Hidetaka Sakai

Rational surfaces and geometry of the Painlevé equations

Abstract. Hamiltonian systems with biquadratic Hamiltonian can be solved in terms of elliptic functions. These systems are closely related to rational elliptic surfaces. The Painlevé equations are "good" non-autonomous analogue of these systems. The non-autonomous systems are also related to rational surfaces, but they are not elliptic surfaces. When we investigate these surfaces, we easily find affine Weyl group symmetry of the systems. Particular solutions called "Riccati solutions" are also caught by this consideration.

Rational Surfaces and Geometry of the Painlevé Equations

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Content

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Hamilton system with biquadratic Hamiltonian

We consider a Hamiltonian system

$$\mathcal{H}: \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dq}{dt} = -\frac{\partial H}{\partial q}$$

with biquadratic Hamitonian

$$H = (p^2, p, 1) \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} q^2 \\ q \\ 1 \end{pmatrix}.$$

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Proof.

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\begin{aligned}
\frac{dq}{dt} &= (2p, 1, 0)\, M \begin{pmatrix} q^2 \\ q \\ 1 \end{pmatrix} \\
&= 2(m_{11}q^2 + m_{12}q + m_{13})p + m_{21}q^2 + m_{22}q + m_{23},
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\begin{aligned}
\left(\frac{dq}{dt}\right)^2 &= (m_{21}q^2 + m_{22}q + m_{23})^2 \\
&\quad - (m_{11}q^2 + m_{12}q + m_{13})(m_{31}q^2 + m_{32}q + m_{33} - C).
\end{aligned}
$$

□

- Look at the case $M = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & m_{23} \\ 0 & m_{32} & 0 \end{pmatrix}$, $m_{11}m_{12}m_{21} \neq 0$.

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Putting $Q = 1/q$, we have

$$\frac{dQ}{dt} = -(2p, 1, 0)M \begin{pmatrix} 1 \\ Q \\ Q^2 \end{pmatrix}, \quad \frac{dp}{dt} = -(p^2, p, 1)M \begin{pmatrix} 2/Q \\ 1 \\ 0 \end{pmatrix}.$$

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$$\theta_1 := dQ + (2p, 1, 0)M\begin{pmatrix} 1 \\ Q \\ Q^2 \end{pmatrix}dt, \quad \theta_2 := Qdp + (p^2, p, 1)M\begin{pmatrix} 2 \\ Q \\ 0 \end{pmatrix}dt,$$

$$\theta_1 \wedge \theta_2 = QdQ \wedge dp - Q(2p, 1, 0)M\begin{pmatrix} 1 \\ Q \\ Q^2 \end{pmatrix}dp \wedge dt$$
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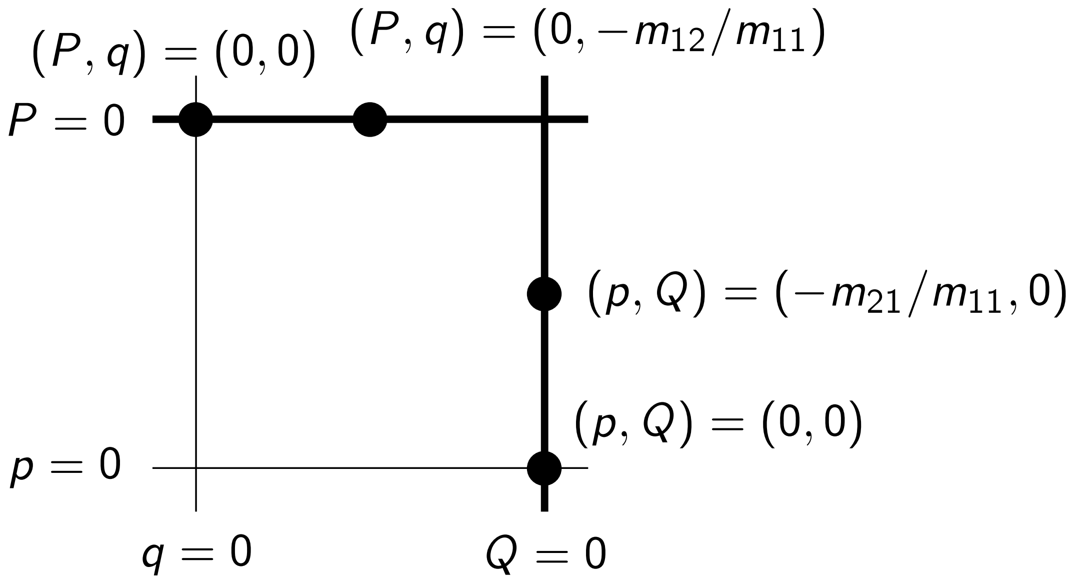
vertical leaf : $Q = 0$, singular points : $(p, Q) = (0, 0)$, $(-m_{21}/m_{11}, 0)$.


$(P,q)=(0,0)$
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$P=0$
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$(p,Q)=(0,0)$
$p=0$
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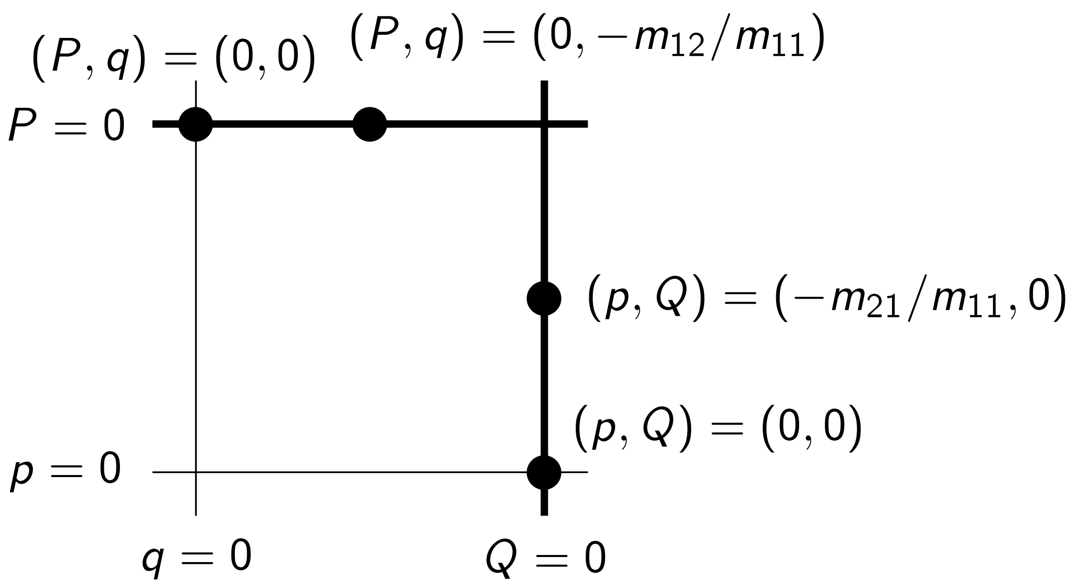
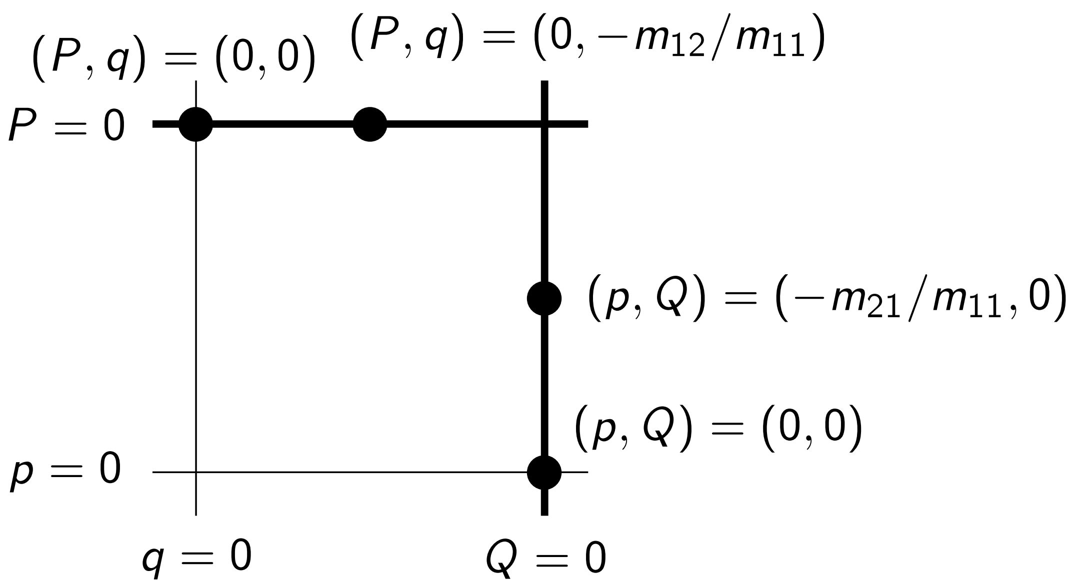


These 4 points are “accessible singular” points.



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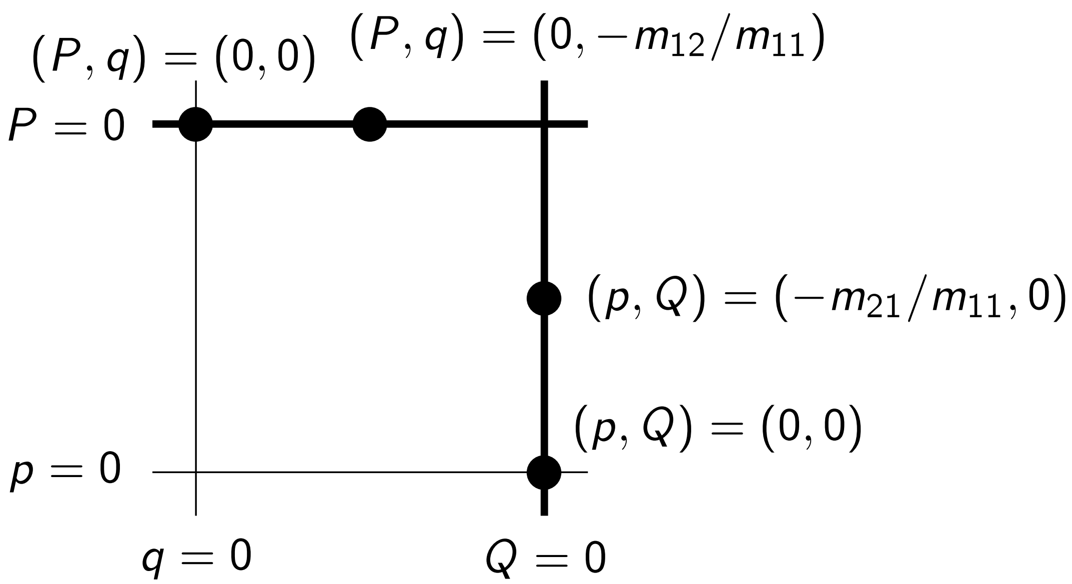

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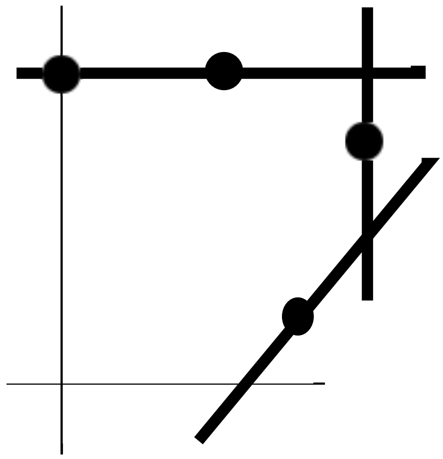
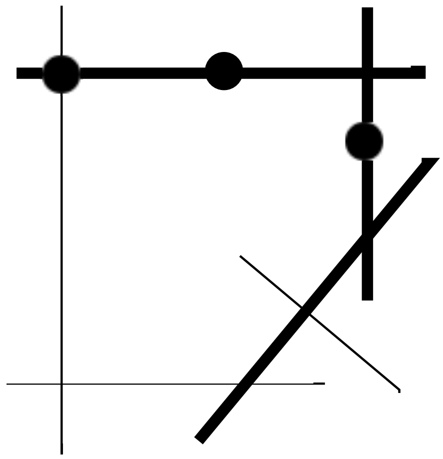
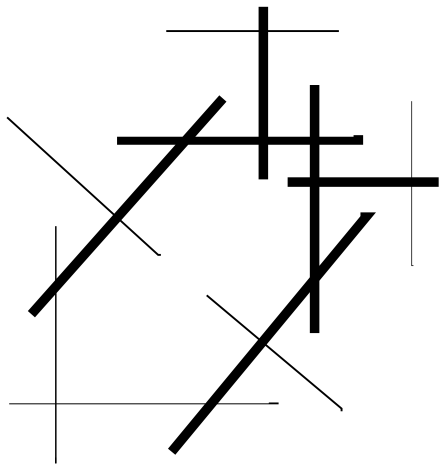
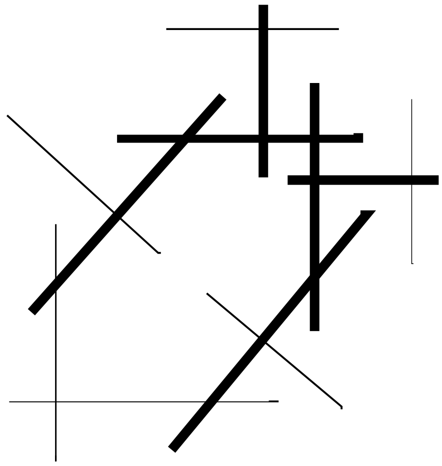
$\leftarrow$ 1-parameter family of solutions is passing through each point.

$\leftarrow$ We need "bowing-ups" to draw distinction between each solutions.









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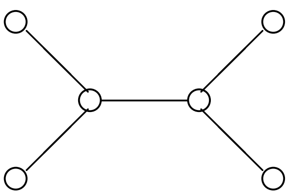
dual diagram



$D_5^{(1)}$

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- When $m_{11} \neq 0$, you can take $m_{13} = m_{31} = 0$ by an affine transformation.
  - $m_{12}m_{21} \neq 0 \quad \Rightarrow \quad D_5^{(1)}$
  - $m_{12}m_{21} = 0$ and $(m_{12}, m_{21}) \neq (0, 0) \quad \Rightarrow \quad D_6^{(1)}$
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- $m_{11} = 0$
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Non-autonomous case

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$$m_{32}/m_{21} = \text{const}, \qquad m_{23}/m_{12} = \text{const},$$
$$\frac{m_{32}}{m_{21}} - \frac{m_{22}}{m_{11}} + \frac{m_{12}m_{21}}{{m_{11}}^2} - \frac{1}{m_{21}}\frac{d}{dt}\left(\frac{m_{21}}{m_{11}}\right) = \text{const},$$
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for resolution of accessible singularities. ($\sim$ Painlevé analysis)

$$\Longrightarrow \qquad M = \begin{pmatrix} 1 & e^t & 0 \\ -1 & -(a_1 + a_3 + e^t) & e^t a_2 \\ 0 & a_1 & 0 \end{pmatrix}.$$

$$M_{D_5} = \begin{pmatrix} 1 & e^t & 0 \\ -1 & -(a_1 + a_3 + e^t) & e^t a_2 \\ 0 & a_1 & 0 \end{pmatrix},$$

$$M_{D_6} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -a_1 - b_1 & -e^t \\ 0 & -a_1 & 0 \end{pmatrix}, \quad M_{D_7} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_1 & -1 \\ 0 & e^t & 0 \end{pmatrix},$$

$$M_{E_6} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -t & -a_1 \\ 0 & -a_2 & 0 \end{pmatrix}, \quad M_{E_7} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & t \\ 0 & a_1 & 0 \end{pmatrix}.$$

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Painlevé equations

$\mathcal{H}_{\mathrm{V}} = \mathcal{H}_{D_5}$, $\mathcal{H}_{\mathrm{III}}(D_6) = \mathcal{H}_{D_6}$, $\mathcal{H}_{\mathrm{III}}(D_7) = \mathcal{H}_{D_7}$, $\mathcal{H}_{\mathrm{IV}} = \mathcal{H}_{E_6}$, $\mathcal{H}_{\mathrm{II}} = \mathcal{H}_{E_7}$.

Content

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elliptic	$A_0^{(1)}$
multiplicative	$A_0^{(1)*}$, $A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}, \ldots, A_6^{(1)}$, $A_7^{(1)}$, $A_7^{(1)\prime}$, $A_8^{(1)}$
additive	$A_0^{(1)**}$, $A_1^{(1)*}$, $A_2^{(1)*}$, $D_4^{(1)}$, $D_5^{(1)}$, $D_6^{(1)}$, $D_7^{(1)}$, $D_8^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$

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equations	P_{VI}	P_{V}	$P_{\mathrm{III}}(D_6)$	$P_{\mathrm{III}}(D_7)$	$P_{\mathrm{III}}(D_8)$
geometry	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$
symmetry	$D_4^{(1)}$	$A_3^{(1)}$	$(A_1 + A_1)^{(1)}$	$A_1^{(1)}$	-

P_{IV}	P_{II}	P_{I}
$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
$A_2^{(1)}$	$A_1^{(1)}$	-

For $D_4^{(1)}$, and $D_8^{(1)}$, we take Hamiltonians as

$$H = \frac{1}{f_0 f_1 {g_0}^2} ({g_1}^2, g_0 g_1, {g_0}^2) M \begin{pmatrix} {f_1}^2 \\ f_0 f_1 \\ {f_0}^2 \end{pmatrix} = (g^2, g, 1) M \begin{pmatrix} f \\ 1 \\ 1/f \end{pmatrix},$$

where $(g_0 : g_1), (f_0 : f_1) \in \mathbb{P}^1$ express homogenious coordinate.

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$$M_{D_4} = \begin{pmatrix} 1 & -1 - \frac{1}{1-e^t} & \frac{1}{1-e^t} \\ a_1 + 2a_2 & -a_1 - 2a_2 + \frac{a_3 e^t}{1-e^t} + a_4 & \frac{a_4}{1-e^t} \\ a_2(a_1 + a_2) & 0 & 0 \end{pmatrix},$$

$$M_{D_8} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -t & 0 & -1 \end{pmatrix}.$$

(If $s = \frac{1}{1-e^t}$, then $s(s-1)\frac{d}{ds} = \frac{d}{dt}$.)

But, in this case, we have a symplectic form $\omega = \frac{1}{f} dg \wedge df$, and Hamiltonian system is written as

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Remark (D_4)

If we take $G = g/f$, then $\omega = dG \wedge df$ and

$$\begin{aligned} H =& f(f-1)(f-s)G^2 \\ &+ \{(a_1 + 2a_2)(f-1)f + a_3(s-2)f + a_4 s(f-1)\}G + a_2(a_1+a_2)f. \end{aligned}$$

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Remark (D_8)

If we take $G = -fg$, $F = 1/f$, then $\omega = dG \wedge dF$, $H = F^2G^2 - F - t/F$.

Content

Affine Weyl group symmetry

Example (Bäcklund transformations of P_{II})

- $Q = s_1(q) = q + \frac{a_1}{p}, \quad P = s_1(p) = p;$
  (p, q) is a sol of $\mathcal{H}_{\mathrm{II}}(a_1) \quad \Rightarrow \quad (P, Q)$ is a sol of $\mathcal{H}_{\mathrm{II}}(-a_1)$.
- $Q = \pi(q) = -q, \quad P = \pi(p) = -p - q^2 - t;$
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s_1, π, $s_0 = \pi \circ s_1 \circ \pi$ has relations: ${s_1}^2 = {s_0}^2 = \pi^2 = 1$.

s_1 and π generate affine Weyl group $\widetilde{W}(A_1^{(1)})$.

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X: a surface obtained by 9 points blowing-up from $\mathbb{P}^2$, then

$$Pic(X) = \mathbb{Z}\mathcal{E}_0 \oplus \mathbb{Z}\mathcal{E}_1 \oplus \cdots \oplus \mathbb{Z}\mathcal{E}_9,$$

where $\mathcal{E}_0$ is the class of the total transform of a line in $\mathbb{P}^2$, $\mathcal{E}_k$, $k \neq 0$ is the class of the total transform of the exceptional curve.
We have the intersection form with

$$\mathcal{E}_0 \cdot \mathcal{E}_0 = 1, \quad \mathcal{E}_k \cdot \mathcal{E}_k = -1 (k \neq 0), \quad \mathcal{E}_k \cdot \mathcal{E}_l = 0 (k \neq l).$$

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$\Uparrow$

Lorentzian lattice of rank $= 10$

Cremona isometry

Bäcklund transformation

birational, complicated,
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Cremona isometry

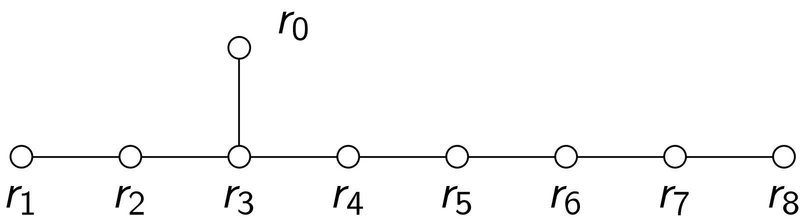
An automorphism σ of $Pic(X)$ is called a Cremona isometry, when

- σ preserves the intersection form in $Pic(X)$,
- σ leaves the canonical class $\mathcal{K}_X$ fixed,
- σ preserves the semi-group of effective classes invariant.

We denote the group of Cremona isometries as $Cr(X)$.

$$\mathcal{K}_X = -3\mathcal{E}_0 + \mathcal{E}_1 + \cdots + \mathcal{E}_9$$

$$\begin{aligned}(\mathcal{K}_X)^\perp &= \{\mathcal{F} \in Pic(X) \mid \mathcal{F} \cdot \mathcal{K}_X = 0\} \\ &= \mathbb{Z}r_0 \oplus \mathbb{Z}r_1 \oplus \cdots \oplus \mathbb{Z}r_8 \quad \simeq \quad Q(E_8^{(1)})\end{aligned}$$



$$r_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \qquad r_k = \mathcal{E}_k - \mathcal{E}_{k+1} \ (k = 1, \ldots 8)$$

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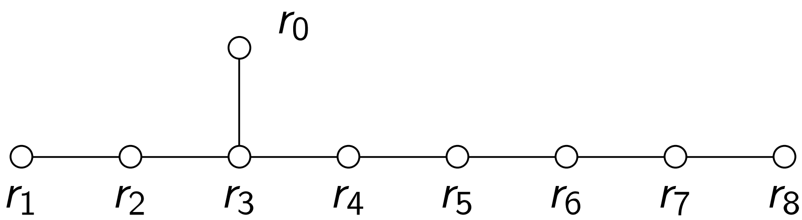

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$$\downarrow$$

$$Cr(X) \subset Aut(Q(E_8^{(1)})) \simeq \widetilde{W}(E_8^{(1)})$$

$$Pic^+(X) \quad = \quad \text{the semi-group of effective classes}$$

is generated by

- $NEFF \cap Pic^+(X)$, $NEFF$: set of numerically effective classes
- EX: set of exceptional classes
- $Comp(D)$: set of classes of irred. comp. of $D \in |-\mathcal{K}_X|$
- Δ^{nod}: set of classes of nodal curves disjoint from $Comp(D)$

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- $NEFF \cap Pic^+(X)$, $NEFF$: set of numerically effective classes
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$$Cr(X) = \widetilde{W}(R^{\perp})_{\Delta^{nod}}$$

Example (Root system of P_{V})

$R = D_5^{(1)}; \qquad Q(D_5^{(1)}) = \mathbb{Z}D_0 \oplus \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_5,$

$D_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad D_2 = \mathcal{E}_2 - \mathcal{E}_8, \quad D_3 = \mathcal{E}_1 - \mathcal{E}_2,$

$D_4 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad D_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_7, \quad D_0 = \mathcal{E}_8 - \mathcal{E}_9,$

$R^{\perp} = A_3^{(1)}; \qquad Q(A_3^{(1)}) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_3,$

$\alpha_1 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_2 = \mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4 - \mathcal{E}_6, \quad \alpha_3 = \mathcal{E}_6 - \mathcal{E}_7,$

$\alpha_0 = 2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_4 - \mathcal{E}_6 - \mathcal{E}_8 - \mathcal{E}_9.$

$\delta = -\mathcal{K}_X = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = D_0 + D_1 + 2D_2 + 2D_3 + D_4 + D_5.$

Example (Root system of P_{II})

$$R = E_7^{(1)}; \qquad Q(E_7^{(1)}) = \mathbb{Z}D_0 \oplus \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_7,$$

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$$D_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5,$$

$$R^\perp = A_1^{(1)}; \qquad Q(A_1^{(1)}) = \mathbb{Z}\alpha_0 \oplus \mathbb{Z}\alpha_1,$$

$$\alpha_1 = \mathcal{E}_4 - \mathcal{E}_5, \quad \alpha_0 = 3\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3 - 2\mathcal{E}_4 - \mathcal{E}_6 - \cdots - \mathcal{E}_9.$$

By using $Pic(X)$, π is expressed by a matrix

$$\begin{pmatrix} 4 & -1 & -1 & -1 & -3 & 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 3 & -1 & -1 & -1 & -2 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Discrete Painlevé equations

Symmetry of the surface
= Cremona transformation

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produces

Discrete Painlevé Equations.

Content

Riccati solutions

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$\Longrightarrow$ We like to know when $\Delta^{nod} \neq \emptyset$.

Period map

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and we take $H_2(X - D_{red}, \mathbb{Z})$.
We have exact sequence $0 \to H^1(D_{red}) \to H_2(X - D_{red}) \to Q(R^\perp) \to 0$, so we obtain the map:

$$\chi \ : \ Q(R^\perp) \to \mathbb{C} \quad \text{mod } \hat{\chi}(H_1(D_{red})).$$

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- X is an elliptic surface $\quad \Leftrightarrow \quad \chi(\delta) = 0$.
  (δ is null root and it realized as $-\mathcal{K}_X$.)

Thank you.

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