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Discrete Painlevé equations from the self-dual Yang-Mills equations

Abstract. Many (continuous) integrable systems, including all of the Painlevé equations, are known to be reductions of the self-dual Yang-Mills (SDYM) equations. A general class of Bäcklund transformations for the SDYM equations will be described. The Bianchi permutability of these Bäcklund transformations leads to a very rich discrete integrable system, which will be shown to have reductions to many important discrete equations, including discrete and q-Painlevé equations.

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Discrete Painlevé equations from the self-dual Yang-Mills equations

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The anti-self-dual Yang-Mills equations

Four independent variables: z, $\tilde{z}$, w and $\tilde{w}$.

Four Lie-algebra (in our case $\mathrm{sl}(2;\mathbb{C})$)-valued functions: A_z, A_w, $A_{\tilde{z}}$ and $A_{\tilde{w}}$.

The ASDYM equations are

$$\begin{aligned}
&\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0,\\
&\partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0,\\
&\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0.
\end{aligned}$$

- Connection one-form: $\mathbf{A} := A_z\,\mathrm{d}z + A_w\,\mathrm{d}w + A_{\tilde{z}}\,\mathrm{d}\tilde{z} + A_{\tilde{w}}\,\mathrm{d}\tilde{w}$.
- The ASDYM equations are the compatibility condition for the Lax pair

$$\begin{aligned}
(\partial_z - \zeta\partial_{\tilde{w}})\Psi &= -(A_z - \zeta A_{\tilde{w}})\Psi,\\
(\partial_w - \zeta\partial_{\tilde{z}})\Psi &= -(A_w - \zeta A_{\tilde{z}})\Psi.
\end{aligned}$$

- This Lax pair is equivalent to the statement that the differential operators

$$L = D_w - \zeta D_{\tilde{z}} \qquad \text{and} \qquad M = D_z - \zeta D_{\tilde{w}}$$

commute, where

$$D_z = \partial_z + A_z, \quad D_w = \partial_w + A_w, \quad D_{\tilde{z}} = \partial_{\tilde{z}} + A_{\tilde{z}}, \text{ and } D_{\tilde{w}} = \partial_{\tilde{w}} + A_{\tilde{w}}.$$

Symmetries

- Conformal symmetries (translations, rotations/boosts, dilations, inversions)
- The ASDYM equations are invariant under the gauge transformation

$$A_\mu \mapsto g^{-1}\partial_\mu g + g^{-1}A_\mu g,$$

  for any nonsingular g.
- Symmetry reductions, Ward.
- Non-point symmetries (Bäcklund transformations)

Symmetry reductions: the Nahm equations

- We look for solutions in which the components of A depend only on $t := w + \tilde{w}$.
- Using a gauge transformation, we set $A_w + A_{\tilde{w}} = 0$.
- On writing $A_z = i(T_2 + iT_3)$, $A_{\tilde{z}} = i(T_2 - iT_3)$, $A_w = -iT_1$ and $A_{\tilde{w}} = iT_1$, the ASDYM equations reduce to the Nahm equations:
$$\dot{T}_1 = [T_2, T_3], \quad \dot{T}_2 = [T_3, T_1] \text{ and } \dot{T}_3 = [T_1, T_2].$$
- Making the further restriction that $T_j(t) = \omega_j(t)\sigma_j$, $j = 1, 2, 3$ we obtain
$$\dot{\omega}_1 = \omega_2\omega_3, \; \dot{\omega}_2 = \omega_3\omega_1 \text{ and } \dot{\omega}_3 = \omega_1\omega_2.$$
- We immediately have the first integrals $\omega_1^2 - \omega_2^2 = \mu^2$ and $\omega_1^2 - \omega_3^2 = \lambda^2$ and hence the solution
$$\begin{aligned} \omega_1(t) &= \mu \,\mathrm{sn}\big(\lambda t + \alpha, \lambda^{-1}\mu\big)\,, \\ \omega_2(t) &= i\mu \,\mathrm{cn}\big(\lambda t + \alpha, \lambda^{-1}\mu\big)\,, \\ \omega_3(t) &= -i\lambda \,\mathrm{dn}\big(\lambda t + \alpha, \lambda^{-1}\mu\big)\,. \end{aligned}$$

Symmetry reductions: the sine-Gordon equation

- We look for solutions of the ASDYM equations such that the A_μ's depend on z and $\tilde{z}$ only.
- We choose a gauge such that $A_{\tilde{z}} = 0$.
- The field equations become

$$\partial_z A_w + [A_z, A_w] = 0, \quad \partial_{\tilde{z}} A_{\tilde{w}} = 0, \quad \text{and} \quad \partial_{\tilde{z}} A_z + [A_w, A_{\tilde{w}}] = 0.$$

- Generically, we can use the remaining gauge freedom to put $A_{\tilde{w}}$ in the form

$$k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- For the remaining matrices, we take the form

$$A_w = \begin{pmatrix} 0 & a - ib \\ a + ib & 0 \end{pmatrix} \quad \text{and} \quad A_z = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}.$$

- This leads to the equations

$$a_z = 2ibc, \quad b_z = -2iac \quad \text{and} \quad c_{\tilde{z}} = 2ikb.$$

- We have

$$a_z = 2ibc, \quad b_z = -2iac \quad \text{and} \quad c_{\tilde{z}} = 2ikb.$$

- The first two equations give

$$a^2 + b^2 = \lambda^2,$$

where λ is a constant.

- Introducing the parametrization

$$a = \lambda \cos \omega \quad \text{and} \quad b = \lambda \sin \omega,$$

we find that $c = \frac{i}{2}\omega_z$ and

$$\omega_{z\tilde{z}} = 4k\lambda \sin \omega.$$

- So the sine-Gordon reduction is

$$A_z = \frac{i\omega_z}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_w = \lambda \begin{pmatrix} 0 & \exp(-i\omega) \\ \exp(i\omega) & 0 \end{pmatrix}, \quad A_{\tilde{z}} = 0, \quad A_{\tilde{w}} = k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\omega \equiv \omega(z, \tilde{z})$ solves $\omega_{z\tilde{z}} = 4k\lambda \sin \omega$.

Symmetry reductions: P_{VI} (Mason and Woodhouse)

- New variables: $p = -\log w$, $q = -\log \tilde{z}$, $r = \log(\tilde{w}/\tilde{z})$, and $t = (z\tilde{z})/(w\tilde{w})$.
- We intoduce the functions $P(t)$, $Q(t)$ and $R(t)$ by

$$
\begin{aligned}
\mathbf{A} &= A_z\,\mathrm{d}z + A_w\,\mathrm{d}w + A_{\tilde{z}}\,\mathrm{d}\tilde{z} + A_{\tilde{w}}\,\mathrm{d}\tilde{w} \\
&= P\mathrm{d}p + Q\mathrm{d}q + R\mathrm{d}r \\
&= -\frac{1}{w}P\mathrm{d}w - \frac{1}{\tilde{z}}Q\mathrm{d}\tilde{z} + R\left(\frac{\mathrm{d}\tilde{w}}{\tilde{w}} - \frac{\mathrm{d}\tilde{z}}{\tilde{z}}\right).
\end{aligned}
$$

- Hence $zA_z = 0$, $wA_w = -P$, $\tilde{z}A_{\tilde{z}} = -(Q+R)$ and $\tilde{w}A_{\tilde{w}} = R$.
- The ASDYM equations,

$$
\begin{aligned}
&\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0, \\
&\partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0, \\
&\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0.
\end{aligned}
$$

reduce to

$$
P' = 0, \quad tQ' = [R, Q] \quad \text{and} \quad t(1-t)R' = [tP + Q, R].
$$

- Furthermore, rewriting the ASDYM Lax pair as

$$\left(z\partial_z - \frac{z}{\tilde{w}}\zeta\tilde{w}\partial_{\tilde{w}}\right)\Psi = -\left(zA_z - \frac{z}{\tilde{w}}\zeta\tilde{w}A_{\tilde{w}}\right)\Psi,$$
$$\left(w\partial_w - \frac{w}{\tilde{z}}\zeta\tilde{z}\partial_{\tilde{z}}\right)\Psi = -\left(wA_w - \frac{w}{\tilde{z}}\zeta\tilde{z}A_{\tilde{z}}\right)\Psi,$$

and recalling that

$$zA_z = 0,\ wA_w = -P,\ \tilde{z}A_{\tilde{z}} = -(Q+R),\ \tilde{w}A_{\tilde{w}} = R$$

and

$$p = -\log w,\ q = -\log\tilde{z},\ r = \log(\tilde{w}/\tilde{z}),\ t = (z\tilde{z})/(w\tilde{w}),$$

we let $\lambda = -w/(\tilde{z}\zeta)$, giving

$$\partial_t\Psi = -\left(\frac{R}{\lambda - t}\right)\Psi,$$
$$\partial_\lambda\Psi = \left(\frac{Q}{\lambda} - \frac{P+Q+R}{\lambda - 1} + \frac{R}{\lambda - t}\right)\Psi.$$

Other reductions

- ODEs reductions of the ASDYM equations alone lead to elliptic functions, all six Painlevé equations in full generality.
- KdV, mKdV, sine-Gordon, Boussinesq, Ernst, chiral field, ...
- Darboux-Halphen
- generalised Chazy

Bäcklund transformations

BTs for the (A)SDYM equations have been introduced by several authors:

- Corrigan, Fairlie, Goddard and Yates, 1978
- Ling-Lie Chau Wang, 1980
- Bruschi, Levi and Ragnisco, 1982
- Chau and Chinea, 1986
- Papachristou and Harrison, 1987
- Mason, Chakravarty and Newman, 1987
- Tafel, 1989
- Masuda obtained the affine Weyl group symmetry of P_{II}, P_{III} and P_{IV} from the ASDYM Bäcklund transformation, 2005, 2007

Yang's equation (1)

The first two of the ASDYM equations,

$$\partial_z A_w - \partial_w A_z + [A_z, A_w] = 0, \qquad \text{and}$$
$$\partial_{\tilde{z}} A_{\tilde{w}} - \partial_{\tilde{w}} A_{\tilde{z}} + [A_{\tilde{z}}, A_{\tilde{w}}] = 0,$$

imply the existence of $SL(2;\mathbb{C})$-valued functions H and K respectively such that

$$\partial_z H = -A_z H, \quad \partial_w H = -A_w H, \quad \partial_{\tilde{z}} K = -A_{\tilde{z}} K, \quad \partial_{\tilde{w}} K = -A_{\tilde{w}} K.$$

The final equation,

$$\partial_z A_{\tilde{z}} - \partial_{\tilde{z}} A_z - \partial_w A_{\tilde{w}} + \partial_{\tilde{w}} A_w + [A_z, A_{\tilde{z}}] - [A_w, A_{\tilde{w}}] = 0,$$

then takes the compact form

$$\partial_w (J^{-1} \partial_{\tilde{w}} J) - \partial_z (J^{-1} \partial_{\tilde{z}} J) = 0,$$

where $J = K^{-1} H$. This is known as *Yang's equation*.

Yang's equation has the obvious symmetry

$$J(z, w, \tilde{z}, \tilde{w}) \mapsto M(z, w) J(z, w, \tilde{z}, \tilde{w}) \widetilde{M}(\tilde{z}, \tilde{w}).$$

Yang's equation (2)

Apart from the Ernst equation and the chiral fields models, this form of ASDYM is not usually considered in relation to symmetry reductions.

One reason is that Yang's equation has lost some of the Lie symmetries of the original system.

Also, the matrices A_z, A_w, etc, of the original formalism appear directly in the Lax pair of ASDYM.

If we parametrize $J \in SL(2;\mathbb{C})$ by

$$J = \frac{1}{f}\begin{pmatrix} 1 & g \\ e & f^2 + eg \end{pmatrix},$$

then Yang's equations reduces to three simple second-order PDEs:

$$\partial_z \partial_{\tilde{z}}(\log f) + \frac{(\partial_{\tilde{z}} e)(\partial_z g)}{f^2} = \partial_w \partial_{\tilde{w}}(\log f) + \frac{(\partial_{\tilde{w}} e)(\partial_w g)}{f^2},$$

$$\partial_{\tilde{z}}\left(\frac{\partial_z g}{f^2}\right) = \partial_{\tilde{w}}\left(\frac{\partial_w g}{f^2}\right),$$

$$\partial_z\left(\frac{\partial_{\tilde{z}} g}{f^2}\right) = \partial_w\left(\frac{\partial_{\tilde{w}} g}{f^2}\right).$$

The Standard ASDYM Bäcklund transformation

Writing out Yang's form of the ASDYM eqns in component form gives

$$\partial_z\partial_{\tilde{z}}(\log f) + \frac{(\partial_{\tilde{z}}e)(\partial_z g)}{f^2} = \partial_w\partial_{\tilde{w}}(\log f) + \frac{(\partial_{\tilde{w}}e)(\partial_w g)}{f^2},$$
$$\partial_{\tilde{z}}\left(\frac{\partial_z g}{f^2}\right) = \partial_{\tilde{w}}\left(\frac{\partial_w g}{f^2}\right),$$
$$\partial_z\left(\frac{\partial_{\tilde{z}} g}{f^2}\right) = \partial_w\left(\frac{\partial_{\tilde{w}} g}{f^2}\right).$$

Bäcklund transformation: $$\hat{f} = \frac{1}{f},$$

$$\partial_z\hat{g} = \frac{\partial_{\tilde{w}}e}{f^2}, \qquad \partial_w\hat{g} = \frac{\partial_{\tilde{z}}e}{f^2},$$
$$\partial_{\tilde{z}}\hat{e} = \frac{\partial_w g}{f^2}, \qquad \partial_{\tilde{w}}\hat{e} = \frac{\partial_z g}{f^2}.$$

Nahm reduction in the Yang formalism

- We return to the Nahm reduction with $T_j = \omega_j(t)\sigma_j$, $t = w + \tilde{w}$:

$$A_z = \frac{1}{2}\begin{pmatrix} 0 & \omega_1 + \omega_2 \\ \omega_1 + \omega_2 & 0 \end{pmatrix}, \quad A_w = \frac{\omega_3}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$A_{\tilde{z}} = \frac{1}{2}\begin{pmatrix} 0 & \omega_1 - \omega_2 \\ \omega_1 - \omega_2 & 0 \end{pmatrix}, \quad A_{\tilde{w}} = \frac{\omega_3}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- We need to find H and K such that

$$\partial_z H = -A_z H, \quad \partial_w H = -A_w H, \quad \partial_{\tilde{z}} K = -A_{\tilde{z}} K, \quad \partial_{\tilde{w}} K = -A_{\tilde{w}} K.$$

- In terms of $\mu = \sqrt{\omega_1^2 - \omega_2^2}$, the first two equations give

$$H(z, w, \tilde{z}, \tilde{w}) = \begin{pmatrix} \sqrt{\omega_1(t) + \omega_2(t)} & -\sqrt{\omega_1(t) + \omega_2(t)} \\ \sqrt{\omega_1(t) - \omega_2(t)} & \sqrt{\omega_1(t) - \omega_2(t)} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mu z/2} & 0 \\ 0 & \mathrm{e}^{\mu z/2} \end{pmatrix} \widetilde{M}(\tilde{z}, \tilde{w}).$$

- K has a similar expression, giving

$$J = \frac{1}{\mu} M(z, w) \begin{pmatrix} \omega_1(t)\mathrm{e}^{-\mu(z-\tilde{z})/2} & -\omega_2(t)\mathrm{e}^{\mu(z+\tilde{z})/2} \\ -\omega_2(t)\mathrm{e}^{-\mu(z+\tilde{z})/2} & \omega_1(t)\mathrm{e}^{\mu(z-\tilde{z})/2} \end{pmatrix} \widetilde{M}(\tilde{z}, \tilde{w}), \qquad \mu^2 = \omega_1^2 - \omega_2^2.$$

The sine-Gordon reduction in Yang's formalism

- The sine-Gordon reduction was

$$A_z = \frac{i\theta_z}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad A_w = \lambda\begin{pmatrix}0 & \exp(i\theta)\\ \exp(-i\theta) & 0\end{pmatrix}, \quad A_{\tilde{z}} = 0, \quad A_{\tilde{w}} = k\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

  where $\theta \equiv \theta(z, \tilde{z})$ solves

$$\theta_{z\tilde{z}} = 4k\lambda \sin\theta.$$

- In order to construct the Bäcklund transformation, we first construct J, and hence H and K.

- From $\partial_{\tilde{z}}K = -A_{\tilde{z}}K$ and $\partial_{\tilde{w}}K = -A_{\tilde{w}}K$, we have

$$K = \exp\left\{-k\tilde{w}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\} M(z, w).$$

- Find H from $\partial_w H = -A_w H$ and $\partial_z H = -A_z H$, we have

$$J = M(z, w)\begin{pmatrix}\cosh k\tilde{w} & \sinh k\tilde{w}\\ \sinh k\tilde{w} & \cosh k\tilde{w}\end{pmatrix}\begin{pmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{pmatrix}\begin{pmatrix}\cosh \lambda w & -\sinh \lambda w\\ -\sinh \lambda w & \cosh \lambda w\end{pmatrix}\widetilde{M}(\tilde{z}, \tilde{w}).$$

$P_{\rm VI}$ reduction in Yang's formalism

- $t = \dfrac{z\tilde{z}}{w\tilde{w}}$.
- Recall that the reduced equations are

$$P' = 0, \quad tQ' = [R, Q] \quad \text{and} \quad t(1-t)R' = [tP + Q, R].$$

- The second equation shows that

$$Q(t) = G(t)^{-1}Q_0G(t) \text{ and } R(t) = -tG(t)^{-1}G'(t),$$

  for some function G and constant Q_0.
- The form of J is then

$$J = \tilde{z}^{-Q_0}G(t)w^P.$$

Bäcklund transformations for the ASDYM equations

Starting from the ASDYM Lax pair

$$(\partial_z - \zeta\partial_{\tilde{w}})\Psi = -(A_z - \zeta A_{\tilde{w}})\Psi,$$
$$(\partial_w - \zeta\partial_{\tilde{z}})\Psi = -(A_w - \zeta A_{\tilde{z}})\Psi,$$

we perform a ζ-dependent gauge transformation

$$\Psi \mapsto \hat{\Psi} = (S + \zeta T)\Psi,$$

such that the resulting system has the same form:

$$(\partial_z - \zeta\partial_{\tilde{w}})\hat{\Psi} = -(\hat{A}_z - \zeta \hat{A}_{\tilde{w}})\hat{\Psi},$$
$$(\partial_w - \zeta\partial_{\tilde{z}})\hat{\Psi} = -(\hat{A}_w - \zeta \hat{A}_{\tilde{z}})\hat{\Psi}.$$

This gives

$$\begin{aligned}\big\{(S_w - SA_w + \hat{A}_w S) + \zeta(T_w - S_{\tilde{z}} + SA_{\tilde{z}} - TA_w + \hat{A}_w T - \hat{A}_{\tilde{z}} S)\\ +\zeta^2(-T_{\tilde{z}} + TA_{\tilde{z}} - \hat{A}_{\tilde{z}} T)\big\}\Psi = 0,\\ \big\{(S_z - SA_z + \hat{A}_z S) + \zeta(T_z - S_{\tilde{w}} + SA_{\tilde{w}} - TA_z + \hat{A}_z T - \hat{A}_{\tilde{w}} S)\\ +\zeta^2(-T_{\tilde{w}} + TA_{\tilde{w}} - \hat{A}_{\tilde{w}} T)\big\}\Psi = 0.\end{aligned}$$

- The coefficients of the various powers of ζ yield

$$S_w = SA_w - \hat{A}_w S, \qquad S_z = SA_z - \hat{A}_z S,$$
$$S_{\tilde{z}} - T_w = SA_{\tilde{z}} - \hat{A}_{\tilde{z}} S - TA_w + \hat{A}_w T,$$
$$S_{\tilde{w}} - T_z = SA_{\tilde{w}} - \hat{A}_{\tilde{w}} S - TA_z + \hat{A}_z T,$$
$$T_{\tilde{z}} = TA_{\tilde{z}} - \hat{A}_{\tilde{z}} T, \qquad T_{\tilde{w}} = TA_{\tilde{w}} - \hat{A}_{\tilde{w}} T.$$

- Recall that two of the three ASDYM equations guarantee H and K such that

$$\partial_z H = -A_z H, \quad \partial_w H = -A_w H, \quad \partial_{\tilde{z}} K = -A_{\tilde{z}} K, \quad \partial_{\tilde{w}} K = -A_{\tilde{w}} K.$$

- $C := \hat{K}^{-1} T K$, $\tilde{C} := \hat{H}^{-1} S H \quad \Rightarrow \quad C \equiv C(z, w)$ and $\widetilde{C} \equiv C(\tilde{z}, \tilde{w})$.
- In terms of $J = K^{-1} H$, the remaining equations become

$$\hat{J}\left(\hat{J}^{-1} C J\right)_z = \left(\hat{J}\widetilde{C} J^{-1}\right)_{\tilde{w}} J,$$
$$\hat{J}\left(\hat{J}^{-1} C J\right)_w = \left(\hat{J}\widetilde{C} J^{-1}\right)_{\tilde{z}} J,$$

- Bruschi, Levi and Ragnisco (1982): C and $\widetilde{C}$ constant.
- Ling-Lie Chau Wang (1980): C and $\widetilde{C}$ constant multiples of the identity.

The BT for the sine-Gordon equation: $\theta_{z\tilde{z}} = \sin\theta$

Substituting

$$J = \begin{pmatrix} \cosh(\tilde{w}/2) & \sinh(\tilde{w}/2) \\ \sinh(\tilde{w}/2) & \cosh(\tilde{w}/2) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\theta/2} & 0 \\ 0 & \mathrm{e}^{-i\theta/2} \end{pmatrix} \begin{pmatrix} \cosh(w/2) & -\sinh(w/2) \\ -\sinh(w/2) & \cosh(w/2) \end{pmatrix}$$

into the equations defining the ASDYM BT transformation,

$$\hat{J}\left(\hat{J}^{-1}CJ\right)_z = \left(\hat{J}\widetilde{C}J^{-1}\right)_{\tilde{w}} J,$$
$$\hat{J}\left(\hat{J}^{-1}CJ\right)_w = \left(\hat{J}\widetilde{C}J^{-1}\right)_{\tilde{z}} J,$$

where

$$C = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{and} \quad \widetilde{C} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{pmatrix},$$

gives

$$\tilde{a}\partial_z(\hat{\theta}-\theta) = 2b\sin\left(\frac{\hat{\theta}+\theta}{2}\right), \quad \tilde{b}\partial_{\tilde{z}}(\hat{\theta}+\theta) = 2a\sin\left(\frac{\hat{\theta}-\theta}{2}\right),$$
$$a\partial_z(\hat{\theta}-\theta) = 2\tilde{b}\sin\left(\frac{\hat{\theta}+\theta}{2}\right), \quad b\partial_{\tilde{z}}(\hat{\theta}+\theta) = 2\tilde{a}\sin\left(\frac{\hat{\theta}-\theta}{2}\right).$$

Compatibility implies that either $a = \tilde{b} = 0$ or $b = \tilde{a} = 0$.

Schlesinger transformations for P_{VI}

- $J = U(\tilde{z})^{-1} G(t) V(w)$, where $t = \dfrac{z\tilde{z}}{w\tilde{w}}$.

$$U(\tilde{z}) = \tilde{z}^{Q_0} = \begin{pmatrix} \tilde{z}^{\theta_0/2} & 0 \\ 0 & \tilde{z}^{-\theta_0/2} \end{pmatrix}, \qquad V(w) = w^P = \begin{pmatrix} w^{\theta_\infty/2} & 0 \\ 0 & w^{-\theta_\infty/2} \end{pmatrix}.$$

- We obtain all 12 Schlesinger transformations for P_{VI} derived in Muğan and Sakka. For example

(a) $\hat{\theta}_0 = \theta_0 + 1$, $\hat{\theta}_1 = \theta_1$, $\hat{\theta}_t = \theta_t$, $\hat{\theta}_\infty = \theta_\infty + 1$,

$$C_1 = w^{1/2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{C}_1 = \tilde{z}^{1/2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) $\hat{\theta}_0 = \theta_0 - 1$, $\hat{\theta}_1 = \theta_1$, $\hat{\theta}_t = \theta_t$, $\hat{\theta}_\infty = \theta_\infty - 1$,

$$C_2 = w^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{C}_2 = w^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Bianchi permutability for the sine-Gordon BT

$$\omega_{uv} = \sin\omega$$

$$\left(\frac{\hat{\omega}-\omega}{2}\right)_u = \beta \sin\left(\frac{\hat{\omega}+\omega}{2}\right),$$
$$\left(\frac{\hat{\omega}+\omega}{2}\right)_v = \frac{1}{\beta} \sin\left(\frac{\hat{\omega}-\omega}{2}\right).$$

$$\omega \mapsto \hat{\omega} = \mathcal{T}(\beta;\omega)$$

$$\mathcal{T}(\beta_2, \mathcal{T}(\beta_1;\omega)) = \mathcal{T}(\beta_1, \mathcal{T}(\beta_2;\omega))$$

Let $\omega_{00} := \omega$ and $\omega_{m,n} := {\mathcal{T}_1}^m {\mathcal{T}_2}^n \omega$, where $\mathcal{T}_j(\omega) := \mathcal{T}(\beta_j, \omega)$. Then

$$(\beta_2 - \beta_1)\tan\left(\frac{\omega_{m+1,n+1} - \omega_{m,n}}{4}\right) = (\beta_2 + \beta_1)\tan\left(\frac{\omega_{m,n+1} - \omega_{m+1,n}}{4}\right).$$

Bianchi permutability for the ASDYM equations

The equations defining the BT transformation for ASDYM can be written as

$$\left(\hat{J}^{-1}CJ\right)_z = \widetilde{C}_{\tilde{w}} + \hat{J}^{-1}\hat{J}_{\tilde{w}}\widetilde{C} - \widetilde{C}J^{-1}J_{\tilde{w}},$$
$$\left(\hat{J}^{-1}CJ\right)_w = \widetilde{C}_{\tilde{z}} + \hat{J}^{-1}\hat{J}_{\tilde{z}}\widetilde{C} - \widetilde{C}J^{-1}J_{\tilde{z}},$$

where $C \equiv C(z,w)$ and $\widetilde{C} \equiv C(\tilde{z},\tilde{w})$.

Consider two BT transformations corresponding to the pairs of matrices $(C_1, \widetilde{C}_1)$ and $(C_2, \widetilde{C}_2)$. We have

$$\partial_z\left(J_1^{-1}C_1J\right) = (\partial_{\tilde{w}}\widetilde{C}_1) + J_1^{-1}(\partial_{\tilde{w}}J_1)\widetilde{C}_1 - \widetilde{C}_1J^{-1}\partial_{\tilde{w}}J,$$
$$\partial_z\left(J_2^{-1}C_2J\right) = (\partial_{\tilde{w}}\widetilde{C}_2) + J_2^{-1}(\partial_{\tilde{w}}J_2)\widetilde{C}_2 - \widetilde{C}_2J^{-1}\partial_{\tilde{w}}J,$$
$$\partial_z\left(J_{12}^{-1}C_2J_1\right) = (\partial_{\tilde{w}}\widetilde{C}_2) + J_{12}^{-1}(\partial_{\tilde{w}}J_{12})\widetilde{C}_2 - \widetilde{C}_2J_1^{-1}\partial_{\tilde{w}}J_1,$$
$$\partial_z\left(J_{21}^{-1}C_1J_2\right) = (\partial_{\tilde{w}}\widetilde{C}_1) + J_{21}^{-1}(\partial_{\tilde{w}}J_{21})\widetilde{C}_1 - \widetilde{C}_1J_2^{-1}\partial_{\tilde{w}}J_2.$$

If $J_{12} = J_{21}$ and $[\widetilde{C}_1, \widetilde{C}_2] = 0$, it follows that

$$\partial_z\left\{J_{12}^{-1}\left(C_2J_1\widetilde{C}_1 - C_1J_2\widetilde{C}_2\right) + \left(\widetilde{C}_2J_1^{-1}C_1 - \widetilde{C}_1J_2^{-1}C_2\right)J\right\} = 0.$$

We take $\quad J_{12}^{-1}\left(C_2J_1\widetilde{C}_1 - C_1J_2\widetilde{C}_2\right) + \left(\widetilde{C}_2J_1^{-1}C_1 - \widetilde{C}_1J_2^{-1}C_2\right)J = 0.$

Lax pair for ASDYM Bianchi system (1)

Starting from the ASDYM Lax pair

$$(\partial_z - \zeta\partial_{\tilde{w}})\Psi = -(A_z - \zeta A_{\tilde{w}})\Psi,$$
$$(\partial_w - \zeta\partial_{\tilde{z}})\Psi = -(A_w - \zeta A_{\tilde{z}})\Psi,$$

we perform a ζ-dependent gauge transformation

$$\Psi \mapsto \hat{\Psi} = (S + \zeta T)\Psi,$$

such that the resulting system has the same form:

$$(\partial_z - \zeta\partial_{\tilde{w}})\hat{\Psi} = -(\hat{A}_z - \zeta \hat{A}_{\tilde{w}})\hat{\Psi},$$
$$(\partial_w - \zeta\partial_{\tilde{z}})\hat{\Psi} = -(\hat{A}_w - \zeta \hat{A}_{\tilde{z}})\hat{\Psi}.$$

In terms of H and K given by

$$\partial_z H = -A_z H, \quad \partial_w H = -A_w H, \quad \partial_{\tilde{z}} K = -A_{\tilde{z}} K, \quad \partial_{\tilde{w}} K = -A_{\tilde{w}} K,$$

we have

$$\Psi \mapsto \hat{\Psi} = (S + \zeta T)\Psi = (\hat{H}\widetilde{C}H^{-1} + \zeta \hat{K} C K^{-1})\Psi.$$

Set $\Phi = K^{-1}\Psi$. Then

$$\Phi \mapsto \hat{\Phi} = \left(\hat{J}\widetilde{C}J^{-1} + \zeta C\right)\Phi,$$

where $J = K^{-1}H$.

Lax pair for ASDYM Bianchi system (2)

$$\Phi \mapsto \hat{\Phi} = \left(\hat{J} \widetilde{C} J^{-1} + \zeta C \right) \Phi,$$

$$\Phi_{m+1,n} = \left(J_{m+1,n} \widetilde{C}^{(1)} J_{m,n}^{-1} + \zeta C^{(1)} \right) \Phi_{m,n},$$
$$\Phi_{m,n+1} = \left(J_{m,n+1} \widetilde{C}^{(2)} J_{m,n}^{-1} + \zeta C^{(2)} \right) \Phi_{m,n}.$$

Compatibility gives

$$J_{m+1,n+1}^{-1} \left(C^{(2)} J_{m+1,n} \widetilde{C}^{(1)} - C^{(1)} J_{m,n+1} \widetilde{C}^{(2)} \right)$$
$$+ \left(\widetilde{C}^{(2)} J_{m+1,n}^{-1} C^{(1)} - \widetilde{C}^{(1)} J_{m,n+1}^{-1} C^{(2)} \right) J_{m,n} = 0,$$

where

$$[C^{(1)}, C^{(2)}] = [\widetilde{C}^{(1)}, \widetilde{C}^{(2)}] = 0.$$

Sine-Gordon permutability as a reduction

Substituting

$$J = \begin{pmatrix} \cosh k\tilde{w} & \sinh k\tilde{w} \\ \sinh k\tilde{w} & \cosh k\tilde{w} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\theta/2} & 0 \\ 0 & \mathrm{e}^{-i\theta/2} \end{pmatrix} \begin{pmatrix} \cosh \lambda w & -\sinh \lambda w \\ -\sinh \lambda w & \cosh \lambda w \end{pmatrix} =: \widetilde{W} F W.$$

into

$$J_{12}^{-1}\left(C_2 J_1 \widetilde{C}_1 - C_1 J_2 \widetilde{C}_2\right) + \left(\widetilde{C}_2 J_1^{-1} C_1 - \widetilde{C}_1 J_2^{-1} C_2\right) J = 0$$

with

$$C_j = b_j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \widetilde{C}_j = \tilde{a}_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives

$$\sin\left(\frac{\theta_1 + \theta}{2}\right) - \sin\left(\frac{\theta_2 + \theta_{12}}{2}\right) + \kappa\left[\sin\left(\frac{\theta_1 + \theta_{12}}{2}\right) - \sin\left(\frac{\theta_2 + \theta}{2}\right)\right] = 0.$$

Discrete Painlevé equations as Bäcklund transformations

- This example is from Fokas, Grammaticos and Ramani.
- The third Painlevé equation is
$$w'' = \frac{w'^2}{w} - \frac{w'}{x} + \frac{1}{x}(\alpha w^2 + \beta) + \gamma + \frac{\delta}{w},$$
where α, β, γ and δ are constants.
- If $\gamma = 0$ but α and δ are non-zero, then rescaling gives
$$w'' = \frac{w'^2}{w} - \frac{w'}{x} - \frac{1}{x}(w^2 - \beta) - \frac{1}{w}. \tag{1}$$
- If $w \equiv w(x, \beta)$ is a solution of equation (1) then
$$w(x; \beta + 2) = \frac{x(1 + w'(x; \beta))}{w(x; \beta)^2} - \frac{\beta + 1}{w(x; \beta)}, \text{ and} \tag{2}$$
$$w(x; \beta - 2) = \frac{x(1 - w'(x; \beta))}{w(x; \beta)^2} - \frac{\beta - 1}{w(x; \beta)}, \tag{3}$$
are also solutions with β replaced by $\beta + 2$ and $\beta - 2$ respectively.
- Adding equations (2) and (3) gives
$$w(x; \beta + 2) + w(x; \beta - 2) = \frac{2x}{w(x; \beta)^2} - \frac{2\beta}{w(x; \beta)}.$$

Non-autonomous ASDYM Bianchi system

$$\Phi \mapsto \hat{\Phi} = \left(\hat{J}\widetilde{C}J^{-1} + \zeta C\right)\Phi,$$

$$\begin{aligned}\Phi_{m+1,n} &= \left(J_{m+1,n}\widetilde{C}^{(1)}_{m,n}J^{-1}_{m,n} + \zeta C^{(1)}_{m,n}\right)\Phi_{m,n},\\ \Phi_{m,n+1} &= \left(J_{m,n+1}\widetilde{C}^{(2)}_{m,n}J^{-1}_{m,n} + \zeta C^{(2)}_{m,n}\right)\Phi_{m,n}.\end{aligned}$$

Compatibility gives

$$\begin{aligned}&J^{-1}_{m+1,n+1}\left(C^{(2)}_{m+1,n}J_{m+1,n}\widetilde{C}^{(1)}_{m,n} - C^{(1)}_{m,n+1}J_{m,n+1}\widetilde{C}^{(2)}_{m,n}\right)\\ &+\left(\widetilde{C}^{(2)}_{m+1,n}J^{-1}_{m+1,n}C^{(1)}_{m,n} - \widetilde{C}^{(1)}_{m,n+1}J^{-1}_{m,n+1}C^{(2)}_{m,n}\right)J_{m,n} = 0,\end{aligned}$$

where

$$C^{(1)}_{m,n+1}C^{(2)}_{m,n} = C^{(2)}_{m+1,n}C^{(1)}_{m,n} \quad \text{and} \quad \widetilde{C}^{(1)}_{m,n+1}\widetilde{C}^{(2)}_{m,n} = \widetilde{C}^{(2)}_{m+1,n}\widetilde{C}^{(1)}_{m,n}.$$

Reduction of Bianchi system to the non-autonomous lattice mKdV equation

Let us consider the system

$$J_{m+1,n+1}^{-1}\left(C_{m+1,n}^{(2)}J_{m+1,n}\widetilde{C}_{m,n}^{(1)}-C_{m,n+1}^{(1)}J_{m,n+1}\widetilde{C}_{m,n}^{(2)}\right)$$
$$+\left(\widetilde{C}_{m+1,n}^{(2)}J_{m+1,n}^{-1}C_{m,n}^{(1)}-\widetilde{C}_{m,n+1}^{(1)}J_{m,n+1}^{-1}C_{m,n}^{(2)}\right)J_{m,n}=0,$$
$$C_{m,n+1}^{(1)}C_{m,n}^{(2)}=C_{m+1,n}^{(2)}C_{m,n}^{(1)}\quad\text{and}\quad\widetilde{C}_{m,n+1}^{(1)}\widetilde{C}_{m,n}^{(2)}=\widetilde{C}_{m+1,n}^{(2)}\widetilde{C}_{m,n}^{(1)},$$

independently of any connection with the ASDYM equations.

Let

$$C_{m,n}^{(1)}=\frac{1}{\alpha_m}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad C_{m,n}^{(2)}=\frac{1}{\beta_n}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \widetilde{C}_{m,n}^{(1)}=\widetilde{C}_{m,n}^{(2)}=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

and

$$J_{m,n}=\begin{pmatrix}u_{m,n}&0\\0&1/u_{m,n}\end{pmatrix}.$$

Then the Bianchi system reduces to

$$\alpha_m(u_{m,n}u_{m+1,n}-u_{m,n+1}u_{m+1,n+1})-\beta_n(u_{m,n}u_{m,n+1}-u_{m+1,n}u_{m+1,n+1})=0.$$

Factorisation

Symmetry reductions of the ASDYM equations in the original variables (A_μ) lead to reductions in Yang's form where J has the "dressed" form $J_{m,n} = AG_{m,n}B$.

Substituting this into

$$J_{m+1,n+1}^{-1}\left(C_{m+1,n}^{(2)}J_{m+1,n}\widetilde{C}_{m,n}^{(1)} - C_{m,n+1}^{(1)}J_{m,n+1}\widetilde{C}_{m,n}^{(2)}\right) + \left(\widetilde{C}_{m+1,n}^{(2)}J_{m+1,n}^{-1}C_{m,n}^{(1)} - \widetilde{C}_{m,n+1}^{(1)}J_{m,n+1}^{-1}C_{m,n}^{(2)}\right)J_{m,n} = 0,$$

gives

$$G_{m+1,n+1}^{-1}\left(D_{m+1,n}^{(2)}G_{m+1,n}\widetilde{D}_{m,n}^{(1)} - D_{m,n+1}^{(1)}G_{m,n+1}\widetilde{D}_{m,n}^{(2)}\right) + \left(\widetilde{D}_{m+1,n}^{(2)}G_{m+1,n}^{-1}D_{m,n}^{(1)} - \widetilde{D}_{m,n+1}^{(1)}G_{m,n+1}^{-1}D_{m,n}^{(2)}\right)G_{m,n} = 0,$$

where $D^{(j)} = A^{-1}C^{(j)}A$ and $\widetilde{D}^{(j)} = B\widetilde{C}^{(j)}B^{-1}$.

Furthermore, the conditions

$$C_{m,n+1}^{(1)}C_{m,n}^{(2)} = C_{m+1,n}^{(2)}C_{m,n}^{(1)} \quad \text{and} \quad \widetilde{C}_{m,n+1}^{(1)}\widetilde{C}_{m,n}^{(2)} = \widetilde{C}_{m+1,n}^{(2)}\widetilde{C}_{m,n}^{(1)}$$

become

$$D_{m,n+1}^{(1)}D_{m,n}^{(2)} = D_{m+1,n}^{(2)}D_{m,n}^{(1)} \quad \text{and} \quad \widetilde{D}_{m,n+1}^{(1)}\widetilde{D}_{m,n}^{(2)} = \widetilde{D}_{m+1,n}^{(2)}\widetilde{D}_{m,n}^{(1)}.$$

dmKdV in terms of BTs

- The ASDYM Bianchi system with

$$J_{m,n} = \begin{pmatrix} \cosh(\tilde{w}/2) & \sinh(\tilde{w}/2) \\ \sinh(\tilde{w}/2) & \cosh(\tilde{w}/2) \end{pmatrix} \begin{pmatrix} u_{m,n}(z,\tilde{z}) & 0 \\ 0 & \frac{1}{u_{m,n}(z,\tilde{z})} \end{pmatrix} \begin{pmatrix} \cosh(w/2) & -\sinh(w/2) \\ -\sinh(w/2) & \cosh(w/2) \end{pmatrix}$$

and

$$C^{(1)}_{m,n} = \frac{1}{\alpha_m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C^{(2)}_{m,n} = \frac{1}{\beta_n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{C}^{(1)}_{m,n} = \widetilde{C}^{(2)}_{m,n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

again gives the dmKdV

$$\alpha_m(u_{m,n}u_{m+1,n} - u_{m,n+1}u_{m+1,n+1}) - \beta_n(u_{m,n}u_{m,n+1} - u_{m+1,n}u_{m+1,n+1}) = 0.$$

- However this is now a statement about BTs of a reduction of the ASDYM equations (specifically the sine-Gordon equation with $u_{m,n}(z,\tilde{z}) = \mathrm{e}^{i\theta_{m,n}(z,\tilde{z})/2}$).
- Ormerod has shown that dmKdV has a reduction to $\mathrm{qP_{VI}}$.

Summary

- Many integrable equations are know to be reductions of the ASDYM equations:

$$\partial_w(J^{-1}\partial_{\tilde{w}}J) - \partial_z(J^{-1}\partial_{\tilde{z}}J) = 0,$$

- We have derived the following form of the Bäcklund transformation

$$\hat{J}\left(\hat{J}^{-1}CJ\right)_z = \left(\hat{J}\widetilde{C}J^{-1}\right)_{\tilde{w}}J,$$
$$\hat{J}\left(\hat{J}^{-1}CJ\right)_w = \left(\hat{J}\widetilde{C}J^{-1}\right)_{\tilde{z}}J,$$

  where $C \equiv C(z,w)$ and $\widetilde{C} \equiv \widetilde{C}(\tilde{z},\tilde{w})$

- Bianchi permutability gives

$$J_{12}^{-1}\left(C_2J_1\widetilde{C}_1 - C_1J_2\widetilde{C}_2\right) + \left(\widetilde{C}_2J_1^{-1}C_1 - \widetilde{C}_1J_2^{-1}C_2\right)J = 0,$$

  which is a source for many discrete integrable systems.

- The richness of reductions of the ASDYM equations comes form the large (conformal) group of symmetries.

- The richness of discrete reductions of the ASDYM BT equations comes from the choices of $C(z,w)$, $\widetilde{C}(\tilde{z},\tilde{w})$ as well as the form of J.

Nevanlinna theory

- The *proximity function* is $m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta$, where $\log^+ x := \max(\log x, 0)$.
- The *enumerative function* is $N(r,f) := \int_0^r \frac{n(t,f)}{t} dt$, where $n(r,f)$ is the number of poles of f (counting multiplicities) in $|z| \le r$.
- The *Nevanlinna characteristic function* $T(r,f) = m(r,f) + N(r,f)$ measures "the affinity" of f for infinity.
- Nevanlinna's First Main Theorem
  For $a \in \mathbf{C}$,
  $$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1), \qquad r \to \infty.$$
- We use $S(r,f)$ to denote any function of r that is $o(T(r,f))$ outside some set of finite linear measure.
- Lemma on the Logarithmic Derivative: $m(r, f'/f) = S(r,f)$.

Joint work with Galina Filipuk and Risto Korhonen

Theorem

If there exist a pair x, y of non-rational finite order meromorphic solutions of system

$$\begin{aligned} x(z+1)+x(z) &= \frac{\alpha(z)+\beta(z)y(z)+\gamma(z)y(z)^2}{y(z)^2-1}, \\ y(z)+y(z-1) &= \frac{a(z)+b(z)x(z)+c(z)x(z)^2}{x(z)^2-1}, \end{aligned}$$

then $\gamma \equiv c \equiv 0$ and either x and y satisfy first-order difference equations or the system is reduced to the coupled d-$\mathrm{P_{II}}$ equations

$$\begin{aligned} x(z+1)+x(z) &= \frac{\alpha+(\beta+\delta z)y(z)}{y(z)^2-1}, \\ y(z)+y(z-1) &= \frac{a+(\beta-\delta/2+\delta z)x(z)}{x(z)^2-1}, \end{aligned} \tag{4}$$

where a, α, β and δ are constants.

Extending Painlevé analysis to find particular solutions

- Suppose that a solution of
$$\frac{d^2y}{dz^2} = 6y^2 + f(z)$$
has a pole at a point z_0 where f is analytic.

- The series expansion of the solution is necessarily of the form
$$y(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^{n-2}, \qquad a_0 = 1.$$

- Substituting and equating coeffs gives $a_1 = a_2 = a_3 = 0$ and the recurrence relation
$$(n + 1)(n - 6)a_n = 6 \sum_{m=1}^{n-1} a_m a_{n-m} + \frac{1}{(n - 4)!} f^{(n-4)}(z_0).$$

- There is a resonance at $n = 6$ which gives $f''(z_0) = 0$. If this is true for "enough" z_0 then
$$\frac{d^2y}{dz^2} = 6y^2 + Az + B,$$
where A and B are constants.

Can we find all meromorphic solutions of solutions of

$$ww'' - w'^2 = \alpha(z)w + \beta(z)w' + \gamma(z)?$$

- The constant coefficient case was solved in work with Chiang.
- Let Φ be the set of all zeros and poles of the coefficients α, β and γ.
- w has no poles in $\Omega := \mathbb{C} \setminus \Phi$.
- If $\beta \equiv \gamma \equiv 0$, then any zero of w on Ω is a double zero. Otherwise it is simple.
- If $\gamma \not\equiv 0$ then at each zero $z_0 \in \Omega$ of w,

  $$w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1},$$

  where $a_0^2 + \beta(z_0)a_0 + \gamma(z_0) = 0$ and the a_ns satisfy a recurrence relation of the form

  $$(n+1)(n-r)a_0 a_r = P_n(a_0, \dots, a_{n-1}),$$

  where r depends on the coefficients α, β and γ.
- Finiteness property (work of Hille, Eremenko, Conte).

General meromorphic coefficients

Joint work with Jun Wang.

- An admissible solution of $ww'' - w'^2 = \alpha(z)w + \beta(z)w' + \gamma(z)$ satisfies $T(r,\alpha) + T(r,\beta) + T(r,\gamma) = S(r,w)$.
- Consider the case $\alpha \not\equiv 0$, $\beta \equiv \gamma \equiv 0$.
- In the neighbourhood of any zero $z_0 \in \Omega$ of w,
$$w(z) = -\frac{\alpha(z_0)}{2}(z-z_0)^2 - \frac{\alpha'(z_0)}{2}(z-z_0)^3 + O\left((z-z_0)^4\right).$$
- Together with the fact that w is analytic on Ω, it follows that
$$f(z) := \left(\frac{w'}{w} - \frac{\alpha'}{\alpha}\right)^2 + 2\frac{\alpha}{w} = \left(\frac{w'}{w} - \frac{\alpha'}{\alpha}\right)^2 + 2\left(\frac{w'}{w}\right)'$$
is also analytic on Ω. Also $m(r,f) = S(r,w)$.
- Furthermore
$$\begin{aligned} N(r,f) &= N_\Phi(r,f) \le 2N_\Phi\left(r, \frac{w'}{w}\right) + 2N_\Phi\left(r, \frac{\alpha'}{\alpha}\right) + N_\Phi\left(r, \left(\frac{w'}{w}\right)'\right) \\ &= 4\left\{\bar{N}_\Phi(r,w) + \bar{N}_\Phi\left(r, \frac{1}{w}\right)\right\} + 2N_\Phi\left(r, \frac{\alpha'}{\alpha}\right) = S(r,w). \end{aligned}$$

Now

$$f' = 2\Big(\frac{\alpha'}{\alpha}\Big)'\Big(\frac{\alpha'}{\alpha} - \frac{w'}{w}\Big). \tag{5}$$

When $(\frac{\alpha'}{\alpha})' \not\equiv 0$, we obtain

$$\frac{w'}{w} = \frac{\alpha'}{\alpha} - \frac{f'}{2}\Big[\Big(\frac{\alpha'}{\alpha}\Big)'\Big]^{-1}.$$

Substituting this into the definition of f gives

$$f = \frac{f'^2}{4}\Big[\Big(\frac{\alpha'}{\alpha}\Big)'\Big]^{-2} + 2\frac{\alpha}{w}. \tag{6}$$

We must have $(\alpha'/\alpha)' \equiv 0$, so from Eq. (5), $f' \equiv 0$. Thus,

$$\alpha(z) \equiv k_1 e^{k_2 z} \quad \text{and} \quad f(z) \equiv c_1^2,$$

where $k_1 \neq 0$, k_2 and c_1 are constants.

In terms of $u = w/\alpha$, the definition of f becomes

$$u'^2 = c_1^2 u^2 - 2u.$$

Other generalisations with Khadija Al-Amoudi (including branching at fixed singularities).

Algebroid solutions

- A function f is called *algebroid* if it is algebraic over the meromorphic functions, i.e., it satisfies

$$a_0(z) + a_1(z)f(z) + \cdots + a_{n-1}(z)f(z)^{n-1} + f(z)^n = 0,$$

for meromorphic functions $a_0, \ldots, a_{n-1}$.

- Malmquist actually showed that if $F(z, y, y') = 0$ has an algebroid solution, where F is rational, then the equation can be reduced to either a Riccati equation or the equation for the Weierstrass elliptic function.

- Thomas Kecker and I have shown that the only *admissible* degree 2 algebroid solutions of

$$y'' = c_0(z) + \cdots + c_4(z)y^4 + y^5$$

can be expressed in terms of either admissible solutions of Riccati equations or the fourth Painlevé equation (or its degenerations).