Tetsuji Tokihiro

Irreducibility and co-primeness of terms in discrete equations with respect to initial variables.

Abstract. We study the Laurent property, the irreducibility and co-primeness of discrete integrable and non-integrable equations. First we study a discrete integrable equation related to the Somos-4 sequence, and also a non-integrable equation as a comparison. We prove that the conditions of irreducibility and co-primeness hold only in the integrable case. Then, we generalize our previous results on the singularities of the discrete Korteweg-de Vries equation and the discrete Toda lattice equation with several boundary conditions. We find that irreducibility and co-primeness equally hold in these integrable discrete equations and conclude that they provide a criterion of integrability for discrete nonlinear equations.

Irreducibility and co-primeness of terms in discrete equations with respect to initial variables

Tetsuji Tokihiro (Univ. Tokyo, Japan)

Coworkers: **Masataka Kanki** (Rikkyo Univ.)
Jun Mada (Nihon Univ.)
Takafumi Mase (Univ. Tokyo)

Three days on Painlevé equations and their applications
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Research Purposes

- Clarify *initial value dependence* of solutions to integrable equations

  We investigate solutions of initial value problems in terms of *Laurent property, irreducibility* and *co-primeness*.

- Give a *new discrete integrability criterion*

  In particular, we wish to define an analogue of *singularity confinement test (SCT)** for partial difference equations**.

* Grammaticos-Ramani-Papageorgiou (1991)

** Ramani-Grammaticos-Satsuma (1992)

Today's topic*

* We examine simple mappings related to SOMOS4 sequence and a QRT mapping.
  - ✓ Define the notion of irreducibility and co-primeness of solutions associated with Laurent property.
  - ✓ Irreducibility and co-primeness are conjectured to be a mathematical reinterpretation of SCT.
* We investigate several examples of discrete soliton equations (dKdV eq., dToda eq.) with several boundary conditions.
  - ✓ Show that irreducibility and co-primeness are naturally extended and applied to partial difference equations as an integrability criterion.
* We also show recent results on generalised Hietarinta-Viallet equations [Hietarinta-Viallet 1998].

* M. Kanki, J. Mada, TT, J. Phys. A: Math. Theor. **47**, 065201 .
M. Kanki , T. Mase, J. Mada, TT, J. Phys. A : Math. Theor. **46**, 5204.
M. Kanki, J. Mada, TT, arXiv: 1412.1167

Example : nonlinear mappings with a parameter

A mapping related to the QRT mappings [Quispel-Roberts-Thompson 1989].
For γ=1,2, (#) is integrable, but not integrable for γ≥3.
It also passes the SCT iff γ=1,2.

$$(\#): \qquad x_{n+1} = \frac{x_n + 1}{x_{n-1}(x_n)^\gamma} \qquad (\gamma = 1,2,3,\ldots)$$

Ex.) γ=2: $x_0 = u,\ x_1 = -1,\ x_2 = 0,\ x_3 = \infty,\ x_4 = \infty / 0 \times \infty^2 \ldots ?$

SCT: $x_0 = u,\ x_1 = -1 + \varepsilon,\ x_2 = \frac{\varepsilon}{u} + O(\varepsilon^2),\ x_3 = -\frac{u^2}{\varepsilon^2} + O(\varepsilon^{-1}),$

$$x_4 = -\frac{\varepsilon}{u} + O(\varepsilon^2),\ x_5 = -1 - \varepsilon + O(\varepsilon^2),\ x_6 = u + O(\varepsilon),$$

$$x_7 = -\frac{1+u}{u^2} + O(\varepsilon),\ \ldots.$$

Initial value dependence (γ=1)

$$x_0 = u,\ x_1 = t,\ x_2 = \frac{P_2}{tu},\ x_3 = \frac{P_3}{tP_2},\ x_4 = \frac{tuP_4}{P_2P_3},\ x_5 = \frac{P_2P_5}{uP_3P_4},\ x_6 = \frac{P_3P_6}{tuP_4P_5},\ x_7 = \frac{uP_4P_7}{P_5P_6},\ x_8 = \frac{tP_5P_8}{P_6P_7},\ \ldots\ldots$$

$$P_2 = 1+t,\quad P_3 = 1+t+tu,\quad P_4 = 1+2t+t^2+tu,\quad P_5 = 1+2t+t^2+2tu+3t^2u+t^3u+t^2u^2$$

$$P_6 = 1+3t+3t^2+t^3+u+5tu+8t^2u+5t^3u+t^4u+2tu^2+4t^2u^2+2t^3u^2+t^2u^3$$

$$P_7 = 1+3t+3t^2+t^3+u+7tu+13t^2u+9t^3u+2t^4u+3tu^2+11t^2u^2+12t^3u^2+5t^4u^2+t^5u^2 \\ +3t^2u^3+5t^3u^3+2t^4u^3+t^3u^4$$

$$P_8 = 1+4t+6t^2+4t^3+t^4+2u+11tu+23t^2u+23t^3u+11t^4u+2t^5u+u^2+11tu^2+31t^2u^2 \\ +38t^3u^2+23t^4u^2+7t^5u^2+t^6u^2+3tu^3+16t^2u^3+24t^3u^3+14t^4u^3+3t^5u^3+3t^2u^4 \\ +7t^3u^4+3t^4u^4+t^3u^5$$

Initial value dependence (γ=2)

$$x_0 = u,\ x_1 = t,\ x_2 = \frac{Q_2}{t^2 u},\ x_3 = \frac{tuQ_3}{{Q_2}^2},\ x_4 = \frac{Q_2 Q_4}{u{Q_3}^2},\ x_5 = \frac{Q_3 Q_5}{t{Q_4}^2},\ x_6 = \frac{tuQ_4 Q_6}{{Q_5}^2},\ x_7 = \frac{Q_5 Q_7}{tu^2{Q_6}^2},\ x_8 = \frac{uQ_6 Q_8}{{Q_7}^2},\ \ldots\ldots$$

$$Q_2 = 1+t,\ Q_3 = 1+t+t^2u,\ Q_4 = 1+2t+t^2+tu+t^2u+t^3u^2$$

$$Q_5 = 1+3t+3t^2+t^3+u+3tu+3t^2u+t^3u+2t^2u^2+3t^3u^2+t^4u^2+t^4u^3$$

$$Q_6 = 1+4t+6t^2+4t^3+t^4+u+3tu+6t^2u+7t^3u+3t^4u+3t^2u^2+6t^3u^2+5t^4u^2+2t^5u^2 \\ +3t^4u^3+3t^5u^3+t^6u^4$$

$$Q_7 = 1+5t+10t^2+10t^3+5t^4+t^5+2u+11tu+24t^2u+26t^3u+14t^4u+3t^5u+u^2+6tu^2 \\ +\cdots+9t^6u^4+3t^7u^4+4t^6u^5+3t^7u^5+t^8u^6$$

$$Q_8 = 1+6t+15t^2+20t^3+15t^4+6t^5+t^6+3u+18tu+45t^2u+60t^3u+45t^4u+18t^5u+3t^6u \\ +3u^2+\cdots+4t^9u^6+5t^8u^7+6t^9u^7+t^{10}u^8$$

Initial value dependence (γ=3)

$$x_0 = u,\ x_1 = t,\ x_2 = \frac{R_2}{t^3 u},\ x_3 = \frac{t^5 u^2 R_3}{R_2{}^3},\ x_4 = \frac{R_2{}^5 R_4}{t^{12} u^5 R_3{}^3},\ x_5 = \frac{t^{19} u^8 R_3{}^5 R_5}{R_2{}^{12} R_4{}^3},\ x_6 = \frac{R_2{}^{19} R_4{}^5 R_6}{t^{45} u^{19} R_3{}^{19} R_5{}^2},\ \ldots\ldots$$

$$R_2 = 1+t,\quad R_3 = 1+t+t^3 u,\ R_4 = 1+3t+3t^2+t^3+t^5u^2+t^6u^2+t^8u^3$$

$$\begin{aligned}R_5 = {} & 1+8t+28t^2+56t^3+70t^4+56t^5+28t^6+8t^7+t^8+t^5u^2+6t^6u^2+15t^7u^2+20t^8u^2+15t^9u^2 \\ & +6t^{10}u^2+t^{11}u^2+t^8u^3+5t^9u^3+10t^{10}u^3+10t^{11}u^3+5t^{12}u^3+t^{13}u^3+t^{12}u^5+3t^{13}u^5+3t^{14}u^5 \\ & +t^{15}u^5+3t^{15}u^6+6t^{16}u^6+3t^{17}u^6+3t^{18}u^7+3t^{19}u^7+t^{21}u^8\end{aligned}$$

$$\begin{aligned}R_6 = {} & 1+21t+210t^2+1330t^3+5985t^4+20349t^5+54264t^6+116280t^7+203490t^8+293930t^9 \\ & +352716t^{10}+352716t^{11}+293930t^{12}+203490t^{13}+\cdots(216\text{terms})\cdots \\ & +28t^{51}u^{19}+8t^{52}u^{20}+8t^{53}u^{20}+t^{55}u^{21}\end{aligned}$$

Observation - Proposition

* The terms x_n is expressed as $x_n = t^a u^b \frac{P_n P_{n-3}}{P_{n-1} p_{n-2}}$ for $\gamma = 1,$ and $x_n = t^a u^b \frac{Q_n Q_{n-2}}{(Q_{n-1})^2}$ for $\gamma = 2,$ where $a, b \in \mathbb{Z}$ and polynomials, $P_n(t,u)$, $Q_n(t,u)$ are irreducible and co-prime.
* For $\gamma \geq 3,$ x_n consists of all the polynomials R_k $(1 \leq k \leq n)$: $x_n = t^a u^b \frac{R_n (R_{n-2})^{l_2} (R_{n-4})^{l_4} (R_{n-6})^{l_6} \cdots}{(R_{n-1})^{l_1} (R_{n-3})^{l_3} (R_{n-5})^{l_5} (R_{n-7})^{l_7} \cdots}$, and hence x_n contains all factors in x_j $(j < n)$.

☆ Proposition 1 [Kanki-Mada-Mase-T 2014, Kanki 2014]

The above observation is true.

Relation to SCT

- Singularities (0 or ∞) will not propagate for $\gamma = 1,2$.

  ∵) If there exists an infinite sequence, $x_k = x_{k+l_1} = x_{k+l_2} = \cdots = \infty$ (for $\gamma = 2$), that implies that $Q_{k-1} = Q_{k+l_1-1} = Q_{k+l_2-1} = \cdots = 0$.

  But there is no common zero* for all these distinct irreducible polynomials

- Singularities propagate for $\gamma \geq 3$.

  ∵) $x_k = \infty \Longrightarrow R_{k-1} = 0 \Longrightarrow x_{k+2} = x_{k+4} = x_{k+6} = \cdots = \infty$

Observation: SCT is strongly related to the irreducibility and co-primeness.

* Precisely speaking, no 1D manifold on which $Q_i = Q_j = 0$ $(i \neq j)$.

Claim

* Co-primeness of the terms as functions of initial values can be used as a new integrability criterion.
* This new criterion is a mathematical reinterpretation of SCT.

Hereafter we give the precise definition of the notions, irreducibility, co-primeness etc., and show results on discrete soliton equations

definition1

* The Laurent polynomial ring, $K \coloneqq \mathbb{Z}[a^{\pm}, b^{\pm}, \cdots]$, is a set of polynomials of $a^{\pm 1}, b^{\pm 1}, \cdots$ with coefficients in $\mathbb{Z}$. And a unit in K is a reversible element of K.
* $a^2 + b + 1$, $\frac{b^3+c^3+1}{a^5}$, $\frac{a^2+b+2}{a^3b^2d^5} \cdots$ are Laurent polynomials.
* $\frac{a}{1+b}$, $\frac{a^2-1}{a+2b^2+3c}$, $\frac{a^2+3b+2}{a^3+b^2} \cdots$ are not Laurent polynomials.
* $\pm 1, a^2 b, \frac{1}{a^3b^2}, \frac{b}{a^5} \cdots$ are units. (only monomials are units.)
* 2, $\frac{3b}{a^2}$, $a + b$, $\frac{1-c^3}{a^5} \cdots$ are not units in K.

definition2

- In a commutative ring K, $f \in K$ is called irreducible when it holds that if $f = gh$ $(g, h \in K)$, then g or h is a unit of K[#].
- Ex.) In the ring $K = \mathbb{Z}[a^{\pm}, b^{\pm}]$, the Laurent polynomials $a^2, \frac{a}{b^2}$, 3, $a^2 - 2b^2$, $\frac{a+b}{a^2}$, $\frac{a^3+2b^2}{a^2b^3}$, $\frac{3a^2+2ab+b^2+3}{ab^2}$ are irreducible.
- Ex.) In the same ring K, the Laurent polynomials 6, $a^2 - b^2$, $\frac{a^2+2ab+b^2}{a^3b^2}$, $\frac{a^3+b^3-3ab+1}{ab^5}$ are not irreducible (reducible), because $6 = 3 \cdot 2$, $a^2 - b^2 = (a+b)\cdot(a-b)$, $\frac{a^2+2ab+b^2}{a^3b^2} = (a+b)\cdot\frac{a+b}{a^3\,b^2}$ and $\frac{a^3+b^3-3ab+1}{ab^5} = (a+b+1)\cdot\frac{a^2+b^2-2a-2b-2ab+1}{ab^5}$.
- [#] By this definition, a unit is irreducible in this talk.

definition3

* In a commutative ring K, $f \in K$ and $g \in K$ are said to be co-prime when the following condition is satisfied: if f and g is decomposed as $f = f_1 h$ and $g = g_1 h$ $(f_1, g_1, h \in K)$, then h is a unit.
* Ex.) In the Laurent polynomial ring $K = \mathbb{Z}[a^{\pm}, b^{\pm}]$, $\frac{1+2b^2}{a^2 b}$ and $\frac{1-a^2}{ab^2}$ are co-prime, but $\frac{1+a+b+ab}{a^2 b}$ and $\frac{1-a^2}{ab^2}$ are not co-prime because $\frac{1+a+b+ab}{a^2} = (1+a)\frac{1+b}{a^2}$ and $\frac{1-a^2}{b^2} = (1+a)\frac{1-a}{b^2}$.
* Ex.) A unit is co-prime to any element of K.

definition4

- A rational function $f \in \mathbb{Q}(a, b, c, \ldots)$ is decomposed as $f = \frac{f_1}{f_2}$ ($f_1, f_2 \in K :=\mathbb{Z}[a^{\pm}, b^{\pm}, c^{\pm}, \ldots]$) where f_1 and f_2 are co-prime in K. This decomposition is unique up to units. Suppose that $f = \frac{f_1}{f_2}$ and $g = \frac{g_1}{g_2}$ with this decomposition, then we say that f and g are co-prime when f_1, f_2, g_1, g_2 are mutually co-prime in K.
- Ex.) Let $f := \frac{a^2+ab}{1+a^2}$, $g := \frac{a^2+ab+b^2}{ab^2}$, $h := \frac{2+2a^2}{1+2ab^2}$, then f and g are co-prime, but f and h are not co-prime because they have common factor $1 + a^2$.

Sketch of the proof of proposition 1 ($\gamma = 2$)

1. By variable transformation $x_n = \frac{y_{n+3}y_{n+1}}{{y_{n+2}}^2}$, (#) turns to the SOMOS4* sequence: $y_{n+2}y_{n-2} = y_{n+1}y_{n-1} + (y_n)^2$
2. SOMOS 4 has Laurent property in connection with cluster algebra**.
3. Using the key lemma (shown in the next slide), we can prove the irreducibility of terms as functions of initial variables.
4. By inverse transformation of variables, we complete the proof.

NOTE: For γ=1, (#) turns to the SOMOS5 sequence.

*cf.) [Hone-Swart 2008], **[Fomin-Zelevinsky 2002]

key lemma for the proof of irreducibility

- **Lemma 1** [Mase 2013, Kanki-Mase-T 2014]

Let $\{p_1, p_2, \dots, p_n\}$ and $\{q_1, q_2, \dots, q_n\}$ be the sets of variables:

(i) $^\forall j\ \ p_j \in \mathbb{Z}[q_1^\pm, \cdots, q_n^\pm]$ as a function of $\{q_1, q_2, \dots, q_n\}$,

(ii) $^\forall j\ q_j \in \mathbb{Z}[p_1^\pm, \cdots, p_n^\pm]$ as a function of $\{p_1, p_2, \dots, p_n\}$,

(iii) q_j is irreducible as an element of $\mathbb{Z}[p_1^\pm, \cdots, p_n^\pm]$.

If $f \in \mathbb{Z}[p_1^\pm, \cdots, p_n^\pm]$ is an irreducible Laurent polynomial and another Laurent polynomial $g \in \mathbb{Z}[q_1^\pm, \cdots, q_n^\pm]$ satisfies $g(q_1, \dots, q_n) = f(p_1, \dots, p_n)$. Then g is decomposed as

$$g = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} \bar{g}(q_1, \dots, q_n)$$

where $r_1, r_2, \dots, r_n \in \mathbb{Z}$ and $\bar{g}(q_1, \dots, q_n)$ is an irreducible Laurent polynomial.

NOTE: When $\{p_j\}$ and $\{q_j\}$ are cluster variables which are mutually convertible by mutation, they satisfy conditions (i) and (ii).

Example of the key lemma (SOMOS4)

* $p := (p_1, p_2, p_3, p_4) = (y_2, y_3, y_4, y_5)$,
* $q := (q_1, q_2, q_3, q_4) = (y_1, y_2, y_3, y_4)$,
* $p \in \mathbb{Z}[q^{\pm}]$ and, due to the symmetry of SOMOS4, $q \in \mathbb{Z}[p^{\pm}]$. Furthermore q is irreducible w.r.t. p.
* $f(p) = g(q) = y_6$
* $f(p) = \frac{y_5 y_3 + y_4^2}{y_2}$ is irreducible in $\mathbb{Z}[p^{\pm}]$
* From the lemma1, $g(q) = y_5^a g'(q)$. ($g'(q) = g'(y_1, \dots, y_4)$ irrd. and a is a nonnegative integer.)
* Putting $y_1 = y_2 = y_3 = y_4 = 1$, $y_5 = 2$, $y_6 = 3$. Hence, we have $3 = 2^a \times integer$, which means y_6 is irrd. w.r.t. q.

Conjecture

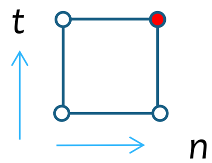
- Conjecture
  For a rational mapping $x_{n+1} = R(x_n, x_{n-1}, \dots, x_{n-l})$, to pass SCT is equivalent to the following property;
  There exists a positive integer M and whenever $|n - m| \geq M$, x_n and $x_m$are co-prime as rational functions of initial variables $\{x_0, x_1, \dots x_l\}$.

NOTE: (1) The property in the above conjecture is claimed to be a mathematical reinterpretation or definition of singularity confinement.
(2) We also claim that it gives a criterion related to integrability.
(3) The above conjecture is extended to partial difference equation in a straightforward manner.

Nonlinear dKdV equation

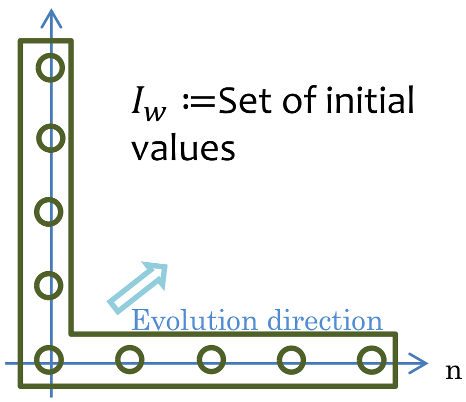
Discrete Korteweg-de Vries (dKdV) equation:

$$\frac{1}{w_{n+1}^{t+1}} - \frac{1}{w_n^t} + \frac{\delta}{1+\delta}\left(w_n^{t+1} - w_{n+1}^t\right) = 0 \quad \left((m,t) \in \mathbb{Z}^2,\ \delta, w_n^t \in K\right)$$



- time evolution:

Red point is determined by 3 white points.



I_w :=Set of initial values

$I_w = \{w_0^t, w_n^0 \mid t \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{>0}\}$

Example) $w_0^0 = a, w_1^0 = b, w_2^0 = c, w_0^1 = d, w_0^2 = e, \delta = 1,$

$$w_1^1 = \frac{2a}{2-ad+ab}, w_1^2 = \frac{2d(2-ad+ab)}{(2-de)(2-ad+ab)+2ad},$$

$$w_2^1 = \frac{2b(2-ad+ab)}{(2+bc)(2-ad+ab)+2ad}$$

We do not have Laurent property.

Theorem 1 (dKdV eq.)

* Theorem 1 [Kanki-Mada-Mase-T2014]

Two terms w_n^t, w_m^s of (nonlinear) dKdV has the following property: They are rational functions of I_w and co-prime with each other if $|n-m| \geq 2$ or $|s-t| \geq 2$ is satisfied.

(This means the only factors they can have in common are monomials of initial variables in I_w .)

(Key point) If two terms are separated by more than one cell, information on the zeros or poles do not propagate.

i.e., Re-formulation of SC in terms of "co-primeness".

(Note1) The result is also true on the field of positive characteristic.

(Note2) It holds for the b.c. $\{w_0^t = 1\}$, where $I_w = \{w_n^0, n \in \mathbb{Z}_{>0}\}$.

Discrete Toda equations

* Discrete Toda equation:

$$\begin{cases} I_n^{t+1} = I_n^t + V_n^t - V_{n-1}^{t+1} \cdots (1) \\ \quad V_n^{t+1} = \frac{I_{n+1}^t V_n^t}{I_n^{t+1}} \cdots (2) \end{cases} \qquad \left((n,t) \in \mathbb{Z}^2, \;\; I_n^t, V_n^t \in K \right)$$

Boundary conditions:

1. Semi-infinite ($V_0^t = 0$),

   the set of initial variables : $I_{IV} = \{ I_n^0, V_n^0; n \in \mathbb{Z}_{>0} \}$

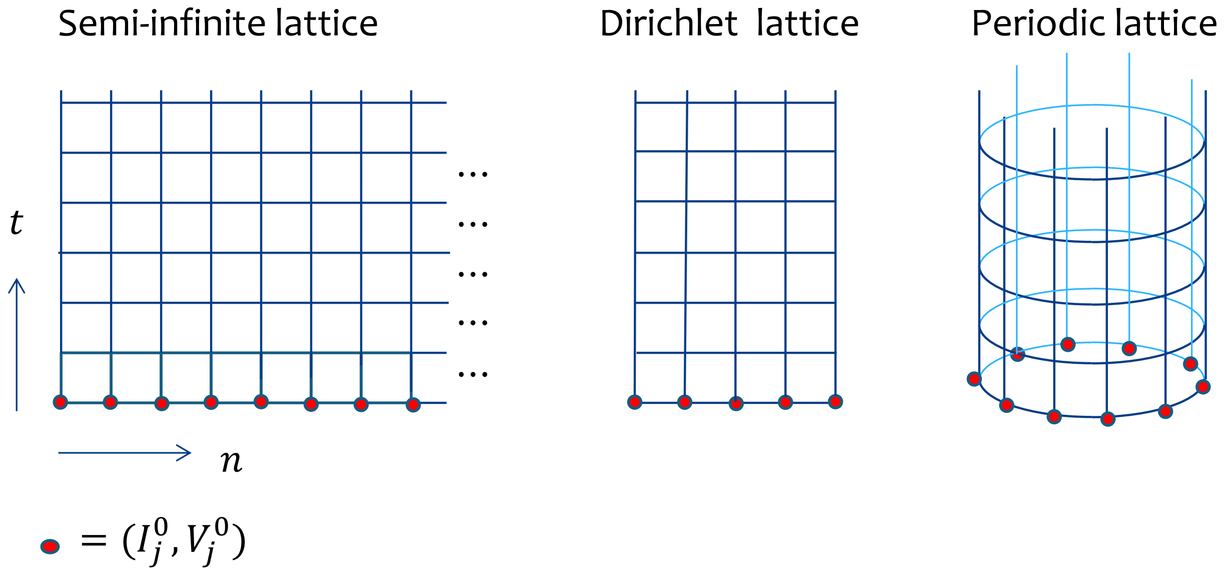
2. Dirichlet ($V_0^t = V_{N+1}^t = 0$),

   $I_{IV} = \{ I_m^0, V_n^0; 1 \le m \le N+1, \; 1 \le n \le N \}$

3. Periodic ($V_n^t = V_{n+N}^t, I_n^t = I_{n+N}^t$),

   $I_{IV} = \{ I_n^0, V_n^0; 1 \le n \le N+1 \}$

Boundary conditions and the set of initial variables



remark 1

* Toda equation has a bilinear form: $\tau_n^{t+1}\tau_n^{t-1} = \tau_{n-1}^{t+1}\tau_{n+1}^{t-1} + (\tau_n^t)^2$ with $I_n^t = \frac{\tau_{n-1}^t \tau_n^{t+1}}{\tau_n^t \tau_{n-1}^{t+1}}$ and $V_n^t = \frac{\tau_{n+1}^t \tau_{n-1}^{t+1}}{\tau_n^t \tau_n^{t+1}}$.
* For the periodic b.c., to avoid a trivial solution, we have to use the following equation instead of Eq.(1) : $I_n^{t+1} = V_n^t + I_n^t Y_n^t$ where
  $$Y_n^t = \left(1 - \frac{\prod_{i=1}^N V_i^t}{\prod_{i=1}^N I_i^t}\right) \Big/ \left(1 + \frac{V_{n-1}^t}{I_{n-1}^t} + \frac{V_{n-1}^t V_{n-2}^t}{I_{n-1}^t I_{n-2}^t} + \cdots + \frac{V_{n-1}^t V_{n-2}^t \cdots V_{n+1}^t}{I_{n-1}^t I_{n-2}^t \cdots I_{n+1}^t}\right)$$
* For Dirichlet b.c. and periodic b.c., the direct relation to cluster algebra is not clear.

remark 2 (τ function for the periodic boundary condition)

- For the periodic boundary condition, we have to consider the following equation for the τ- function:
  $\tau_n^{t+1}\tau_n^{t-1} = \left(1 - \frac{\lambda^2}{\mu}\right)\tau_{n-1}^{t+1}\tau_{n+1}^{t-1} + (\tau_n^t)^2$
- with boundary condition $\tau_{n+N}^t = K\mu^n\lambda^t\tau_n^t$
- where $K = \prod_{i=1}^N \left(V_i^0 I_i^0\right)^{N-i}$, $\lambda = \prod_{i=1}^N V_i^0 I_i^0$ and $\mu = \prod_{i=1}^N I_i^0$.
- To avoid a trivial solution, we have to consider
- $\tau_n^{t+1} = \tau_{n+1}^{t-1}\left[\frac{\lambda^2}{\mu}\sum_{i=0}^n \frac{\left(\tau_i^t\right)^2}{\tau_i^{t-1}\tau_{i+1}^{t-1}} + \sum_{i=n+1}^{N-1} \frac{\left(\tau_i^t\right)^2}{\tau_i^{t-1}\tau_{i+1}^{t-1}}\right]$.

Theorem (dToda eq.)

- Theorem 2 [Kanki-Mada-T 2014]

Terms $I_{n_1}^{t_1}, I_{n_2}^{t_2}, V_{n_3}^{t_3}, V_{n_4}^{t_4}$ of dToda eqs have the following property:

They are rational functions of I_w and co-prime with each other if $|n_i - n_j| \geq 3$ or $|t_i - t_j| \geq 2$ is satisfied for those three types of boundary conditions.

(This means co-primeness holds for the boundary conditios where we do not have the direct relation to the cluster algebra.)

Hietarinta-Viallet equation

- Hietarinta- Viallet (HV) eq. : $x_{n+1} = -x_{n-1} + x_n + \frac{1}{(x_n)^2}$
- HV eq. passes SC test, but is proved nonintegrable (chaotic). Algebraic entropy, S , is proposed as a measure of integrability[Hietarinta-Viallet1998].

$$S \coloneqq \lim_{n\to\infty} \frac{1}{n} \log d_n, \qquad d_n\text{: degree of } x_n \ (n\text{th iterate})$$

- If $S = 0$, the map is integrable w.r.t. algebraic entropy, and non-integrable if $S > 0$.

$$S = \log \frac{3+\sqrt{5}}{2} > 0 \quad \text{for HV eq.}$$

Generalisation of HV equation

* Generalisation of HV equation (GHV eq.):

$$x_{n+1} = -x_{n-1} + x_{n+1} + \frac{1}{(x_n)^k} \qquad (k = 2,3,4,\dots)$$

* Theorem 3 [Kanki-Mase-T]

  If k is an even integer, terms x_m, x_n of GHV eqs. satisfy co-primeness when $|n-m| \geq 4$. If k is an odd integer, there is no co-primeness.

  The degree of x_n satisfies

$$d_{n+1} = (k+1)d_n - (k+1)d_{n-2} - d_{n-3} \quad (k: even)$$

$$d_{n+1} = (k+1)d_n - kd_{n-2} \quad (k: odd)$$

* Algebraic entropies are given as $S = \frac{k+1+\sqrt{(k-1)(k+3)}}{2}$ for $k\ even$, and as $S = \frac{k+\sqrt{k(k+4)}}{2}$ for $k\ odd$.

Conclusion

☆We demonstrate that irreducibility and co-primeness could be an integrability criterion by an example of nonlinear mappings.

- ✓ This criterion is a mathematical reinterpretation of SC.
- ✓ We examined typical integrable partial difference equations and found that they have these properties irrespective of the boundary conditions.

➢ Future problems: construction of new PΔEs w.r.t. co-primeness, relation to other integrbility criterion , application to integrable cellular automaton, finite fields etc.

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* Thank you !