\eta = -1$ and $\kappa = c/4$. Moreover:

$$\alpha = \frac{1}{8}\left\{(a+b)^2 - 4c\right\}, \quad \beta = -\frac{1}{8}a^2, \quad \gamma = -\frac{1}{2}b - 1, \quad \delta = -\frac{1}{2}.$$

We have

$$U^* H^{(\alpha,\beta,\gamma)} = \tilde{H}^{(a,b,c)},$$

where the map

$$\begin{aligned} U &= U_{a,b,c} : \\ & (x,y,q,p) = \left(x', y' - f(x') - \frac{q'}{(x'-1)^2}, q', p' + \frac{1}{x'-1} + g(q') \right), \\ f(x) &= \frac{a}{2x} + \frac{b/2}{x-1}, \quad g(q) = \frac{a(a+b)-2c}{4q} - \frac{a+b}{2} \end{aligned}$$

is symplectic. The Hamiltonian $\tilde{H}^{(a,b,c)}$ will be our reference Hamiltonian.

The first two symplectic Bäcklund transformations are the following:

$$T : (x, y, q, p) = \left(x', y', -q', -p'\right), \qquad \tilde{T} = U_{a,b,c}^{-1} \circ T \circ U_{a',b',c'}. \tag{22}$$

We have

$$\begin{gathered} T^* H^{(\alpha,\beta,\gamma)} = -H^{(\alpha,\beta,-\gamma)}, \\ \tilde{T}^* \tilde{H}^{(a,b,c)} = -\tilde{H}^{(a',b',c')}, \\ a' = a, \ b' = -b - 4, \ c' = c + \tfrac{1}{4}\left\{(a - b - 4)^2 - (a + b)^2\right\}. \end{gathered} \tag{23}$$

The symplectic change

$$S_1 : (x, y, q, p) = \left(x', y' + \frac{a}{x'}, q', p' + a\right) \tag{24}$$

gives

$$S_1^* \tilde{H}^{(a,b,c)} = \tilde{H}^{(-a,b,c-ab)}.$$

The maps

$$S_\pm : (x, y, q, p) = \left(x' \frac{x'y' - a}{x'y' - \lambda_\pm}, y' \frac{x'y' - \lambda_\pm}{x'y' - a}, q', p' + \frac{c - c'}{2q'} \right), \tag{25}$$

where

$$\lambda_\pm = \frac{1}{2}\left(a + b \pm \sqrt{\Delta} \right), \quad \Delta = (a+b)^2 - 4c = 8\alpha$$

and $c' = a\lambda_\mp$, are symplectic and satisfy

$$S_\pm^* \tilde{H}^{(a,b,c)} = \tilde{H}^{(a',b',c')}.$$

The **changes of the parameters** can be expressed in terms of the new parameters a, λ_+, λ_- as follows:

$$(a, \lambda_+, \lambda_-) \longmapsto (\lambda_+, a, \lambda_-), \qquad (a, \lambda_+, \lambda_-) \longmapsto (\lambda_-, \lambda_+, a); \tag{26}$$

they are two transpositions between the roots of the equations $(z - \lambda_+)(z - \lambda_-) = 0$ and $z - a = 0$.

It is useful to introduce another parameter, which replaces c:

$$d = \sqrt{\Delta} = \lambda_+ - \lambda_-$$

(or $d = \lambda_- - \lambda_+$). Thus $\alpha = d^2/8$ and $\lambda_\pm = \frac{1}{2}(a + b \pm d)$.

Then the maps S_1 and $S_\pm$ lead to the following linear changes in the (a, b, d) −space:

$$\begin{aligned} s_1 \quad &: \quad (a, b, d) \longmapsto (-a, b, d) \\ s_\pm \quad &: \quad (a, b, d) \longmapsto \left(\frac{a + b \pm d}{2}, a \mp d, \frac{d \pm a \mp b}{2} \right); \end{aligned}$$

here the third component d' in the image of s_+ is $a - \lambda_-$, and $d' = \lambda_+ - a$ in the case s_-. The maps s_1 and $s_\pm$ are involutions and generate the finite Weyl group associated with the root system $A_3 \simeq S(4)$.

To see this, we use the following linear functions:

$$v_1 = \frac{1}{8}(2a + b), \quad v_2 = \frac{1}{8}(b - 2a), \quad v_3 = -\frac{1}{8}(b + 2d), \quad v_4 = \frac{1}{8}(2d - b);$$

they satisfy $v_1 + v_2 + v_3 + v_4 = 0$.

We find the following form of the above involutions:

$$\begin{aligned} s_1 &: v \longmapsto (v_2, v_1, v_3, v_4)\,, \\ s_+ &: v \longmapsto (v_1, v_3, v_2, v_4)\,, \\ s_- &: v \longmapsto (v_1, v_4, v_3, v_2)\,; \end{aligned} \tag{27}$$

They are orthogonal reflections in the space

$$\mathbb{X} = \left\{ v \in \mathbb{R}^4 : v_1 + v_2 + v_3 + v_4 = 0 \right\}$$

with respect to the planes $\{v_1 = v_2\}$, $\{v_2 = v_3\}$ and $\{v_2 = v_4\}$.

In order to get an additional reflection, which generates the affine Weyl group $W_a(A_3)$ (together with s_1 and $s_\pm$), we use the map $\tilde{T}$ from Eq. (22). In the (a, b, d) variables, it induces the change

$$t : (a, b, d) \longmapsto (a, -b - 4, d) \,.$$

Then the map

$$S_3 = \tilde{T} \circ S_+ \circ \tilde{T}$$

yields the change

$$s_3 : (a, b, d) \longmapsto \left(\frac{a - b + d - 4}{2}, d - a - 4, \frac{a + b + d + 4}{2} \right).$$

In the space $\mathbb{X}$ we get the map

$$s_3 : v \longmapsto (v_4 - 1, v_2, v_3, v_1 + 1) \,, \tag{28}$$

i.e., the reflection with respect to the plane $\{v_4 = v_1 + 1\}$.

Equation P_{VI}

The **new extended Hamitonian function** is given by

$$\begin{aligned} H^{(\alpha,\beta,\gamma,\delta)} &= \frac{1}{2q(q-1)}\{x(x-1)(x-q)y^2 - 2\alpha x + 2\beta\frac{q}{x} \\ &+ 2\gamma\frac{q-1}{x-1} + 2\delta\frac{q(q-1)}{x-q} + 2q(q-1)p\}. \end{aligned}$$

The **extended Okamoto Hamiltonian function** is given by

$$\begin{aligned} \tilde{H}^{(a,b,c,d)} &= \frac{1}{2q(q-1)}\{x(x-1)(x-q) \\ &\left[y^2 - \left(\frac{a}{x} + \frac{b}{x-1} + \frac{c}{x-q}\right)y\right] + d(x-q)\} + p. \end{aligned}$$

It equals $-2\tilde{h}_{VI}(x, y/2, q) + p$, where $\tilde{h}_{VI}$ is the Okamoto Hamiltonian with $\kappa_0 = a/2$, $\kappa_1 = b/2$, $\theta = 1 + c/2$ and $\kappa = d/4$.

They are **related by the symplectic change**

$$U \quad : \quad (x,y,q,p) = \left(x', y' - f(x') - \frac{c/2}{x'-q'}, q', p' + \frac{c/2}{x'-q'} + g(q') \right),$$

$$f(x) \quad = \quad \frac{a}{2x} + \frac{b/2}{x-1}, \quad g(q) = \frac{(a+c)^2 - b^2 + q\left[(a+b)^2 - c^2 - 4d\right]}{8q(q-1)}.$$

Moreover, we have

$$\alpha = \frac{(a+b+c)^2 - 4d}{8}, \quad \beta = -\frac{a^2}{8}, \quad \gamma = \frac{b^2}{8}, \quad \delta = -\frac{c(c+4)}{8}.$$

We have the following symplectic Bäcklund transformations

$$
\begin{aligned}
T_1 &: (x,y,q,p) = \left(1-x', -y', 1-q', -p' + \tfrac{\alpha}{q'(q'-1)}\right),\\
T_2 &: (x,y,q,p) = \left(\tfrac{1}{x'}, -(x')^2 y', \tfrac{1}{q'}, -(q')^2 p' + (\gamma+\delta) q'\right),\\
T_3 &: (x,y,q,p) = \left(\tfrac{x'}{q'}, q'y' + f(x',q'), \tfrac{1}{q'}, -(q')^2 p' - q'x'y' + g(x',q')\right),
\end{aligned}
$$

where

$$
f(x,q) = -\frac{q(q-1)}{(x-1)(x-q)} = \frac{q}{x-1} - \frac{q}{x-q}, \quad g(x,q) = \frac{-q}{x-1}.
$$

We have

$$
T_j^* H^{(\alpha,\beta,\gamma,\delta)} = (dq'/dq) \cdot H^{(\alpha',\beta',\gamma',\delta')}.
$$

They induce the following **parameter changes**:

$$
\begin{aligned}
t_1 &: (\alpha,\beta,\gamma,\delta) \longmapsto (\alpha, -\gamma, -\beta, \delta),\\
t_2 &: (\alpha,\beta,\gamma,\delta) \longmapsto (-\beta, -\alpha, \gamma, \delta),\\
t_3 &: (\alpha,\beta,\gamma,\delta) \longmapsto \left(\alpha, \beta, \tfrac{1}{2}-\delta, \tfrac{1}{2}-\gamma\right).
\end{aligned}
\tag{29}
$$

Other symplectic Bäcklund transformations are given by

$$\begin{aligned}
S_1 &: (x,y,q,p) = \left(x', y' + \tfrac{a}{x'}, q', p' + \tfrac{ac}{2q'}\right), \\
S_2 &: (x,y,q,p) = \left(x', y' + \tfrac{b}{x'-1}, q', p' + \tfrac{bc}{2(q'-1)}\right), \\
S_3 &: (x,y,q,p) = \left(x', y' + \tfrac{c+2}{x'-q'}, q', p' - \tfrac{c+2}{x'-q'} + g(q')\right), \\
S_\pm &: (x,y,q,p) = \left(x' \tfrac{xy-\lambda_\pm}{xy-a}, y' \tfrac{xy-a}{xy-\lambda_\pm}, q', p' + \tfrac{d-d'}{2(q'-1)}\right),
\end{aligned} \tag{30}$$

where

$$g(q) = (c+2)\,\frac{(a+b-4)q+2-a}{2q(q-1)}, \quad d' = a\lambda_\mp.$$

They induce the corresponding **parameter changes:**

$$\begin{aligned}
s_1 &: (a,b,c,d) \longmapsto (-a,b,c,d-ab-ac), \\
s_2 &: (a,b,c,d) \longmapsto (a,-b,c,d-ab-bc), \\
s_3 &: (a,b,c,d) \longmapsto (a,b,-c-4,d-(a+b-2)(c+2)), \\
s_+ &: (a,\lambda_+,\lambda_-) \longmapsto (\lambda_+,a,\lambda_-), \\
s_- &: (a,\lambda_+,\lambda_-) \longmapsto (\lambda_-,\lambda_+,a).
\end{aligned} \tag{31}$$

It is natural to introduce a new parameter (replacing d) :

$$e=\sqrt{(a+b+c)^2-4d}=\lambda_+-\lambda_-,$$

such that $\lambda_\pm=\frac{1}{2}(a+b+c\pm e)$. Then the maps (31) take the following form:

$$\begin{aligned}
&s_1:(a,b,c,e)\longmapsto(-a,b,c,e),\\
&s_2:(a,b,c,e)\longmapsto(a,-b,c,e),\\
&s_3:(a,b,c,e)\longmapsto(a,b,-c-4,e),\\
&s_\pm:(a,b,c,e)\longmapsto\left(\tfrac{a+b+c\pm e}{2},\tfrac{a+b-c\mp e}{2},\tfrac{a-b+c\mp e}{2},\tfrac{e\pm a\mp b\mp c}{2}\right).
\end{aligned}$$

Note that

$$s_4:=s_-\circ s_+\circ s_-:(a,b,c,e)\longmapsto(a,b,c,-e).$$

In the variables

$$v_1 = \frac{a+b}{4}, \quad v_2 = \frac{a-b}{4}, \quad v_3 = \frac{c+e}{4}, \quad v_4 = \frac{c-e}{4}$$

we get the following maps:

$$\begin{aligned}
s_1 &: v \longmapsto (-v_2, -v_1, v_3, v_4)\,, \\
s_2 &: v \longmapsto (v_2, v_1, v_3, v_4)\,, \\
s_3 &: v \longmapsto (v_1, v_2, -v_4 - 1, -v_3 - 1)\,, \\
s_+ &: v \longmapsto (v_1, v_3, v_2, v_4)\,, \\
s_4 &: v \longmapsto (v_1, v_2, v_4, v_3)\,.
\end{aligned} \tag{32}$$

They are orthogonal reflections in $\mathbb{R}^4$ with respect to the hyperplanes $\{v_1 + v_2 = 0\}$, $\{v_1 - v_2 = 0\}$, $\{v_3 + v_4 = -1\}$, $\{v_2 - v_3 = 0\}$ and $\{v_3 - v_4 = 0\}$. It is known that they generate the affine Weyl group $W_a(D_4)$.

The maps t_j from Eq. (29) read as

$$\begin{aligned}
t_1 &: (a, b, c, e) \longmapsto (b, a, c, e)\,, \\
t_2 &: (a, b, c, e) \longmapsto (e, b, c, a)\,, \\
t_3 &: (a, b, c, e) \longmapsto (a, c + 2, b - 2, e)\,.
\end{aligned} \tag{33}$$

The group generated by the maps (31)–(33) is isomorphic to the affine Weyl group $W_a(F_4)$ associated with the root system F_4.

Partial Integrability and Classical Solutions

In the Hamiltonian mechanics, besides the notion of a complete integrability (in the Liouville–Arnold sense), there exists the notion of a **partial integrability**.

In the two degrees of freedom case, this is the situation when the Hamiltonian vector field X_H does not have additional first integrals (only H), but each 3–dimensional level space $\{H = h\}$ contains a 2–dimensional surface $\Sigma = \Sigma_h$ invariant with respect to X_H. This family $\{\Sigma_h\}$ of invariant surfaces is defined by

$$\{f = f(x, y, q, p) = 0, \quad H = h\},$$

where f is a function on the phase space (usually rational).

In some sources it is claimed that the Hamiltonian vector field restricted to the invariant surface, $X_H|_\Sigma$, is integrable. That is, there exists a regular non-constant function $G : \Sigma \longmapsto \mathbb{R}$ which is a first integral for $X_H|_\Sigma$. But this is not the case, in general. For example, in the so-called Hess–Appelrot case in the rigid body dynamics there exists an invariant surface, but without any sensible first integral.

There are the cases of partial integrability for the extended Hamiltonian systems associated with the Painlevé equations with the invariant surface of the form:

$$\Sigma = \{y = E(x, q)\},$$

which, together with the relations $y = \dot{x}/D(x, t)$ and $q = t$, lead to the **Riccati equations** of the form

$$\dot{x} = a(t)x^2 + b(t)x + c(t). \tag{34}$$

Here a, b, c are rational functions with poles at $t = \infty$ and/or at $t = 0, 1$.

It is well known that the Riccati Eq. (34) is related to the second order linear equation

$$\begin{aligned} \ddot{z} + d(t)\dot{z} + e(t)z &= 0, \\ x &= g(t)\dot{z}/z, \end{aligned} \tag{35}$$

where

$$g = -1/a, \quad d = -b - \dot{a}/a, \quad e = ac. \tag{36}$$

Equations (35), for different Painlevé equations, are related to the classical hypergeometric equation (or the Riemann equation).

More precisely, we have:

Airy equation for P_{II},

Bessel equation for P_{III},

Hermite–Weber equation for P_{IV},

confluent hypergeometric equation for P_V,

Gauss hypergeometric equation for P_{VI}.

(In fact, all the above equations can be obtained from the hypergeometric equation by some limit process, like the Painlevé equations $P_I - P_V$ are limits of P_{VI}).

Since the general Riccati equation is not integrable in the ‘mechanical’ sense, the Hamiltonian system restricted to Σ is also non-integrable. Similar happens in the Hess–Appelrot case (mentioned above), where the system restricted to the corresponding invariant surface is equivalent to a Riccati equation with periodic coefficients

If, for some parameters $\alpha, \beta, \gamma, \delta$, the system related to $H_J^{(\alpha,\beta,\gamma,\delta)}$ has an invariant surface Σ and S is a Bäcklund transformation, leading to a change $(\alpha, \beta, \gamma, \delta) \longmapsto (\alpha', \beta', \gamma', \delta')$, then the system related with $H_J^{(\alpha',\beta',\gamma',\delta')}$ has the invariant surface $\Sigma' = S(\Sigma)$.

Usually, the parameters, corresponding to partially integrable Hamiltonians, lie on walls of the Weyl chambers (hypersurfaces of fixed points of reflections in the affine Weyl group).

We can analogously interpret the algebraic solutions to the Painlevé equations. They correspond to 1−dimensional submanifolds which are invariant for X_H and are algebraic.

Example: Invariant Surfaces for P_{II}

The value $\alpha = -\frac{1}{2}$ of the parameter is the fixed point of the reflection $s_2 : \alpha \mapsto -\alpha - 1$.

Let

$$f = y + x^2 + q/2.$$

With the Hamiltonian

$$\tilde{H}^{(-1/2)}(x, f, q, p) = \frac{1}{2} f^2 - \left(x^2 + \frac{q}{2} \right) f + p$$

we get

$$\dot{f} = -\partial \tilde{H}^{(-1/2)} / \partial x = 2xf.$$

Therefore, **the surface**

$$\Sigma_{-1/2} = \{f = 0\} = \left\{y = -x^2 - q/2\right\}$$

is **invariant**. Putting $y = \dot{x}$ and $q = t$ we get the Riccati equation

$$\dot{x} = -x^2 - t/2, \tag{37}$$

which is the Hamiltonian system restricted to $\Sigma_{-1/2}$. The corresponding second order linear equation is the **Airy equation**

$$\ddot{z} = -tz/2, \;\; x = \dot{z}/z. \tag{38}$$

By applying the Bäcklund transformations S, i.e., compositions of the maps S_1 and S_2 we find the surfaces $\Sigma_{n+1/2} = S(\Sigma_{-1/2})$, where $n + 1/2 \in \mathbb{Z} + 1/2$ are half-integer values of the parameter α; $n + 1/2 = s(-1/2)$ where s is the action on the parameter space corresponding to S. For $\alpha = 1/2$ we get the surface $\Xi_{1/2} = \left\{y = x^2 + q/2\right\} = S_1(\Sigma_{-1/2})$. Other surfaces $\Sigma_{n+1/2}$ are more complicated.

We also have a series of 1−dimensional algebraic curves corresponding to algebraic solutions to P_{II}. Indeed, for $\alpha = 0$ we get the particular solution $x(t) \equiv 0$. It corresponds to the invariant curve $\Gamma_0 = \{x = y = 0\}$. By applying to it the Bäcklund transformations we get a series of algebraic curves Γ_n invariant for $X_{H^{(n)}}$, $n \in \mathbb{Z}$.

Theorem 1 *Hamiltonian system*

$$\dot{x} = H'_y, \quad \dot{y} = -H'_x, \quad \dot{q} = H'_p = 1, \quad \dot{p} = -H'_q$$

associated with any of the equations $P_I - P_{VI}$ excluding the cases

(a) $\alpha = \gamma = 0$ *in* P_{III},

(b) $\beta = \delta = 0$ *in* P_{III},

(c) $\gamma = \delta = 0$ *in* P_V

does not admit any first integral which is an algebraic function of x, y, q, p and is independent of H.

Theorem 2 *Any of the equations $P_I - P_{VI}$, excluding the cases (a), (b) and (c) in Theorem 1 above, does not admit a first integral which is an elementary function of x, dx/dt and t.*

Remarks.

The cases (a), (b) and (c) in Theorem 1 are well known. Theorem 1 is not new, only its proof is new. It was proved by V. Gromak using the so-called second Malmquist theorem which states that if a solution $x = \varphi(t)$ to some of the Painlevé equation P_J satisfies an algebraic relation between t, φ and $\dot{\varphi}$, then this relation is of special form: $\dot{\varphi}^m + P_1(t,\varphi)\dot{\varphi}^{m-1} + \ldots + P_m(t,\varphi) \equiv 0$ with $P_j \in \mathbb{C}(t)\,[\varphi]$. Therefore, we have a monic polynomial in $\dot{\varphi}$ with polynomial in φ coefficients.

Concerning Theorem 2 we should mention the works of H. Umemura and H. Watanabe. They apply advanced differential Galois theory to prove non-integrability of some of the Painlevé equations in the class of the so-called classical functions. The classical functions are obtained from rational functions by successive applications of the so-called permissible operations. The latter include: derivation, quadrature, algebraic operations, solutions to linear differential equations, solutions to first order algebraic equations $F(x, \dot{x}) = 0$ and compositions with Abelian functions (like the Weierstrass P–function).

One should expect an 'elementary' version of Theorem 1, instead of the restricted statement in Theorem 2. We are convinced that it is true, but the rigorous proof would be highly complicated. In fact, the main difficulty with the proof of Theorem 2 is in dealing with elementary functions. We follow the book of J. Ritt devoted to presentation of some Liouville's theorems in terms of multivalued analytic functions.

Introduction of a small parameter

The extended Hamilton function takes the form

$$H(X,Y,Q,P) = \frac{1}{2}XY^2 - \frac{1}{2}X^3 - 2QX^2 - 2Q^2X + 2\alpha X + \frac{\beta}{X} + P. \tag{39}$$

Consider the following change:

$$X = x/\mu, \quad Y = y/\mu, \quad P = p/\mu, \quad Q = q/\mu. \tag{40}$$

It is semi-symplectic:

$$\Omega \longmapsto \Omega/\mu^2, \quad H \longmapsto H_\varepsilon/\mu^3, \tag{41}$$

where Ω is the symplectic form and

$$H_\varepsilon = H_0 + \varepsilon p = \left(\frac{1}{2}xy^2 - \frac{1}{2}x^3 - 2qx^2 - 2q^2x + ax + \frac{b}{x}\right) + \varepsilon p$$

with

$$\varepsilon = \mu^2, \quad a = 2\alpha\mu^2, \quad b = \beta\mu^4.$$

We assume that μ (and ε) is small and we treat a and b as other small parameters. Note also that $H_0 = h_{IV}|_{t=q}$.

The Hamiltonian system (3) associated with the function (39) is orbitally equivalent (i.e., by a time rescaling) to the Hamiltonian system associated with H_ε, i.e.,

$$\dot{x} = xy, \quad \dot{y} = -\frac{1}{2}y^2 + \frac{3}{2}x^2 + 4qx + 2q^2 - a + \frac{b}{x^2}, \quad \dot{p} = 2x^2 + 4qx, \quad \dot{q} = \varepsilon. \tag{42}$$

If system (3) has an additional first integral F, then also system (42) has an integral F_ε independent of H_ε.

Unperturbed system

For $\varepsilon = 0$ system (42) is completely integrable with the functions $H_0 = H_\varepsilon|_{\varepsilon=0}$and $H_1 = q$ playing the role of the first integrals in involution.

The common level sets

$$H_0 = h_0, \quad q = q_0$$

are of the form $\Gamma \times \mathbb{C}$ where

$$\Gamma = \Gamma(q_0, h_0) = \left\{(x, y) : xy^2 = x^3 + 4q_0x^2 + cx - 2b/x + 2h_0\right\} \subset \mathbb{C}^2, \tag{43}$$

$c = 4q_0^2 - 2a$ and the line $\mathbb{C} = \{(p, q) : q = q_0\} \subset \mathbb{C}^2$. After the substitution $y = z/x$ we obtain the curve (birationally equivalent with Γ) :

$$\Delta = \left\{z^2 = f(x)\right\}, \tag{44}$$

$$z = xy, \quad f = x^4 + 4q_0x^3 + cx^2 + 2h_0x - 2b, \tag{45}$$

i.e., Γ is an **elliptic curve** (at least for typical values of h_0 and q_0).

The solutions to equation (42) for $\varepsilon = 0$ are the following:

$$x = \mathcal{X}(\tau - \tau_0), \quad y = \mathcal{Y}(\tau - \tau_0), \quad p = \mathcal{P}(\tau - \tau_0), \quad q = q_0.$$

Here $\mathcal{X}(\tau)$, $\mathcal{Y}(\tau) = \dot{\mathcal{X}}/\mathcal{X}$ and $\mathcal{P}(\tau)$ are defined by the following formulas:

$$\tau = \int_{(x_0,y_0)}^{(\mathcal{X},\mathcal{Y})} \frac{\mathrm{d}x}{xy} = \int_{w_0}^{\mathcal{W}} \frac{\mathrm{d}x}{z}, \tag{46}$$

$$\begin{aligned} \mathcal{P} - p_0 &= 2\int_0^{\tau} \left\{ \mathcal{X}^2(s) + 2q_0\mathcal{X}(s) \right\} \mathrm{d}s = 2\int \frac{x + 2q_0}{y} \mathrm{d}x \\ &= 2\int_{w_0}^{\mathcal{W}} \frac{x^2 + 2q_0 x}{z} \mathrm{d}x, \end{aligned} \tag{47}$$

where the integral $\int_{(x_0,y_0)}^{(\mathcal{X},\mathcal{Y})}$ runs along a path in the complex curve Γ from some initial point (x_0, y_0) to the point $(x, y) = (\mathcal{X}(\tau), \mathcal{Y}(\tau))$ (the second integral $\int_{w_0}^{\mathcal{W}}$ in Eq. (46) runs along a path in Δ from $w_0 = (x_0, z_0) = (x_0, x_0 y_0)$ to $\mathcal{W}(\tau) = (\mathcal{X}(\tau), \mathcal{Z}(\tau)) = (\mathcal{X}, \mathcal{X}\mathcal{Y})$). p

Below we fix the initial conditions for (x, y) by putting $\tau_0 = 0$, $y_0 = 0$ and x_0 as some root of the equation $f(x) = 0$; p_0 is the initial value for p.

The second integral in Eq. (46), i.e., $\int \frac{\mathrm{d}x}{z} = \tau$, demonstrates that $\mathcal{X}(\tau)$ can be expressed via the Weierstrass P–function.

Equation in variations with respect to ε

Take the above special solution of the unperturbed system: $x = \mathcal{X}(\tau)$, $y = \mathcal{Y}(\tau)$, $p = \mathcal{P}(\tau)$, $q = q_0$. We consider the **equation in variations with respect to the parameter along this solution**. We substitute

$$x = \mathcal{X}(\tau) + \varepsilon x_1(\tau),\ \ y = \mathcal{Y}(\tau) + \varepsilon y_1(\tau),\ \ p = \mathcal{P}(\tau) + \varepsilon p_1(\tau),\ \ q = q_0 + \varepsilon q_1(\tau), \tag{48}$$

$x_1(0) = y_1(0) = p_1(0) = q_1(0) = 0$, into system (42) and solve it modulo $O(\varepsilon^2)$. It is easy to see that

$$q_1(\tau) = \tau.$$

Therefore, we have the following (linear in ε) relations:

$$\begin{array}{rcl} H_0(x, y, q) + \varepsilon p &=& h_0 + \varepsilon h_1 + \dots \\ q &=& q_0 + \varepsilon \tau. \end{array} \tag{49}$$

Expansion of an independent first integral

Suppose that the Hamiltonian vector field has an elementary first integral $F(X, Y, Q, P)$ independent of H. Then system (42) has the first integral

$$F_\varepsilon(x, y, q, p) = F(x/\mu, y/\mu, q/\mu, p/\mu),$$

$\mu = \sqrt{\varepsilon}$, independent of H_ε.

Lemma. *There exists an elementary first integral $G_\varepsilon(x, y, q, p)$, independent of H_ε and obtained from F_ε by elementary operations involving F_ε, H_ε and ε, such that:*

(i) G_ε has a uniform, with respect to (x, y, q, p) in an open domain $U \subset \mathbb{C}^4$ and ε in a sectorial domain $V \subset (\mathbb{C}, 0)$ with vertex at $\varepsilon = 0$, expansion

$$G_\varepsilon = G_0 + G_1(\varepsilon) + \ldots, \tag{50}$$

where $G_0 = G_0(x, y, q, p)$, $G_j(\varepsilon) = G_j(\varepsilon; x, y, q, p)$ are elementary functions such that $G_{j+1}/G_j \to 0, \ldots$ as $\varepsilon \to 0$;

(ii) the first term in the right-hand side of Eq. (3.12) is of the form $G_0 = \Psi_0(H_0, q)$ and satisfies

$$\frac{\partial \Psi_0}{\partial q}(h_0, q_0) \neq 0$$

for typical (h_0, q_0) ;

(iii) the condition $F_\varepsilon = \mathrm{const}$ *along solutions becomes the condition*

$$G_\varepsilon(x(\tau), y(\tau), q(\tau), p(\tau)) = g_0 + g_1(\varepsilon) + \dots, \tag{51}$$

where g_0 *and* $g_j(\varepsilon)$ *do not depend on* τ *and are of the same order as* G_0 *and* $G_j(\varepsilon)$ *respectively.*

Substituting H_0 and q from Eqs. (49) into Ψ_0 in the equation

$$\Psi_0(H_0, q) + G_1(\varepsilon; x(\tau), y(\tau), q(\tau), p(\tau)) + \ldots = g_0 + g_1(\varepsilon) + \ldots,$$

we obtain the following identity (as a function of τ) :

$$\begin{array}{c} \{G_1(\varepsilon; \mathcal{X}, \mathcal{Y}, q_0, \mathcal{P}) - g_1(\varepsilon)\} + \ldots \\ +\varepsilon \cdot \{A\tau + B\mathcal{P}(\tau) + \Phi(\mathcal{X}, \mathcal{Y}, \mathcal{P})\} + \ldots \quad \equiv 0, \end{array} \tag{52}$$

where

$$A = \frac{\partial \Psi_0}{\partial q}(h_0, q_0) \neq 0, \quad B = -\frac{\partial \Psi_0}{\partial H_0}(h_0, q_0)$$

are constants (see item (ii) in Lemma 2). The term $\Phi(\mathcal{X}, \mathcal{Y}, \mathcal{P})$ is an elementary function corresponding to the term

$$\frac{\partial \Psi_0}{\partial H_0}(h_0, q_0) \cdot \varepsilon h_1 + G_k(\varepsilon; \mathcal{X}, \mathcal{Y}, \mathcal{P}, q_0) - g_k(\varepsilon) = \varepsilon \cdot \{C + \Psi_k(\mathcal{X}, \mathcal{Y}, \mathcal{P}, q_0) - c_k\},$$

when $G_k(\varepsilon) = \varepsilon \cdot \Psi_k(x, y, q, p)$ and $g_k(\varepsilon) = \varepsilon \cdot c_k$.

If F (and G) is algebraic then Φ in (52) is also algebraic. If G depends only on x, y, q then Φ depends only on $\mathcal{X}, \mathcal{Y}$.

Suppose $G_1(\varepsilon) > \varepsilon$ as $\varepsilon \to 0$. Then the term $G_1(\varepsilon; \mathcal{X}, \mathcal{Y}, q_0, \mathcal{P}) - g_1(\varepsilon)$ in Eq. (52) is dominating and, hence, it vanishes, it defines some relation between the functions $\mathcal{X}, \mathcal{Y}, \mathcal{P}$. The same statement holds for other terms in Eq. (3.14) which dominate ε.

However, for the first power of ε Eq. (52) implies the following relation:

$$A\tau + B\mathcal{P}(\tau) \equiv \Phi(\mathcal{X}(\tau), \mathcal{Y}(\tau), \mathcal{P}(\tau)), \qquad A \neq 0. \tag{53}$$

Here we can say the following about $\Phi(x, y, p)$:

either it is algebraic (in the assumptions of Theorem 1) or it is an elementary function of only x, y (in the assumptions of Theorem 2).

Incomplete elliptic integrals

We have

$$\tau = I(x), \quad \mathcal{P}(\tau) = p_0 + J(x),$$

where

$$I(x) = \int_{x_0}^{x} \frac{\mathrm{d}u}{\sqrt{f(u)}}, \quad J(x) = 2\int_{x_0}^{x} \frac{u^2 + 2q_0 u}{\sqrt{f(u)}}\mathrm{d}u \tag{54}$$

are **incomplete elliptic integrals**.

These integrals have **periods** (called also the *complete elliptic integrals*):

$$\omega_1 = \oint_{\gamma_1} \frac{\mathrm{d}x}{z}, \qquad \omega_2 = \oint_{\gamma_2} \frac{\mathrm{d}x}{z}, \tag{55}$$

$$\eta_1 = 2\oint_{\gamma_1} \frac{x^2 + 2q_0 x}{z}\mathrm{d}x, \quad \eta_2 = 2\oint_{\gamma_2} \frac{x^2 + 2q_0 x}{z}\mathrm{d}x. \tag{56}$$

The curves $\gamma_{1,2} \subset \Delta$ generate the first homology group of the Riemann surface Δ. If x_1, x_2, x_3, x_4 are zeroes of the polynomial $f(x)$, then γ_1 (respectively γ_2) is a lift to the Riemann surface of the function $\sqrt{f(x)}$ of a loop which surrounds the points x_1, x_2 (respectively x_1, x_3) in the $x-$plane.

Hence, the Weierstrass function $\mathcal{X}(\tau)$, inverse to $I(x)$, is **doubly periodic**,

$$\mathcal{X}(\tau+\omega_1) = \mathcal{X}(\tau), \quad \mathcal{X}(\tau+\omega_2) = \mathcal{X}(\tau). \tag{57}$$

Also $\mathcal{Y}(\tau) = \dot{\mathcal{X}}(\tau)/\mathcal{X}(\tau)$ is doubly periodic with the same periods.

The function $\mathcal{P}(\tau)$ is not periodic, but it satisfies the following relations:

$$\mathcal{P}(\tau+\omega_1) = \mathcal{P}(\tau) + \eta_1, \quad \mathcal{P}(\tau+\omega_2) = \mathcal{P}(\tau) + \eta_2. \tag{58}$$

We have

$$\begin{vmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{vmatrix} \neq 0$$

for typical values of the parameters q_0 and h_0.

For $A \neq 0$ and any B the incomplete elliptic integral

$$K(x) = AI(x) + BJ(x)$$

is not an elementary function of x.

Proof of Theorem 1 for P_{IV}

Assume relation (53), where Φ is an algebraic function of its arguments. In other words, τ is an algebraic function of the functions $\mathcal{X}(\tau)$, $\mathcal{Y}(\tau)$ and $\mathcal{P}(\tau)$. Let us rewrite the corresponding algebraic equation in the following form:

$$\sum_{m,n} a_{m,n}(\mathcal{X},\mathcal{Y})\tau^m\mathcal{P}^n = 0, \tag{59}$$

where $a_{m,n}$ are polynomials of $\mathcal{X}$ and $\mathcal{Y}$.

Let us replace the function τ with

$$\mathcal{R}(\tau) = \tau - \frac{\omega_1}{\eta_1}\mathcal{P}(\tau). \tag{60}$$

It has the following properties:

$$\mathcal{R}(\tau+\omega_1) = \mathcal{R}(\tau), \quad \mathcal{R}(\tau+\omega_2) = \mathcal{R}(\tau)+\sigma, \quad \sigma = \omega_2 - \omega_1(\eta_2/\eta_1) \neq 0. \tag{61}$$

Equation (59) takes the form

$$\sum_{m,n} b_{m,n}(\mathcal{X},\mathcal{Y})\mathcal{R}^m\mathcal{P}^n \equiv 0. \tag{62}$$

Since only the function $\mathcal{P}$ is not invariant with respect to the translation by ω_1, n must be equal to 0 in the above formula. But then also $m = 0$, because otherwise the left hand side is not invariant with respect to the translation by ω_2.

On the other hand, the degree with respect to $\mathcal{R}$ of the polynomial in equation (62) must be ≥ 1 since equation (59) defines τ as an algebraic function of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{P}$.

The latter contradiction proves Theorem 1 for the fourth Painlevé equation.

Proof of Theorem 2 for P_{IV}

Here Eq. (53) means that the function $A\tau + B\mathcal{P}(\tau)$, $A \neq 0$, is an elementary function of $\mathcal{X}(\tau)$ and $\mathcal{Y}(\tau)$. Taking into account the algebraic nature of $y = \sqrt{f(x)}$, this implies that the function $K(x)$ is an elementary function of x, which is not the case.

Remark. In Differential Galois Theory, besides the class of elementary functions, there exists a class of **generalized Liouvillian functions** (also called the functions expressed in generalized quadratures). Such class is obtained from the field of rational functions on $\mathbb{C}^n$ using the following operations: (a) adding an exponent ($f \longmapsto \exp f$), (b) adding an integral ($f \longmapsto \int f \mathrm{d}x_j$) and (c) adding a solution of an algebraic equation. In the case of elementary functions the operation (b) is replaced by the weaker operation: adding a logarithm ($f \longmapsto \log f$).

Since, in our approach to the integrability/non-integrability problem of the Painlevé equations via the equation in variations, we encounter incomplete elliptic integrals (like $K(x)$) which are evidently primitives of algebraic function, we cannot claim non-integrability of Painlevé equations in the class of generalized Liouvillian functions.

Thank you very much for your attention!