# Distributions of poles to the Painlevé transcendents: theory and experiment

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All solutions to the Painlevé I, II and IV equations are meromorphic functions. The method of isomonodromic deformations provides an exhaustive description of their asymptotic behavior at infinity. In contrast, it is little known about analytic structure of the solutions in bounded domains of the complex plane. Namely, there are non-trivial facts about dynamics of the movable poles, emergence of rational and truncated solutions and various patterns formed by the poles.

A number of numeric experiments has been done recently to reveal and visualize these phenomena. Here we discuss a method based on Padé approximations which exploits the invariance of Painlevé equations under Möbius transformations [1]. The method gives recurrent continued fraction representation avoiding traditional Taylor series expansion of the solution. It works pretty fast on a PC computer and produces hundreds of poles in reasonable time. Moreover, since all solutions are meromorphic, this continued fraction is convergent and gives arbitrary good approximation in any compact domain.

The higher-order approximations allow to check asymptotic expansions at infinity and estimate the range of asymptotic domains. The Coulomb gas interpretation of the pole ensembles is discussed in view of the patterns arising in Painlevé IV transcendents [2].

## References

[1] V.Yu.Novokshenov, Theor. Math. Phys., **159**(3) (2009) 852-861

[2] V.Yu.Novokshenov, Constructive Approximation, **39**(1) (2014) 85-99.

# Distribution of poles to the Painlevé transcendents: theory and experiment

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**Three days on Painlevé equations and their applications**

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## Painlevé equations

$$u_{zz} = 6u^2 + z, \qquad \text{PI}$$

$$u_{zz} = 2u^3 + zu + \alpha, \qquad \text{PII}$$

$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - \alpha\right)u + \frac{\beta}{u}, \qquad \text{PIV}$$

where $u_z = \dfrac{du}{dz}$ and $z, u \in \mathbb{C}$. $\alpha, \beta$ - arbitrary constants.

**Theorem** [1]. *Any solution* $u(z)$ *of PI, PII and PIV is a meromophic function in the complex plane* $\mathbb{C}$ .

The poles are "movable", i.e. their positions depend on the constants of integration, or initial data $u(z_0), u_z(z_0)$ .

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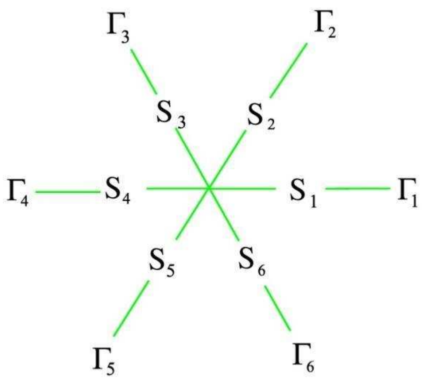
[1] P. Painlevé, Bull.Soc.France, 28, 201–261 (1900); A.Bolibruch, A.Its, A.Kapaev, Algebra and Analysis, 16, 121-161 (2004); A.Veselov, J.Phys. A, 34, 3511-3519 (2001)

## Isomonodromic deformations and Painlevé property




$$\Psi_+ = \Psi_- S_k, \quad \lambda \in \Gamma_k,$$
$$\Psi(\lambda, x) \to e^{-i\lambda\sigma_3}, \quad \lambda \to \infty$$
$$S_k = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}$$
$$s_{k+3} = -s_k, \quad k = 1, 2, 3,$$
$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$$

$$\boxed{u_{zz} = zu + 2u^3}$$

$$u(z;\vec{s}) = \lim_{\lambda\to\infty} \Psi(\lambda, z) e^{i\lambda\sigma_3}$$

$$u(z;\vec{s}) \Leftrightarrow \{\vec{s} = (s_1, s_2, s_3) : s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0\}$$

**Theorem.** (*Bolibruch-Its-Kapaev*) For any point $\vec{s}$ there exits a countable subset $X_{\vec{s}}$ of the z-plane such that

- RH is solvable for $z \notin X_{\vec{s}}$
- $\Psi_k(\lambda, z) e^{i\lambda z}$ are holomorphic in $\overline{\Omega}_k \times (\mathbb{C} \setminus X_{\vec{s}})$ and meromorphic along $\overline{\Omega}_k \times X_{\vec{s}}$
- $\Psi_k(\lambda, z) e^{i\lambda z} = I + \sum_{j=1}^{\infty} \frac{u_j(z)}{\lambda^j}, \quad \lambda \to \infty,$
- $u_j(z)$ are holomorphic in $\mathbb{C} \setminus X_{\vec{s}}$

**Corollary.** Any solution $u(z;\vec{s})$ is meromorphic in *z.* The set of poles coincide with $X_{\vec{s}}$

## Numeric tool: Padé approximation

Let $f(z)$ be a meromorphic function, $z=0$ is not a pole. Then it has

*main diagonal Padé approximation*

$$f(z)=\sum_{k=0}^{\infty} f_k z^k \approx \frac{P_N(z)}{Q_N(z)}, \quad P_N(z)-f(z)Q_N(z)=O\left(z^{2N+1}\right), z\to 0$$

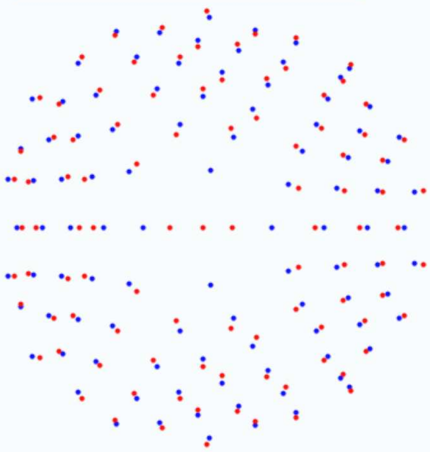
**Theorem** [2]. *For any compact domain $\mathcal{D}$ and $\varepsilon,\delta>0$ there exists N, such that*

$$\left|P_N(z)-f(z)Q_N(z)\right|<\varepsilon, \quad z\in\mathcal{D}$$

*except a set of measure $\delta$ .*

Starting from initial point $z_0$ one can find solution of Painlevé equation in any compact domain $\mathcal{D}$.

Why main diagonal?



Poles (blue) and zeroes (red) of a PII function

[2] J. Nuttall, J. Math. Anal. Appl., 31, 147–153 (1970)..

## Continued fractions for Painlevé functions

Q: How to find singularities (poles) of a Painlevé function?
A: For large $|z|$ by Isomonodromy Method, else - numerically

Main diagonal Padé approximation as continued fraction:

$$u(z)=\cfrac{\alpha_0}{1+\cfrac{\alpha_1 z}{1+\cfrac{\alpha_2 z}{1+\ldots}}}$$

The idea of algorithm: an invariance of Painlevé equations under Möbius transformation.

$$(A+Bu)u''_{zz}+(C+Du)u'_z-2B(u'_z)^2+E+Fu+Gu^2+Hu^3=0,$$

where $A\ldots H$ are polynomials of $z$, and $u\mapsto\dfrac{\alpha u+\beta}{\gamma u+\delta}$ retains the form of equation.

No need to find Tailor expansion!

## The Fair-Luke algorithm [3]

Let $u_n = \dfrac{\alpha_n}{1+zu_{n+1}}$, then

$$\left(A_n + B_n u_n\right)u_n'' + \left(C_n + D_n u_n\right)u_n' - 2B_n\left(u_n'\right)^2 + E_n + F_n u_n + G_n u_n{}^2 + H_n u_n{}^3 = 0,$$

and the recurrence holds

$$A_{n+1} = -A_n - \alpha_n B_n,\quad B_{n+1} = -zA_n,\quad C_{n+1} = -2\frac{A_n + \alpha_n B_n}{z} - C_n - \alpha_n D_n,$$

$$D_{n+1} = 2A_n - zC_n,\quad F_{n+1} = -\frac{C_n + \alpha_n D_n}{z} + 3\frac{E_n}{\alpha_n} + 2F_n + \alpha_n G_n,$$

$$E_{n+1} = z^{-1}\left(\alpha_n^{-1}E_n + F_n + \alpha_n G_n + \alpha_n^2 H_n\right),\quad H_{n+1} = \alpha_n^{-1}z^2 A_n,$$

$$G_{n+1} = 2z^{-1}A_n - C_n + z\left(3\alpha_n^{-1}E_n + F_n\right),$$

where $\alpha_{n+1} = -\dfrac{E_{n+1}(0)}{F_{n+1}(0)}$.

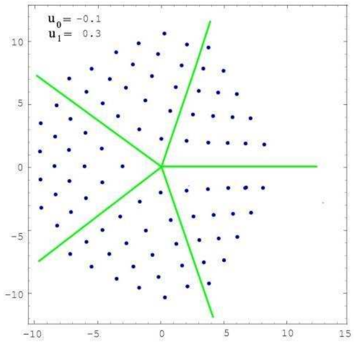
Set $u_{n+1} \equiv 0$, then

$$u(z) \approx u_0(z) = \cfrac{\alpha_0}{1+\cfrac{\alpha_1 z}{1+\cfrac{\alpha_2 z}{1+\dots}}} = \frac{P_n(z)}{Q_n(z)}$$

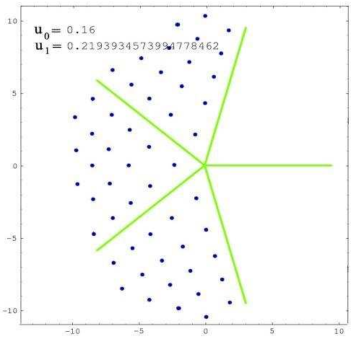
[3] W.Fair, Y.Luke, Math.Comp. v.20 (1968), p.602-605

## Poles of PI solutions in the complex plane

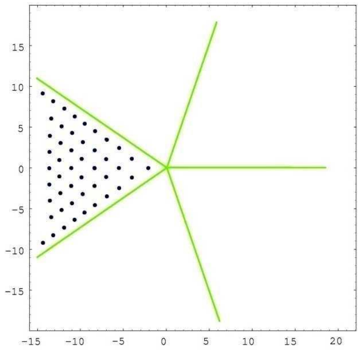
PI general solution




1-truncated solution




3-truncated solution



$$\boxed{u_{zz} = 6u^2 - z}$$

PI equation

- All solutions are meromorphic
- The poles accumulate along the rays

$$\boxed{\arg z = \frac{2\pi i}{5} n, \;\; n = 0, \pm 1, \pm 2. \qquad z \to +\infty}$$

at smooth lines of poles

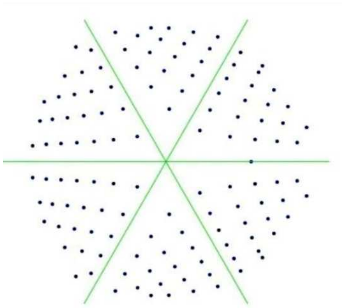
- *Intégrale tronquée*: no poles at infinity along one critical ray (say, along $\arg z = 0$ )
- *Intégrale tritronquée*: no poles at infinity along three consecutive rays [4]
- Asymptotics at infinity:

$$u(z) = \pm\sqrt{\frac{z}{6}}\left(1 + O(z^{-2})\right), \quad z \to +\infty.$$

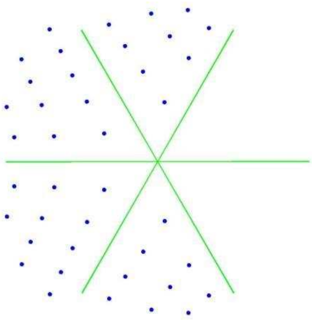
[4] 'P.Boutroux,. Ann.Ecole Norm., 30 (1913) 265 - 375.

# Poles of PII solutions in the complex plane

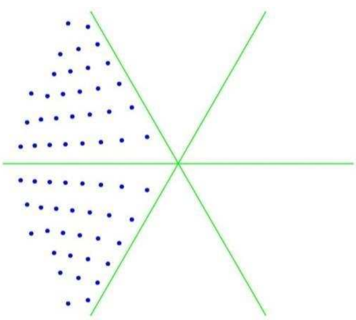
PII generic solution



PII 1-truncated solution



PII 3-truncated solution



$$\boxed{u_{zz} = zu + 2u^3}$$ PII equation

- The poles accumulate along the rays

$$\boxed{\arg z = \frac{\pi i}{3} n, \;\; n = 0, \pm 1, \pm 2. \qquad z \longrightarrow +\infty}$$

- *1-truncated* solution: no poles at infinity along one critical ray
- *2- and 3-truncated* solution: no poles at infinity along 2 or 3 rays [5]
- Asymptotics at infinity:

$$u(z) = \varepsilon \sqrt{\frac{z}{2}} + o(z^{-1}), \quad z \to +\infty, \quad \varepsilon = -1, 0, 1.$$

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[5] P.Boutroux,. Ann.Ecole Norm., 31, (1914) 99--159.

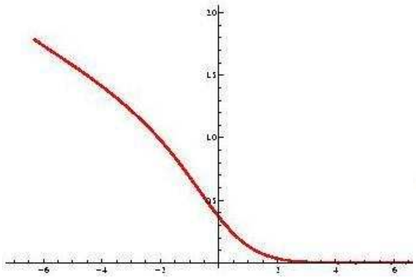
# PII equation. A zoo of truncated solutions

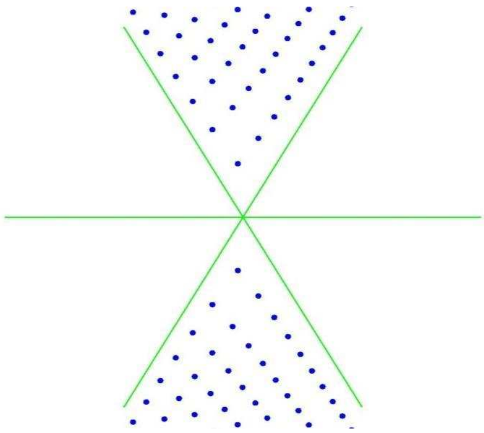
$$u_{zz} = zu + 2u^3$$

Hastings-McLeod 2-truncated

$$u(z) = \sqrt{\frac{-z}{2}} + \frac{(-z)^{5/2}}{8\sqrt{2}} + \ldots, \qquad z \to -\infty$$

$$u(z) \approx \frac{z^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3} z^{3/2}\right), \qquad z \to +\infty$$





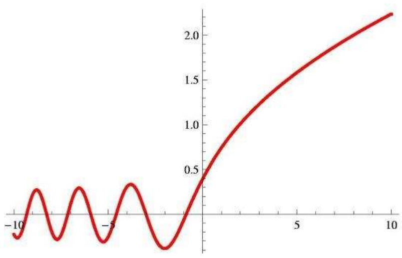
$$u_{zz} = zu + 2u^3$$

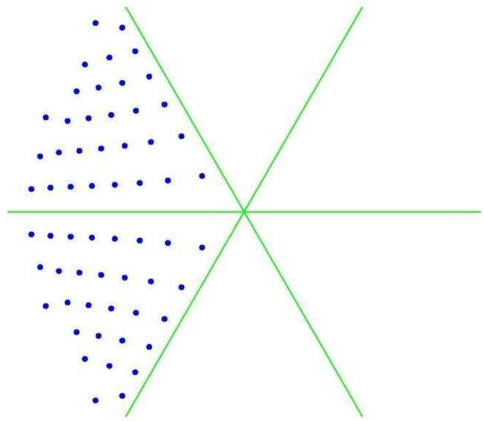
Pure imaginary 3-truncated

$$u(z) \approx \frac{i\alpha}{\sqrt[4]{z}} \sin\left(\frac{2}{3} z^{3/2} + \frac{\alpha^2}{8} \ln z + \beta\right),$$

$z \to -\infty, \qquad \alpha, \beta \in \mathbb{R}$

$$u(z) = i\sqrt{\frac{z}{2}} - \frac{iz^{5/2}}{8\sqrt{2}} + \ldots, \qquad z \to +\infty$$



# PII equation. A zoo of truncated solutions (continued)

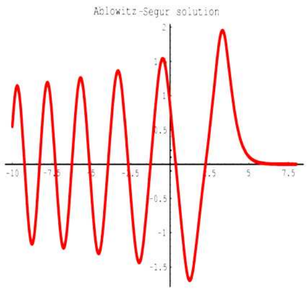
$$u_{zz} = zu - 2u^3$$

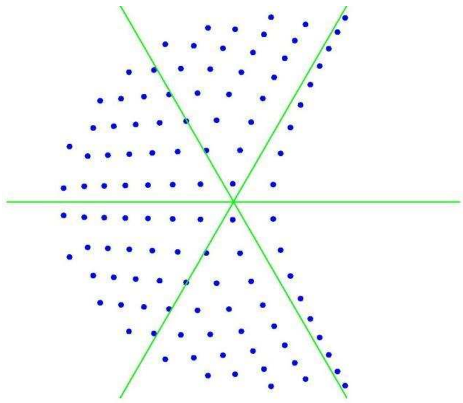
$$u(z) \approx \frac{a}{\sqrt[4]{z}} \exp\left(-\frac{2}{3} z^{3/2}\right), \; z \to +\infty,$$

$$u(z) \approx \frac{b}{\sqrt[4]{-z}} \sin\left(\frac{2}{3}(-z)^{3/2} - \frac{3}{4} b^2 \log(-z) + \theta\right),$$

$$z \to -\infty, \qquad a, b, \theta \in \mathbb{R}$$

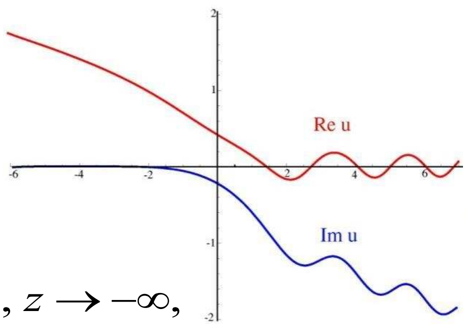
Ablowitz-Segur 1-truncated

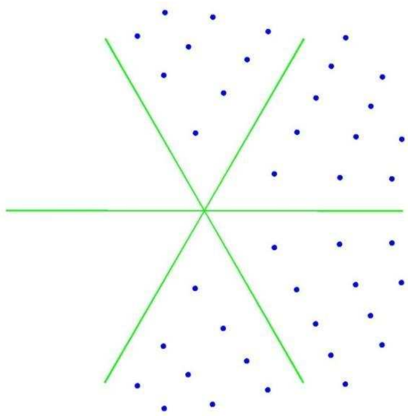


complex-valued 1-truncated

$$u_{zz} = zu + 2u^3$$



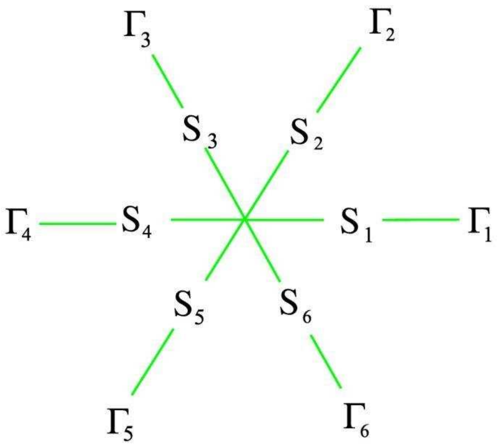
$$u(z) \approx \sqrt{\frac{-z}{2}} + \frac{a}{\sqrt[4]{z}} \exp\left(-\frac{2}{3} z^{3/2}\right), \; z \to -\infty,$$

$$u(z) \approx i\sqrt{\frac{z}{2}} + \frac{b}{\sqrt[4]{z}} \sin\left(-\frac{2}{3} z^{3/2} - \frac{3}{4} b^2 \log z + \theta\right), \; z \to +\infty,$$

## Riemann-Hilbert problem

or Isomonodromic Deformation Method

$$u_{zz} = zu + 2u^3 \quad \Leftrightarrow \quad \begin{cases} \Psi_\lambda = -\left( (4i\lambda^2 + iz + 2iu^2)\sigma_3 + 4u\sigma_2\lambda + 2u_z\sigma_1 \right)\Psi, \\ \Psi_z = \left( -i\lambda\sigma_3 + u(z)\sigma_2 \right)\Psi \end{cases}$$




$$\Psi_j(\lambda, z) \to e^{\frac{4i}{3}\lambda^3 + iz\lambda\sigma_3}, \quad |\lambda| \to \infty, \quad \lambda \in \Gamma_j$$

$$\Psi_{j+1}(\lambda, z) = \Psi_j(\lambda, z) S_j, \quad j = 1, 2, \ldots, 6$$

$$S_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix}, \; S_{2k-1} = \begin{pmatrix} 1 & 0 \\ s_{2k-1} & 1 \end{pmatrix}$$

$$u(z) = -\lim_{\lambda \to \infty} [\lambda \Psi_{(12)}(\lambda, z) e^{-\frac{4i}{3}\lambda^3 - iz\lambda\sigma_3}]$$

$$\boxed{\begin{aligned} & s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0, \\ & s_{j+3} = -s_j, \quad j = 1, 2, 3. \end{aligned}}$$

Hastings-McLeod solution $\quad s_1 = -i,\; s_2 = 0,\; s_3 = \overline{s_1} = i$

## Singularities of the complex manifold PII

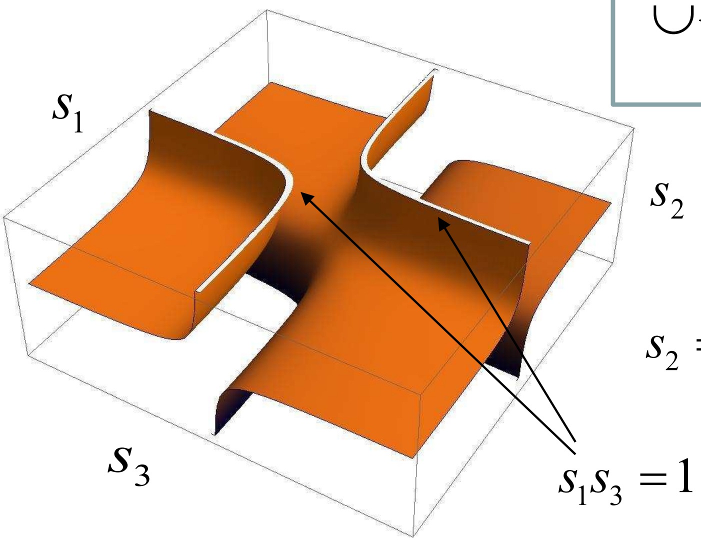
Potential function $F = s_1 - s_2 + s_3 + s_1 s_2 s_3$

Manifold $P_2$ $F = 0$

Singularities $\dfrac{\partial F}{\partial s_j} = 0, \quad j = 1,2,3$

Singularity manifold

$$\tilde{P}_2 = \{\mathbf{s} \mid \{s_1 s_2 = -1\} \cup \{s_1 s_3 = 1\} \cup \{s_2 s_3 = -1\} \cup \{s_j = 0\}, \quad j = 1,2,3\}$$



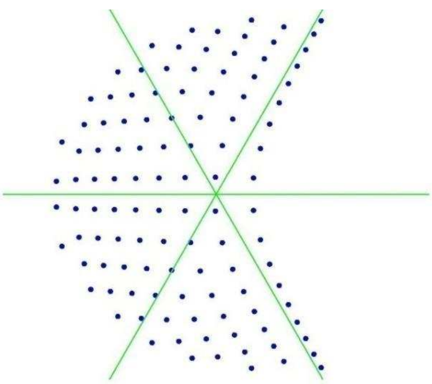

Manifold $P_2$ in local chart is

$s_2 = \dfrac{s_1 + s_3}{1 - s_1 s_3}$ for real-valued $\mathbf{s} = (s_1, s_2, s_3)$

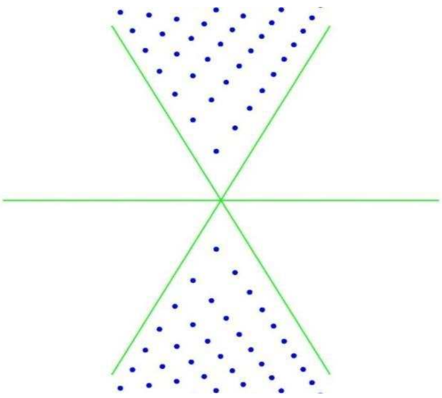
# Examples of singularities

$$\tilde{P}_2 = \{\boldsymbol{s} \mid \{s_1 s_2 = -1\} \cup \{s_1 s_3 = 1\} \cup \{s_2 s_3 = -1\} \cup \{s_j = 0\}, \quad j = 1, 2, 3\}$$

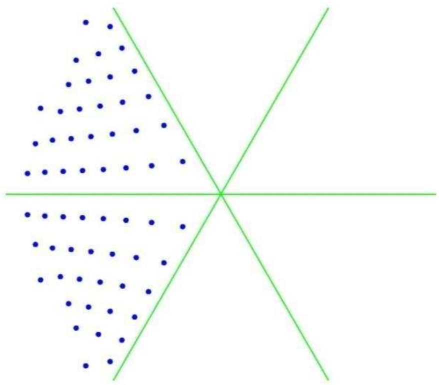
| 1-dim submanifold | 0-dim submanifold | 0-dim submanifold |
|---|---|---|
| $s_2 = 0, s_1 = -s_3 \in \mathbb{C}$ | $s_1 = \pm i, \quad s_3 = \mp i, \quad s_2 = 0$ | $s_1 = s_2 = s_3 = \pm i$ |
| Ablowitz-Segur solution | Hastings-McLeod solution | 3-truncated solution |



1-truncated



2-truncated



3-truncated

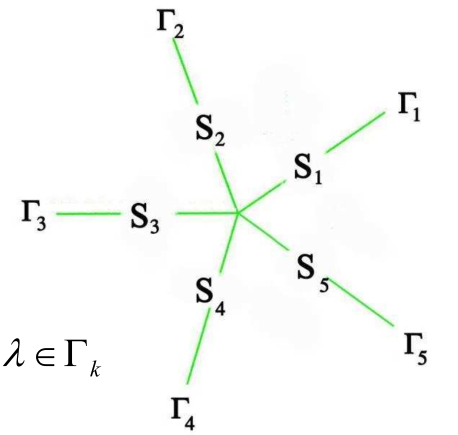
S. Olver & T. Trogdon, Constr. Approx. 39: pp 101-149 (2014).

## Isomonodromic deformations for PI

$$u_{zz} = 6u^2 + z$$ PI equation

$$\begin{cases} \Psi_\lambda = \begin{pmatrix} u_z & 2\lambda^2 + 2u\lambda - z + 2u^2 \\ 2(\lambda - u) & -u_z \end{pmatrix} \Psi, \\ \Psi_z = -\begin{pmatrix} 0 & \lambda + 2u \\ 1 & 0 \end{pmatrix} \Psi. \end{cases}$$

The Lax pair




canonical sectors

$$\Psi_k(\lambda, z) \approx \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda^{1/4} & \lambda^{1/4} \\ \lambda^{-1/4} & -\lambda^{-1/4} \end{pmatrix} e^{\theta(\lambda,z)\sigma_3}, \quad |\lambda| \to \infty, \quad \lambda \in \Gamma_k$$

$$\theta(\lambda, z) = \frac{4}{5}\lambda^{5/2} - z\lambda^{1/2}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Riemann-Hilbert problem

$$\Psi_{k+1}(\lambda, z) = \Psi_k(\lambda, z) S_k, \quad \lambda \in \Gamma_k. \qquad S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \quad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}$$

Inversion formula

$$u(z) = \frac{\partial}{\partial \zeta} \left[ \lim_{\lambda \to \infty} \lambda^{1/2} \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \lambda^{-\frac{1}{4}\sigma_3} \Psi(\lambda, z) e^{-\theta(\lambda,z)\sigma_3} - 1 \right) \right]_{11}$$

Monodromy data

$$s_{j+5} = s_j, \quad s_{j+5} = i\left(1 + s_{j+2}s_{j+3}\right), \quad j = 0, \pm 1$$

## Singularities of the complex manifold PI

Potential function

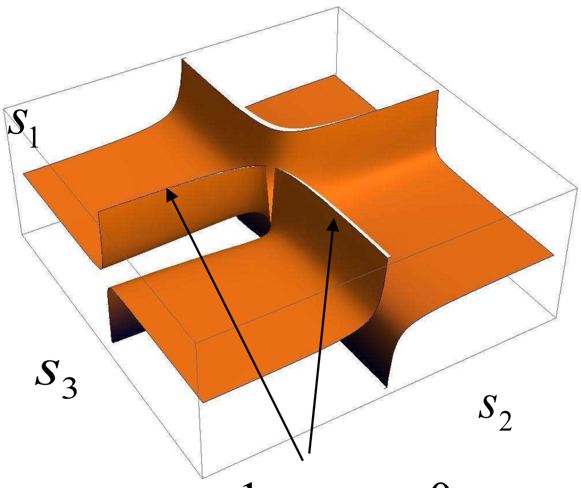
$$\vec{F} = \begin{pmatrix} s_1 + s_3 + s_1 s_2 s_3 - i \\ s_2 + s_4 + s_2 s_3 s_4 - i \\ s_3 + s_5 + s_3 s_4 s_5 - i \end{pmatrix}$$

Manifold $P_1$ $\quad \vec{F} = 0$

Singularities $\quad \dfrac{\partial \vec{F}}{\partial s_j} = 0, \quad j = 1, 2, \ldots, 5$

Singularity manifold

$$\boxed{\begin{aligned} \tilde{P}_1 = \{\mathbf{s} \mid & \{s_1 s_2 = -1\} \cup \{s_2 s_3 = -1\} \\ & \cup \{s_3 s_4 = -1\} \cup \{s_4 s_5 = -1\} \\ & \cup \{s_j = 0\}, \quad j = 1, 2, \ldots, 5\} \end{aligned}}$$




Manifold $P_1$ in local chart $\mathbf{s} = \mathbf{s}(s_2, s_3)$ is

$$s_1 = \frac{i - s_3}{1 + s_2 s_3}, \quad s_4 = \frac{i - s_2}{1 + s_2 s_3}, \quad s_5 = i(1 + s_2 s_3)$$

$s_2 = s_3 = i, \quad s_1 + s_4 = i, \quad s_5 = 0$ if $1 + s_2 s_3 = 0$

## Main Theorem

$$u_{zz} = zu + 2u^3$$

$$u(z) = \frac{1}{\varepsilon} - \frac{a}{6}\varepsilon - \frac{1}{4}\varepsilon^2 + c\varepsilon^3 + O(\varepsilon)^4, \ \ \varepsilon = z - a \to 0$$

**Theorem.** 1) All truncated solutions of PII $\Longleftrightarrow$ singular submanifold $\tilde{P}_2$

2) 1-truncated solutions $\Longleftrightarrow$ 1-dim submanifold

3) 2,3-truncated solutions $\Longleftrightarrow$ 0-dim submanifold

**Idea of the proof** Equations for poles $a$ and parameters $b = \frac{2}{9}a^2 + 40c$

$$\begin{cases} \int_{\lambda_1}^{\lambda_3} \sqrt{16\lambda^4 + 8a\lambda^2 + b}\, d\lambda = 2\pi n + \ln \frac{s_1 + s_3}{1 - s_1 s_3}, \\ \int_{-\lambda_1}^{\lambda_1} \sqrt{16\lambda^4 + 8a\lambda^2 + b}\, d\lambda = 2\pi m + \ln \frac{1 + s_1^2}{1 - s_1 s_3}, \end{cases} \qquad m, n \in \mathbb{Z}^+.$$

If $1 - s_1 s_3 = 0$ or $1 + s_1^2 = 0$ then $a \to \infty$ and poles vanish!

A.V.Kitaev, Russ. Math. Surv., **49,** 77-140 (1994).

## Proof of the Main Theorem

Let $u(z)$ be truncated in $\Omega_3$ and $\boldsymbol{s} \notin P_2$, then in this sector [2]

$$u(z) = (\lambda_1 - \lambda_3) e^{i\varphi/2} \sqrt{z}\, \mathrm{sn}\left( \frac{2}{3} (\lambda_1 + \lambda_3) [e^{\frac{2\pi}{3} - \frac{i\varphi}{2}} z^{3/2} + \chi] \,|\, \kappa \right), \quad |z| \to \infty,$$

$$1 + s_1^2 \neq 0$$
$$1 - s_1 s_3 \neq 0$$

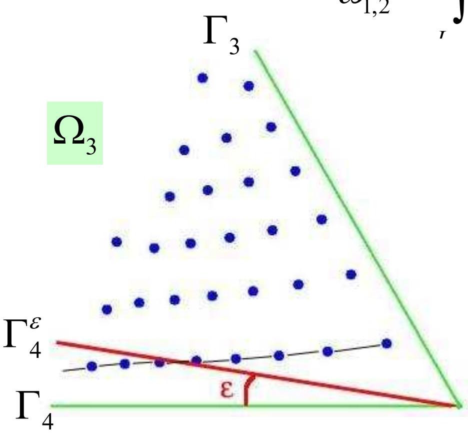
where $\varphi = \arg z, \quad \kappa = \dfrac{\lambda_1 - \lambda_3}{\lambda_1 + \lambda_3}, \quad \chi = \dfrac{i}{2} \omega_2 \mu + i \omega_1 \nu + o(1), \quad \dfrac{2\pi}{3} < \arg z < \pi,$

$$\mu = \frac{1}{2\pi i} \log\left( \frac{1 + s_1^2}{1 - s_1 s_3} \right), \quad \nu = \frac{1}{2\pi i} \log\left( -\frac{s_1 + s_3}{1 - s_1 s_3} \right),$$

$$\omega_{1,2} = \int_{\Gamma_1, \Gamma_2} \frac{d\lambda}{\sqrt{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_3^2)}},$$

**Boutroux problem**

$$\lambda_1^2 + \lambda_3^2 = -\frac{1}{2} e^{i\varphi},$$
$$\mathrm{Im} \int_{L_1, L_2} \sqrt{(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_3^2)}\, d\lambda = 0,$$
$$\lambda_1 \to 0, \quad \lambda_3 \to \frac{i}{\sqrt{2}} e^{\pi i/3}, \text{ as } \quad \varphi \to \pi,$$
$$\lambda_{1,3} \to \frac{1}{2}, \quad \text{as} \quad \varphi \to \frac{2\pi}{3}.$$




Prove that a line of poles crosses $\Gamma_4^\varepsilon$


[2] A.S.Fokas, A.R.Its, A.A.Kapaev and V.Yu.Novokshenov, Painlevé Transcendents. The Riemann-Hilbert Approach, AMS Monograph (2006)

## Proof of the Main Theorem (continued)

Limit case near $\Gamma_4$ $\quad \lambda_1 \to 0, \quad \lambda_3 \to \frac{i}{\sqrt{2}} e^{\pi i/3}, \quad \kappa \to -1, \quad K \to \frac{\pi}{2}, \quad K' \to \ln \frac{4}{\varepsilon}.$

Lattice of poles from Jacobi sine

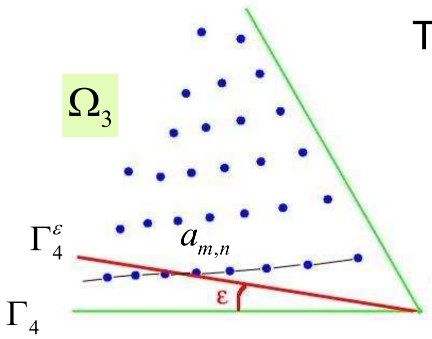
$$\frac{2}{3}(\lambda_1 + \lambda_3)[e^{-i\varphi/2} a_{m,n}{}^{3/2} + \chi] = 2K\, n + 2iK'\, m, \quad n, m \in \mathbb{Z}.$$

where

$$K(\kappa) = \frac{\lambda_1 + \lambda_3}{2} \omega_1, \quad K'(\kappa) = \frac{\lambda_1 + \lambda_3}{2} \omega_2, \quad \chi = \frac{i}{2} \omega_2 \mu + i \omega_1 \nu.$$

Finally

$$a_{m,n} \sqrt{| a_{m,n} |} = -(3m + \mu)\sqrt{2} \ln \frac{4}{\varepsilon} + \left( \frac{3ni}{2} + \nu \right) \pi \sqrt{2} + o(1), \quad \varepsilon \to 0.$$




Take $n = 1, \quad m \to +\infty$ then this line of poles crosses $\Gamma_4^\varepsilon$

Contradiction with assumption that $u(z)$ is truncated in $\Omega_3$

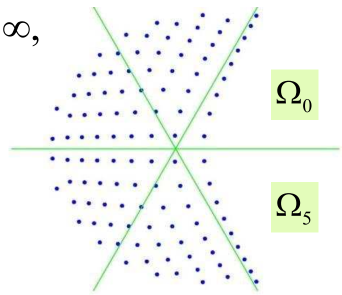
## Proof of the Main Theorem (continued)

Sufficiency follows from Lemmas

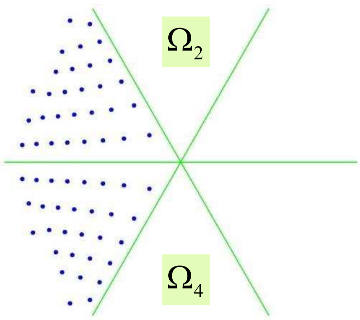
**Lemma 1.** [3] In the case of $s_2 = 0$ the PII solutions have the regular asymptotics in the sectors $\Omega_0 \cup \Omega_5$

$$u(z) = \frac{i\left[\theta(-\varphi)s_1 - \theta(\varphi)s_3\right]}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}} \left(1 + O(|z|^{-3/2})\right), \quad z \to \infty,$$

where $\varphi = \arg z,\ -\frac{\pi}{3} < \arg z < \frac{\pi}{3}$ and $\theta(\varphi) = \begin{cases} 0, & \varphi < 0, \\ \frac{1}{2}, & \varphi = 0, \\ 1, & \varphi > 0, \end{cases}$




**Lemma 2.** [3] In the case of $1 + s_2 s_3 = 0$ the PII solutions have regular asymptotics in the sectors $\Omega_2 \cup \Omega_4$



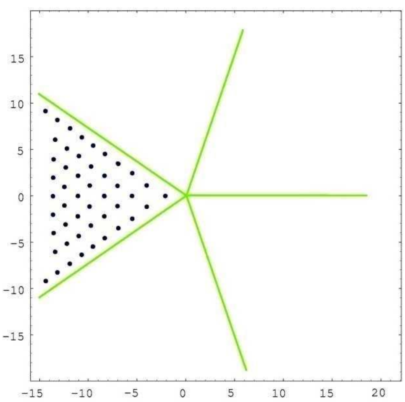

[3] A.A.Kapaev, Phys.Lett. A, **167**, 356-362 (1992)

“No extra poles” conjecture

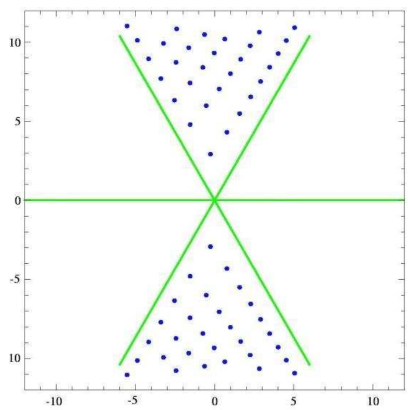
Let the poles of a truncated solution $z_n, |z_n| \gg 1$ lie in the critical sectors $\Omega_j$ .
Are there any extra poles outside of these sectors?

**Conjecture.** *All poles of 2- and 3-truncated solutions lie in the critical sectors*.

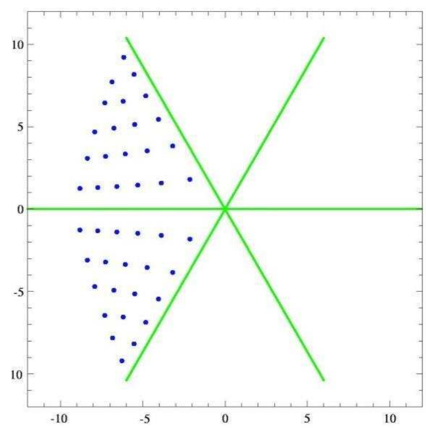
PI 3-truncated




PII H-M 2-truncated




PII 3-truncated




O.Costin, M. Huang & S. Tanveer, Duke J. Math, 163 (4), 665-704 (2014).

## Proof of the (half) conjecture for H-M solution

$$u_{zz} = zu + 2u^3,$$
$$u(z) \to \mathrm{Ai}(z), \quad z \to +\infty$$

Hastings-McLeod solution

where $\mathrm{Ai}(z)$ - Airy function

Fredholm determinant representation

$$u^2(z) = -\frac{d^2}{dz^2} \ln F_2(z), \qquad F_2(z) = \det\left( Id_{L_2[z,\infty)} - K_{\mathrm{Ai}}\big|_{[z,\infty)} \right)$$

$$K_{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$$ - Airy kernel

**Theorem** (M.Bertola, [6]). *The determinant* $F_2(z)$ *does not vanish as* $|\arg z| < \pi/3$ *and the estimate holds*

$$\left\| K_{\mathrm{Ai}}\big|_{[z,\infty)} \right\|_{L_2[z,\infty)} < \exp\left( -\frac{4}{3} \mathrm{Re}\, z^{3/2} \right)$$

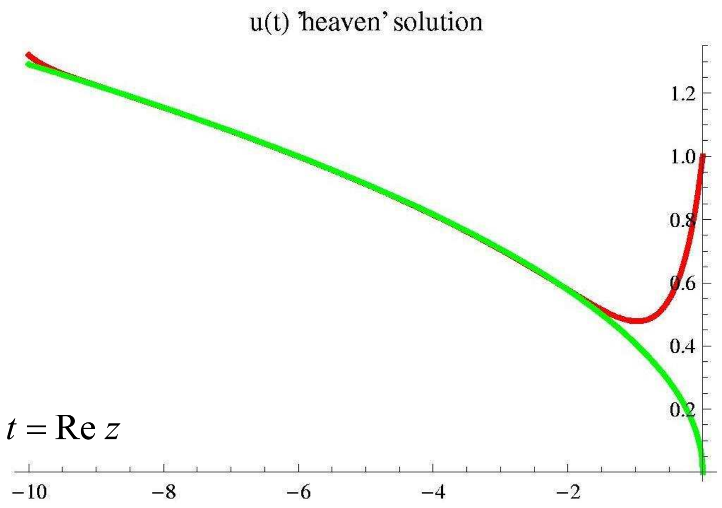
[6] M.Bertola, Nonlinearity 25 (2012) 1179--1185

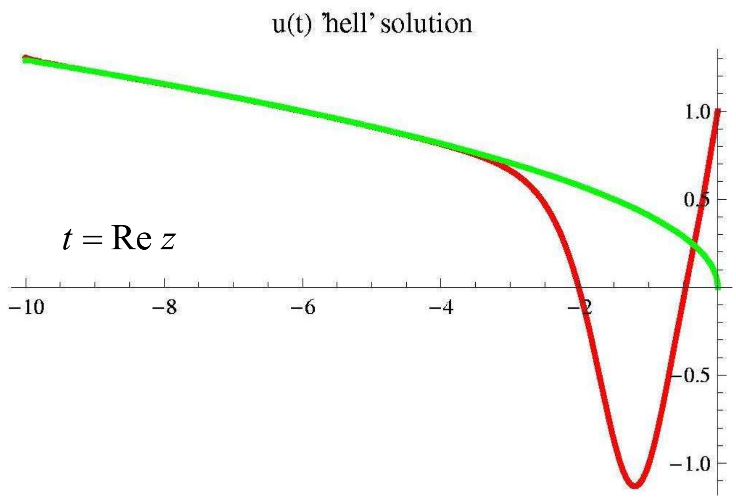
## Formation of patterns: two 1-truncated solutions of PI

$$\boxed{\frac{d^2u}{dz^2} = 6u^2 + z,}$$

$$u(0)=1, \quad u'(0) = u_{heaven} \approx 1.8182750515636$$

$$u(0)=1, \quad u'(0) \nearrow u_{hell} \approx 2.7655981573583$$



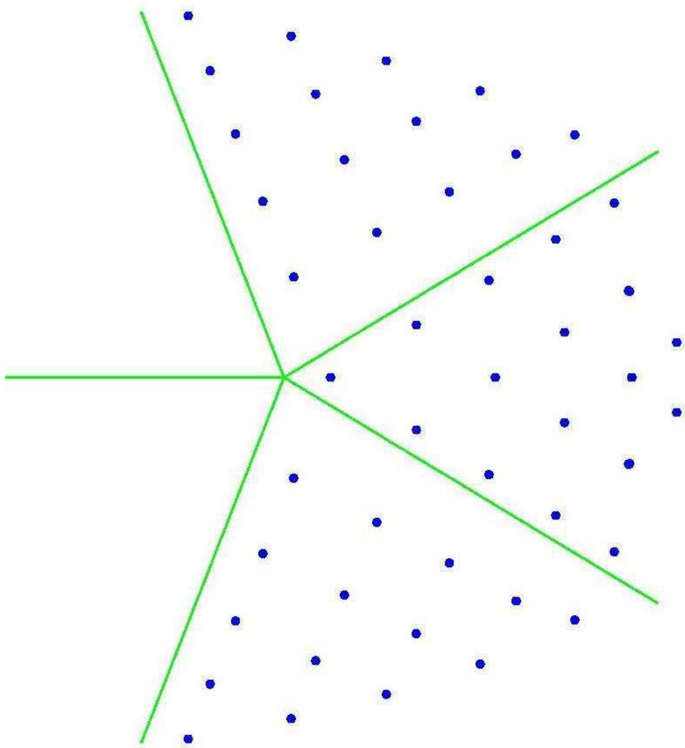




$\sqrt{\frac{-t}{6}}$ $\qquad$ $u(t)$

$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

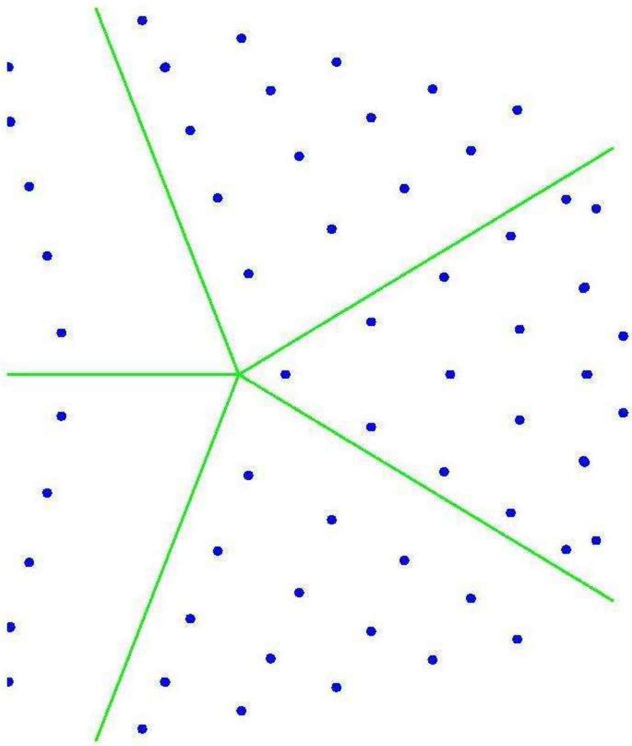
$$u_1(0) = 1$$
$$u_1'(0) = 1.8182750515636$$

'heaven' solution'



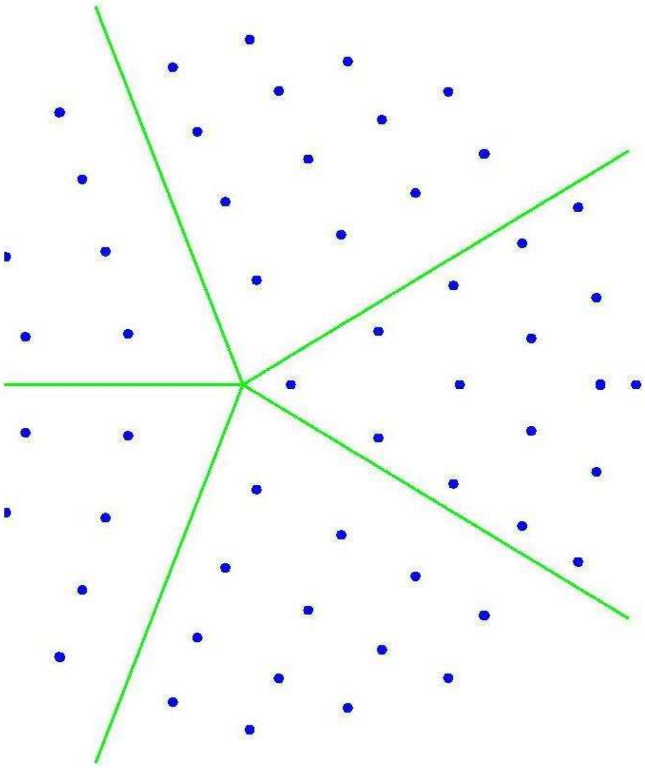
$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

$$u_1(0) = 1$$
$$u_1'(0) = 1.82$$



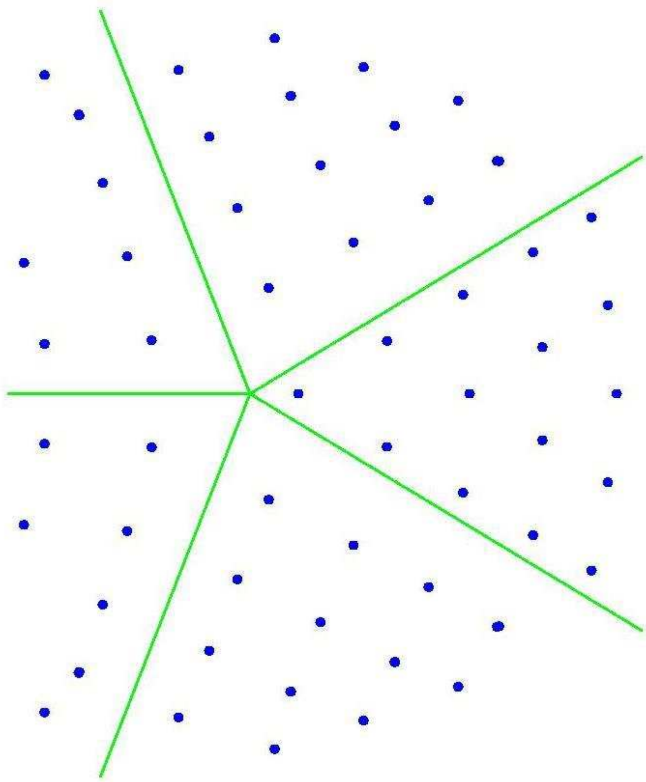
$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

$$u_1(0) = 1$$
$$u_1'(0) = 1.90$$



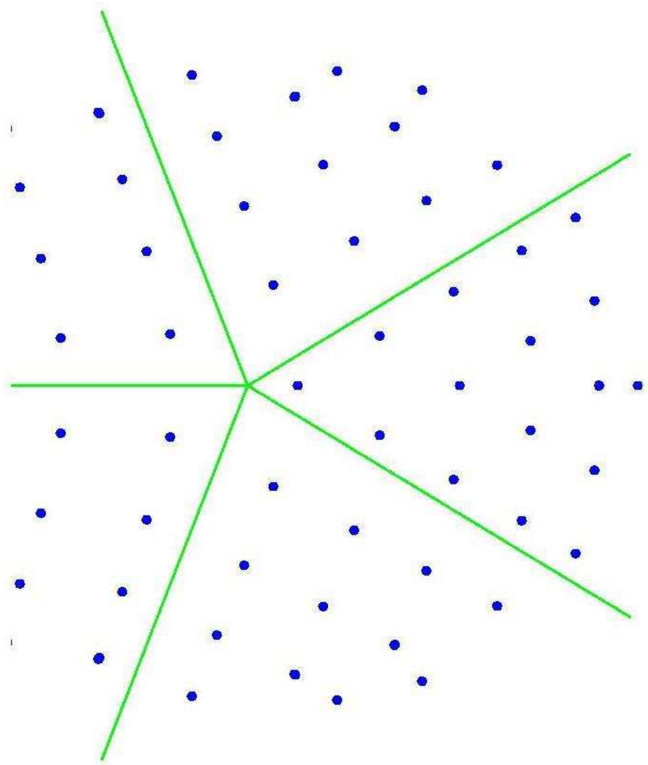
$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

$$u_1(0) = 1$$
$$u_1'(0) = 2.0$$



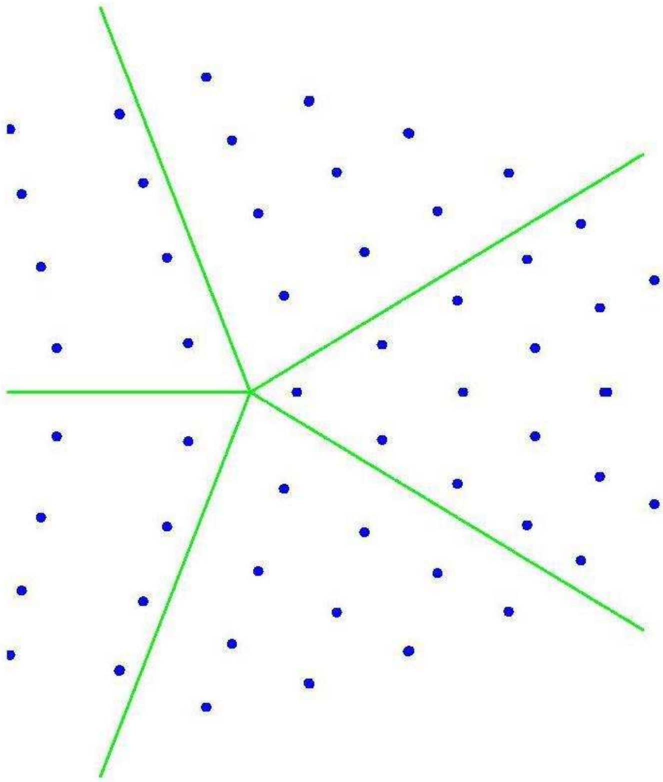
$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

$$u_1(0) = 1$$
$$u_1'(0) = 2.2$$



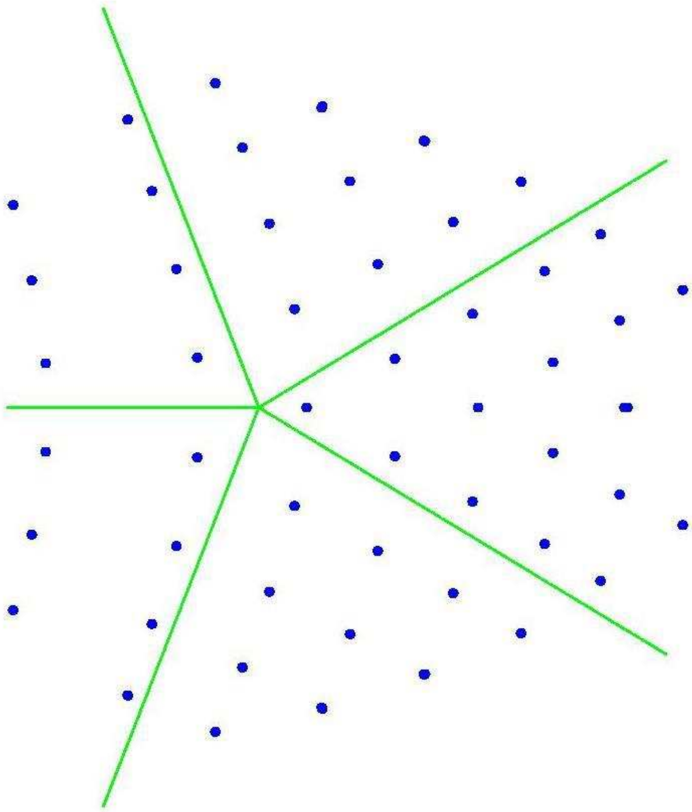
$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

$$u_1(0) = 1$$
$$u_1'(0) = 2.6$$



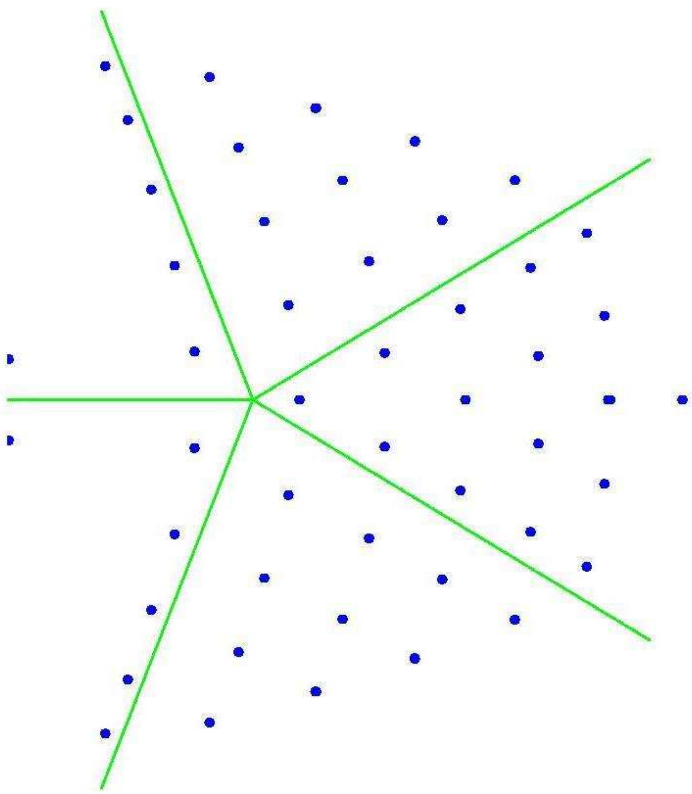
$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

$$u_1(0) = 1$$
$$u_1'(0) = 2.7$$



$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

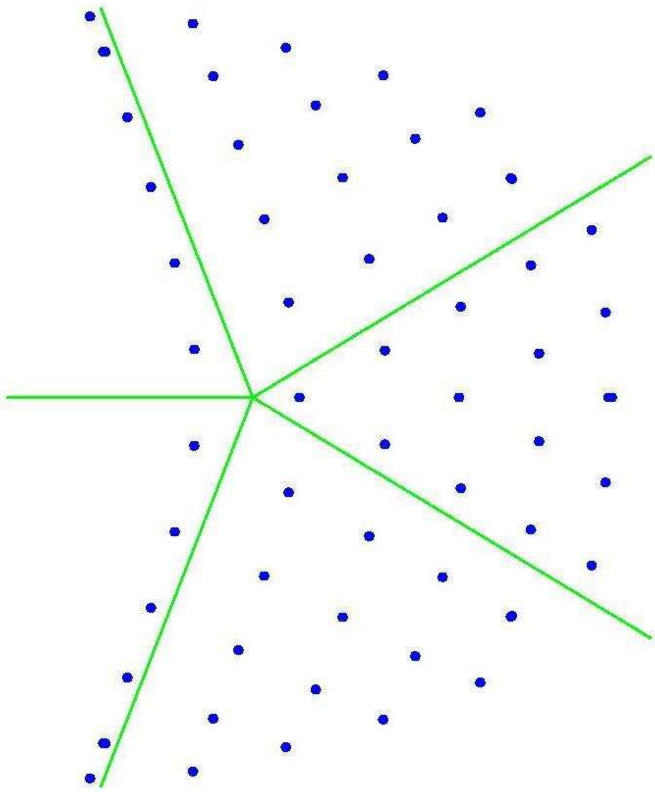
$$u_1(0) = 1$$
$$u_1'(0) = 2.76$$



$$\frac{d^2u}{dz^2} = 6u^2 + z,$$

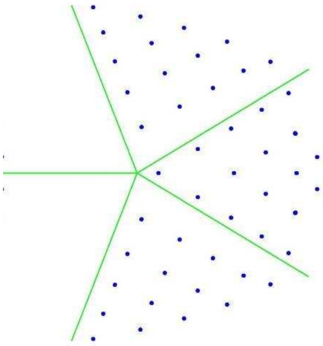
$$u_1(0) = 1$$
$$u_1'(0) = 2.7655981573583$$

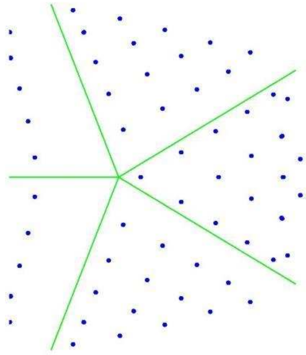
'hell' solution



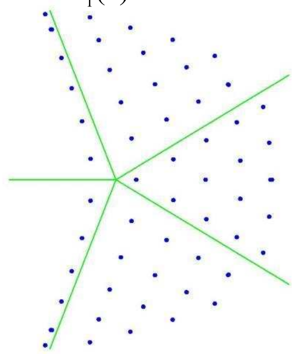
# A family of 1-truncated solutions?

$u_1(0) = 1$
$u_1'(0) = 1.8182750515636$

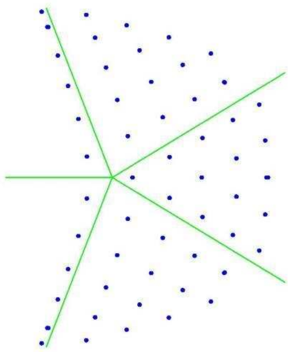




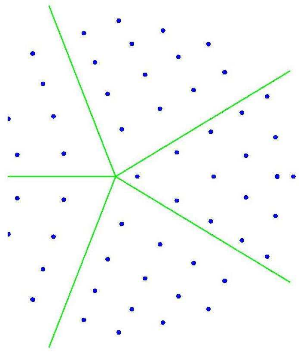
$u_1(0) = 1$
$u_1'(0) = 2.7655981573583$



$u_1(0) = 1$
$u_1'(0) = 2.7655981573583$

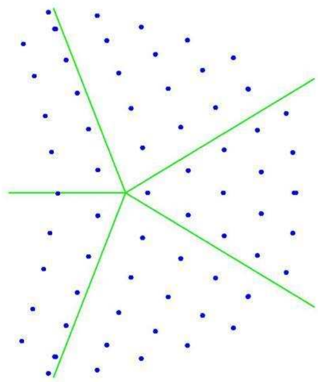


?



?

$u_1(0) = 1$
$u_1'(0) = ?$



… etc

How to find all 1-truncated solutions of PI?

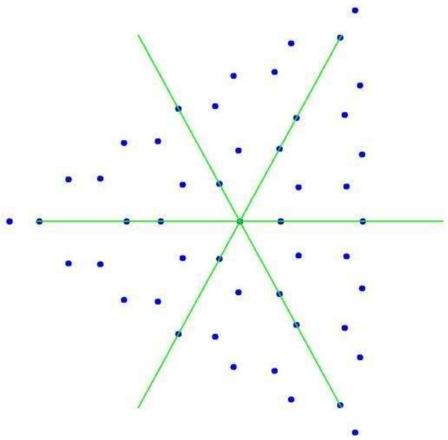
## Transition towards rational solutions

$$u_{zz} = zu + 2u^3 + n$$

Yablonskii-Vorob'ev polynomials [8]

$$u_n(z) = \frac{d}{dz} \ln \frac{y_{n-1}(z)}{y_n(z)},$$

$$y_{n+1} = \frac{z\, y_n^2 - 4\left(y_n y_n'' - y_n' y_n'\right)}{y_{n-1}}, \quad y_0 = 1, \quad y_1 = z.$$



$$y_2(z) = z^3 + 4,$$

$$y_3(z) = z^6 + 20z^3 - 80,$$

$$y_4(z) = (z^9 + 60z^6 - 11200)z,$$

$$\ldots$$

For *n* = odd $\quad u_n(0) = 0, \; u_n'(0) = 0$

Poles of $u_7(z)$ solution

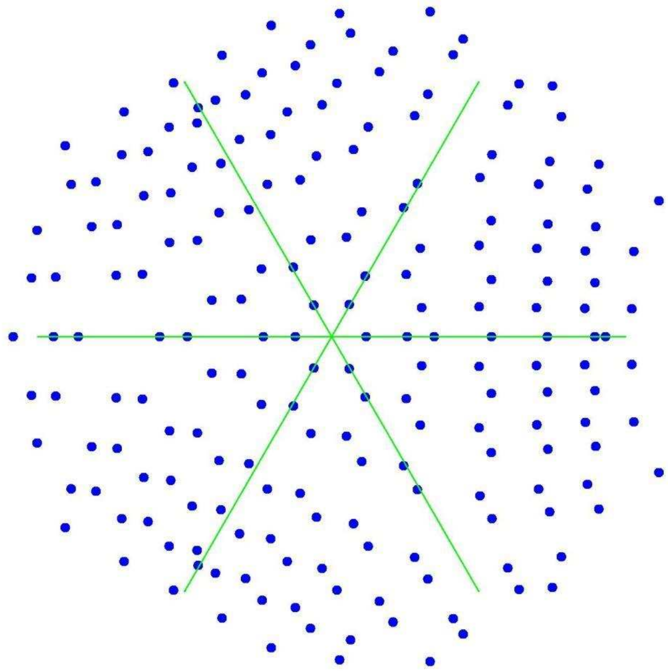
What is the distribution of poles while

$$u(0) = \varepsilon, \; u'(0) = \varepsilon, \quad \varepsilon \to 0$$

[8] A.I.Yablonskii, Vesti Acad.Nauk BSSR, v.3, 30-35 (1959); A. P. Vorob'ev, Diff.Uravneniya, v.1, 79–81 (1965).

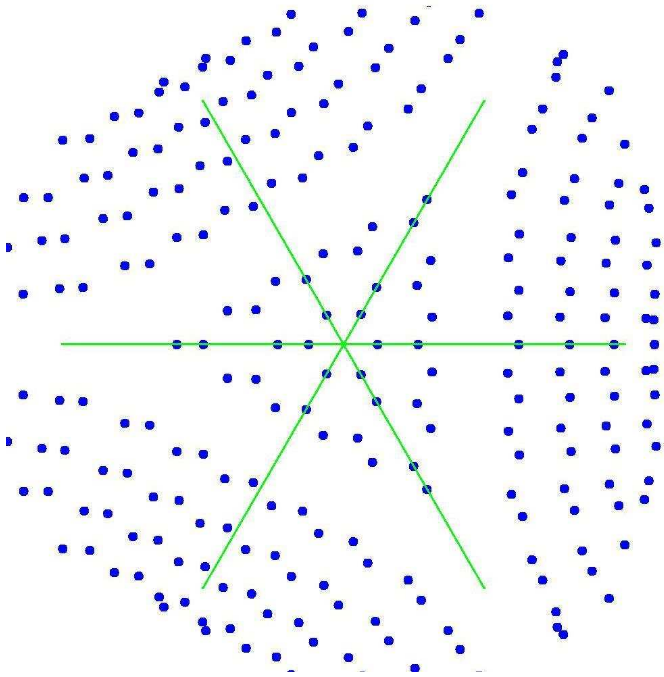
$$u_{zz} = zu + 2u^3 + \alpha$$

$\alpha = 7$
$u(0) = .1$
$u'(0) = .1$



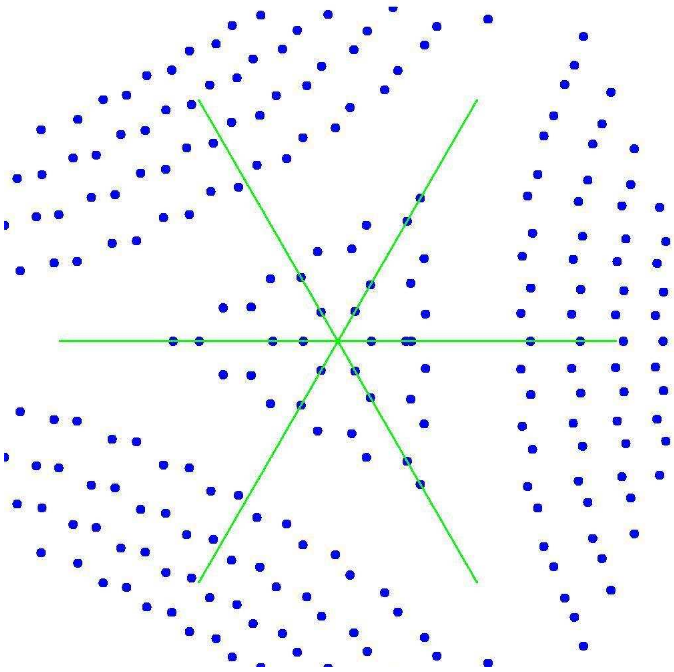
$$u_{zz} = zu + 2u^3 + \alpha$$

$$\alpha = 7$$
$$u(0) = .01$$
$$u'(0) = .01$$



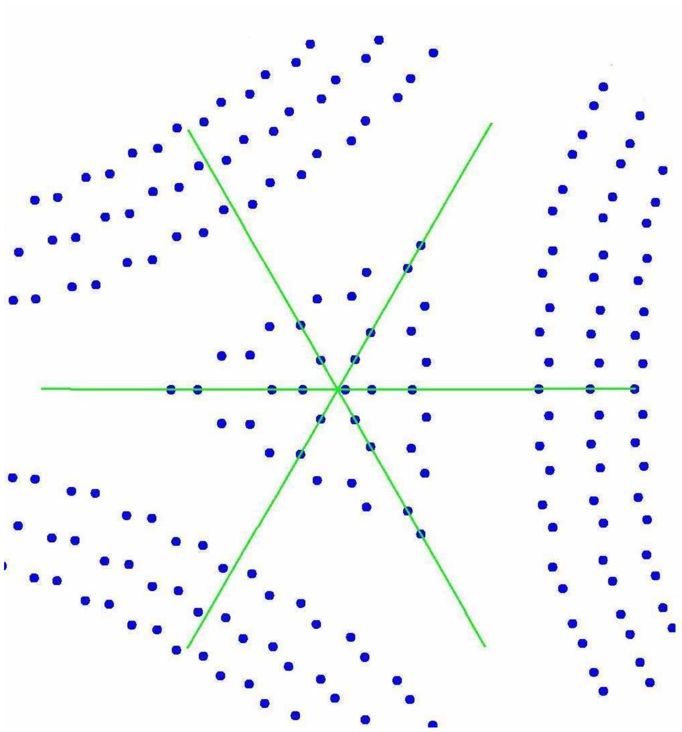
$$u_{zz} = zu + 2u^3 + \alpha$$

$$\alpha = 7$$
$$u(0) = .001$$
$$u'(0) = .001$$



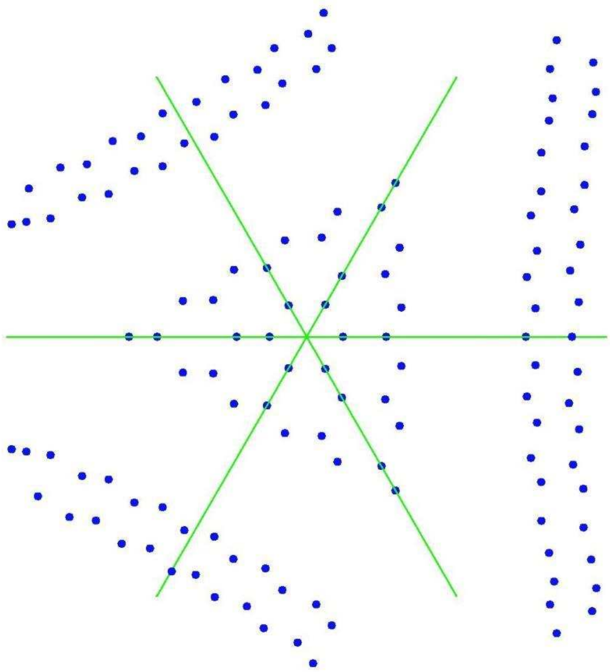
$$u_{zz} = zu + 2u^3 + \alpha$$

$$\alpha = 7$$
$$u(0) = .0001$$
$$u'(0) = .0001$$



$$u_{zz} = zu + 2u^3 + \alpha$$

$$\alpha = 7$$
$$u(0) = .00001$$
$$u'(0) = .00001$$



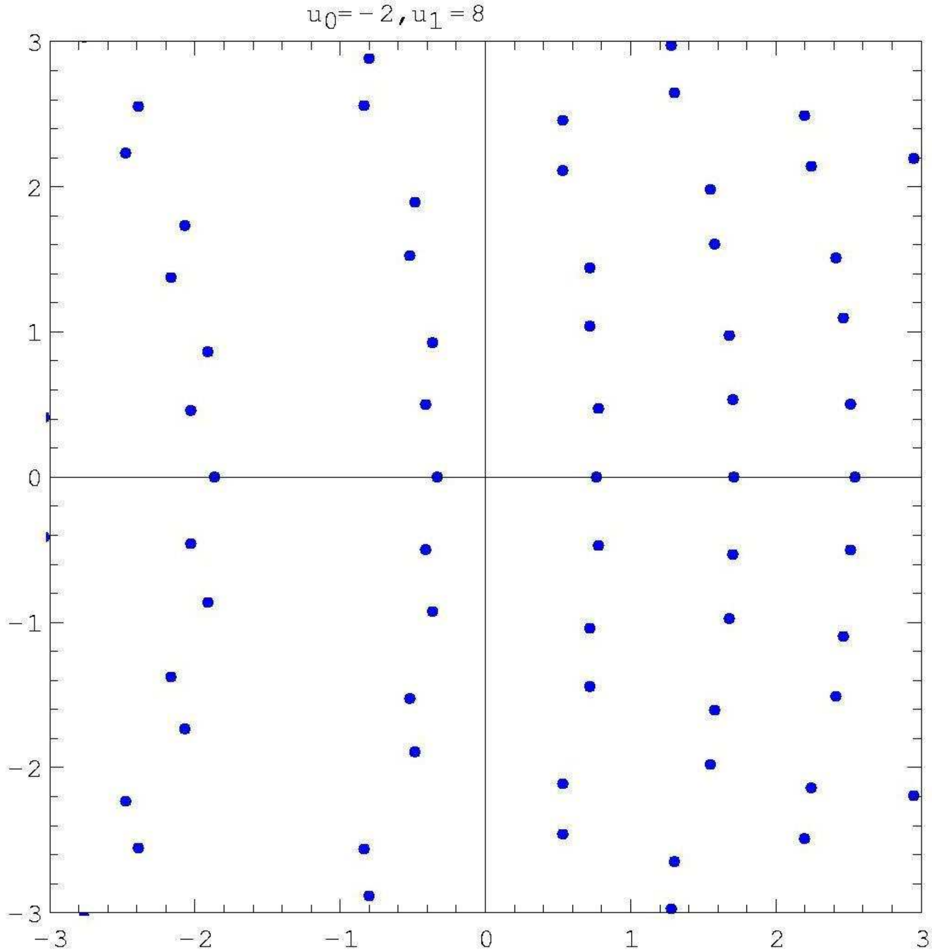
## Formation of patterns: PIV equation

$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

$$u(0) = u_0$$

$$u_z(0) = u_1$$

$$\boxed{\begin{aligned} a &= 10 \\ b &= -1 \\ u_0 &= -2 \\ u_1 &= 8 \end{aligned}}$$



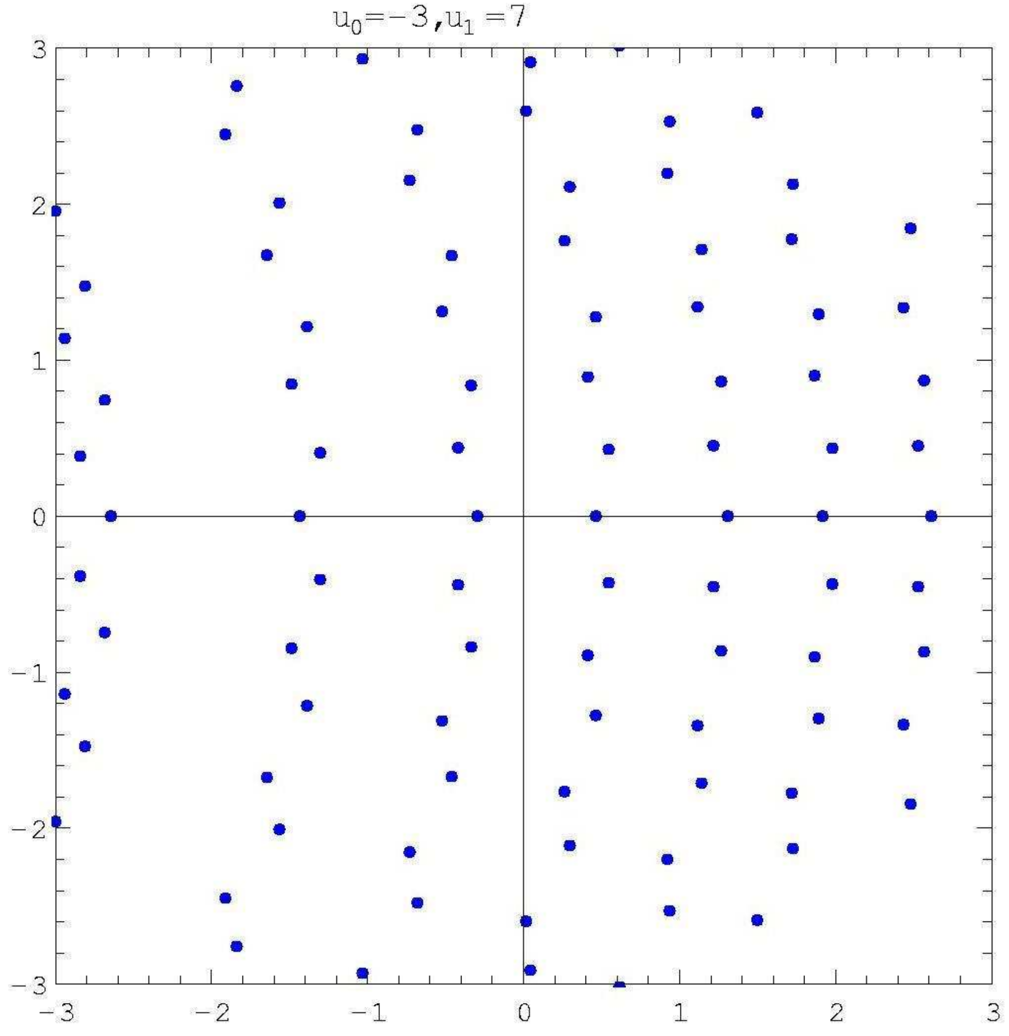

Deformation rule:

$$\boxed{u_1 - u_0 = \text{const}}$$

$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

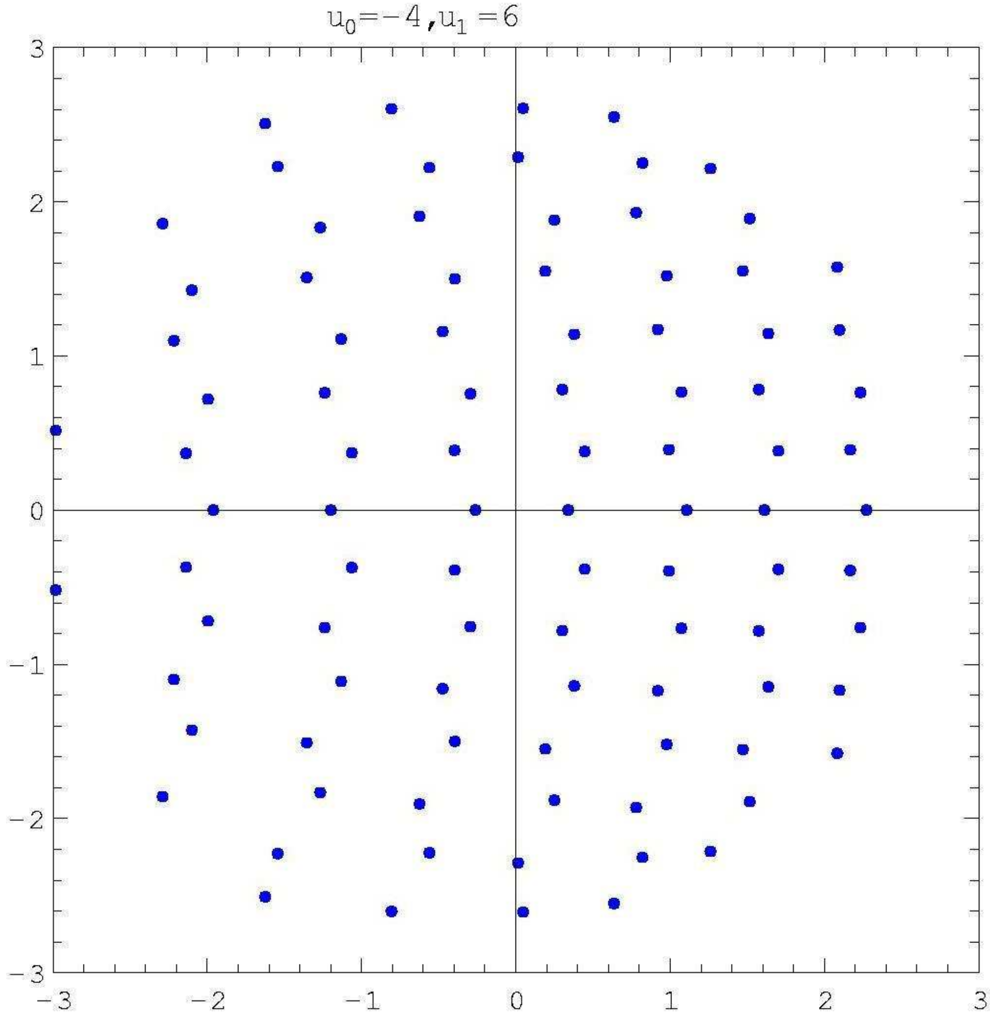
$a = 10$
$b = -1$

$u_0 = -3$
$u_1 = 7$



$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

$a = 10$
$b = -1$
$u_0 = -4$
$u_1 = 6$



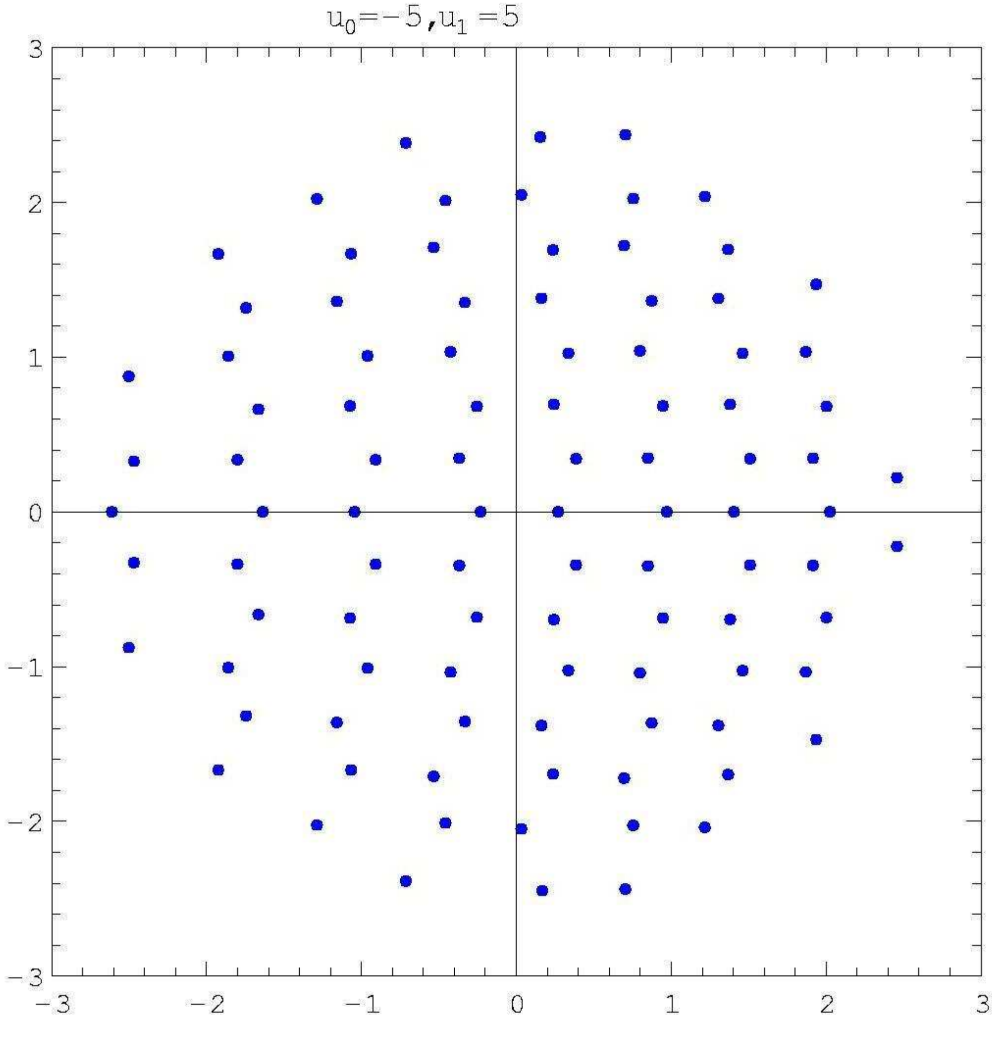
$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

$a = 10$

$b = -1$

$u_0 = -5$

$u_1 = 5$



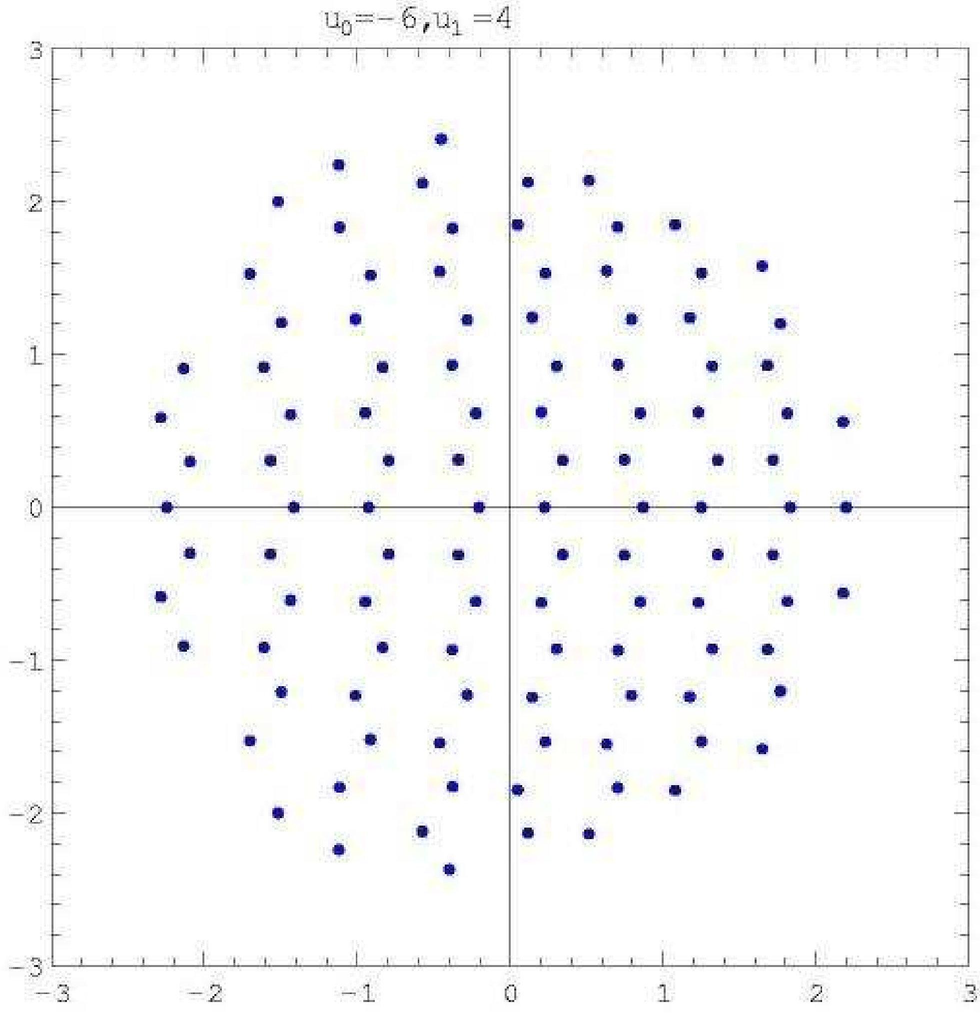
$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

$a = 10$

$b = -1$

$u_0 = -6$

$u_1 = 4$



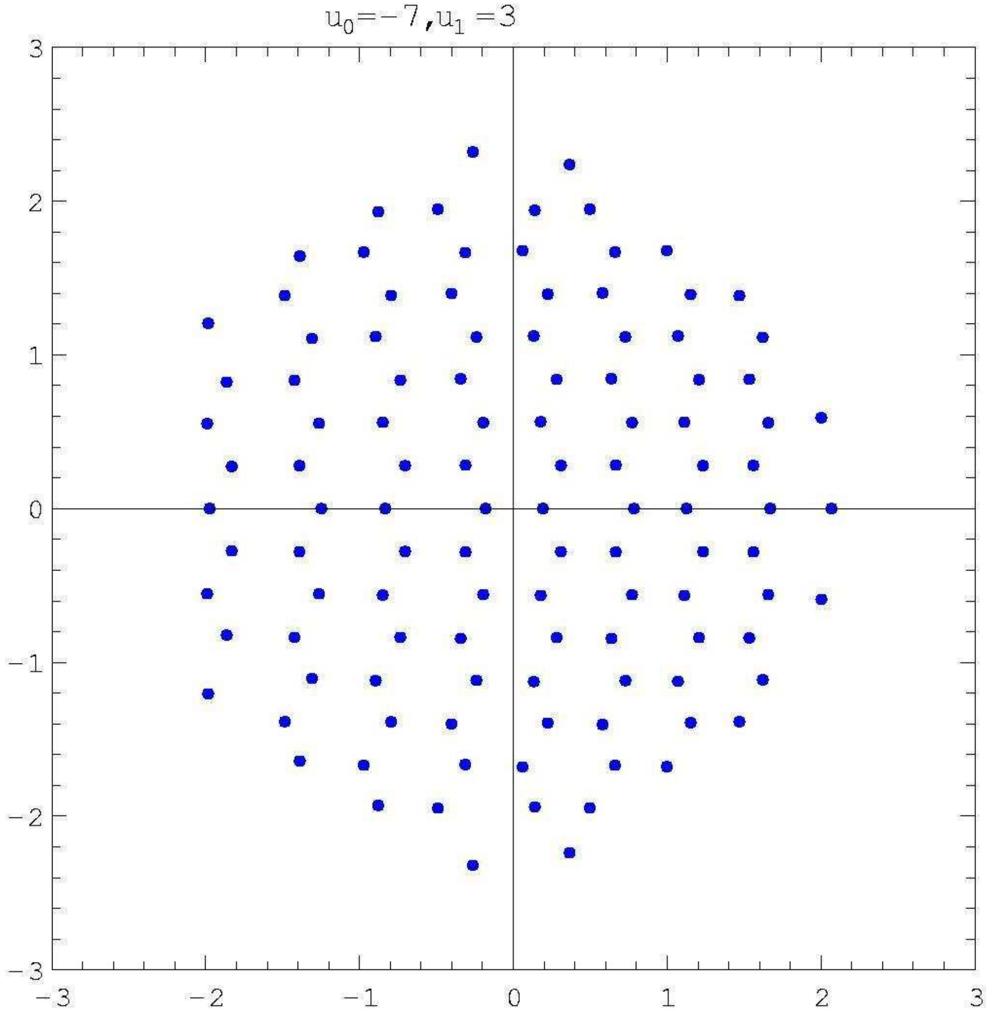
$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

$a = 10$

$b = -1$

$u_0 = -7$

$u_1 = 3$



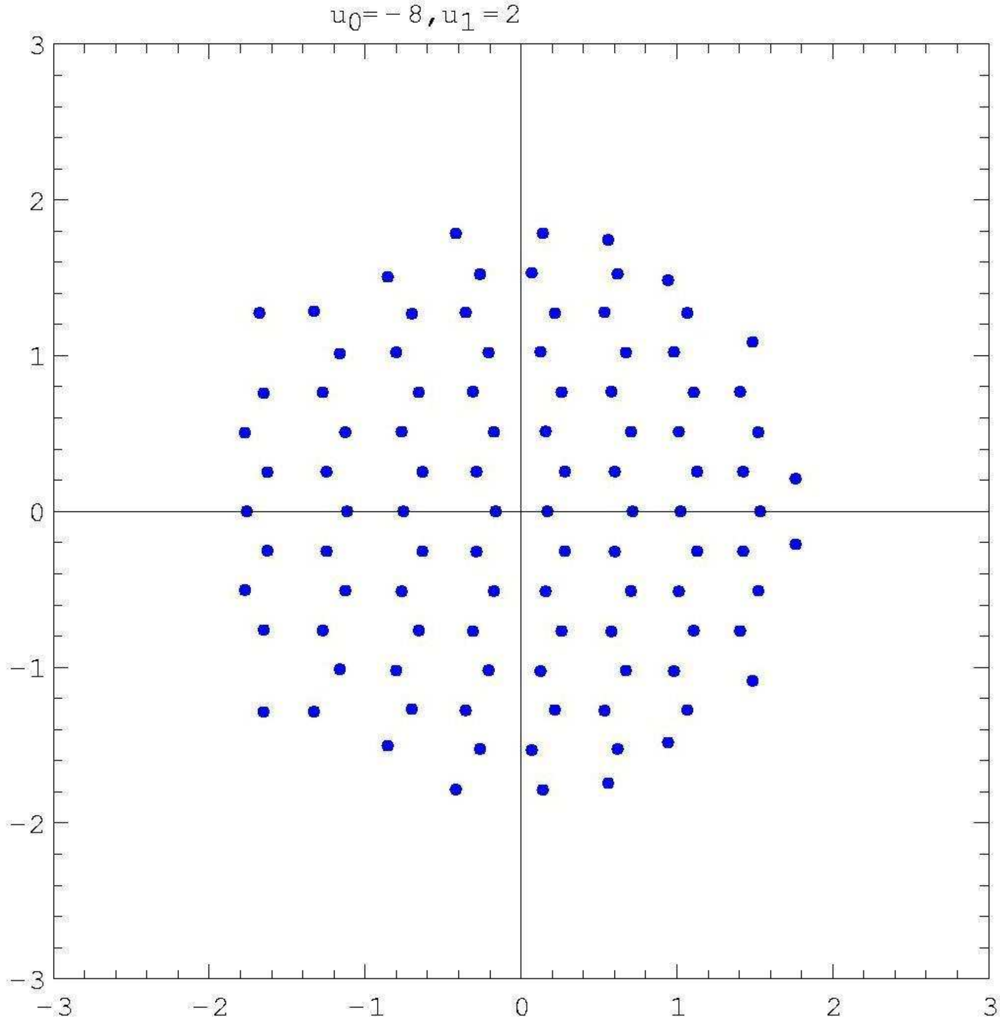
$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

$a = 10$

$b = -1$

$u_0 = -8$

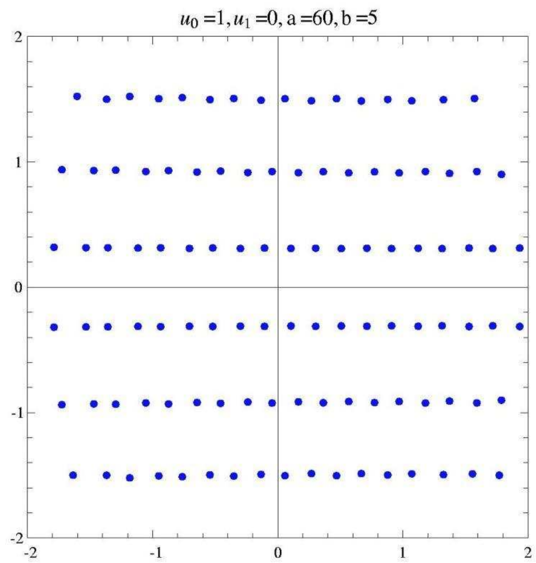
$u_1 = 2$



# Formation of patterns in PIV

$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$

$$u(0) = u_0, \quad u_z(0) = u_1$$




Asymptotics near the origin

$$u(z) = U(z) + zU_1(z) + \ldots, \qquad z \to 0$$

Equation for the leading term

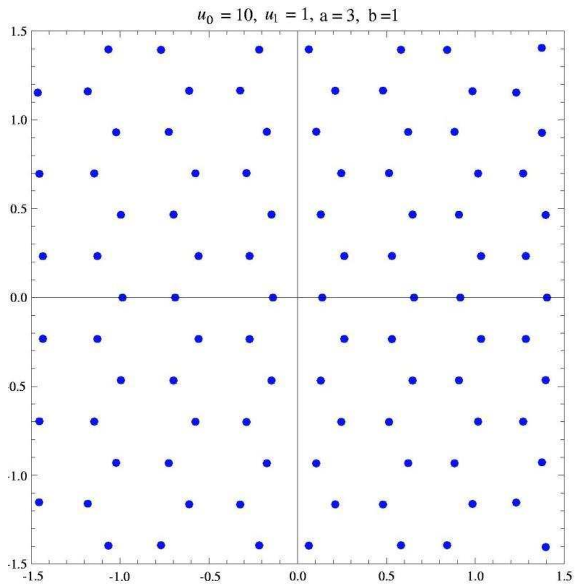
$$U_z^2 = U^4 - 4aU^2 + cU + b, \qquad c = \frac{u_1^2}{u_0} - u_0^3 - au_0 - b$$

Elliptic two-period function

$$z = \int_{u_0}^{u} \frac{dU}{\sqrt{U^4 - 4aU^2 + cU + b}}$$

## Formation of patterns in PIV

$$u_{zz} = \frac{u_z^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2\left(z^2 - a\right)u + \frac{b}{u}$$



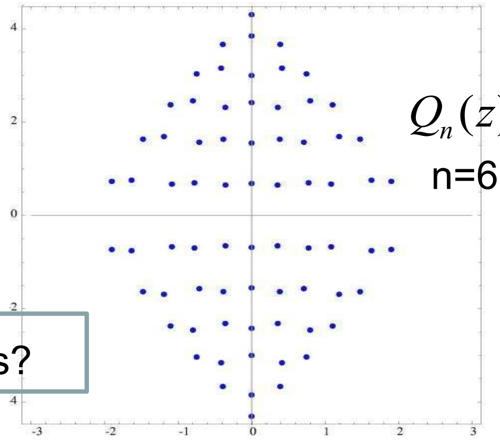

Rational solutions $$u(z) = -\frac{2}{3}z + \frac{Q'_n(z)}{Q_n(z)},$$

$Q_n(z) -$ Okamoto polynomials, $z_k$ - zeroes

Stieltjes relation $$\sum_{j \neq k} \frac{1}{z_k - z_j} - z_k = 0, \quad k = 1, 2, \ldots n$$

Coulomb charges in external field

$$U = -\frac{1}{2}\sum_k z_z^2 + \sum_{k \neq j} \log\left(z_k - z_j\right)$$



$Q_n(z)$

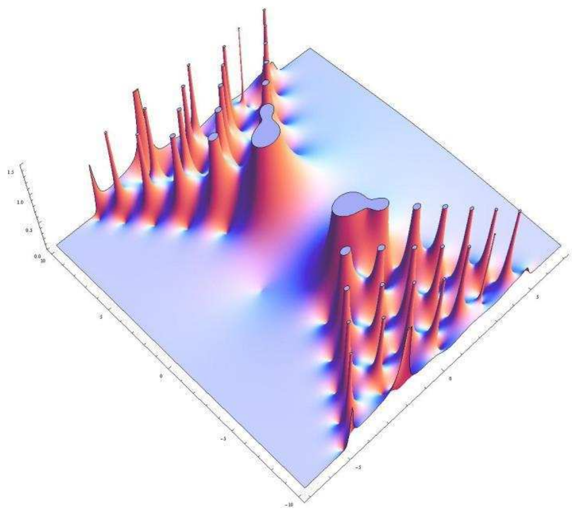
n=6

Equilibrium state - a hexagonal lattice

How to extend Stieltjes relation on infinite set of poles?

## Open questions

- Proof of the “no extra poles” conjectures for 2- and 3-truncated solutions
- Transition between truncated solutions
- Formation of patterns in lattices of poles
- Scaling limits of polynomials $P_n(z)$ which form rational solutions as $n \to \infty$ and $z \to \infty$
- Study of PIII, PV and PVI poles structure



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# Grazie per l'attenzione!