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A Review of the Sixth Painlevé equation. Towards a handbook of non linear special functions.

Abstract. The isomonodromy deformation method provides a unitary description of the critical behaviours of the solutions of the Painlevé 6 equation, their connection formulae and the asymptotic distribution of the poles close to a critical point. I will review the main results achieved by me and others over the years, which allow to write a table of asymptotic behaviours and connection formulae of PVI transcendents.

References

D.G. "A Review of the Sixth Painlevé Equation", Constructive Approximations, DOI 10.1007/s00365-014-9250-6 (2014).

D.G. " Tabulation of Painlevé 6 transcendents". Nonlinearity 25 (2012), no. 12, 3235–3276.

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Rome, December 18, 2014

Introduction

In the 18th, 19th and 20th centuries, several problem of mathematical physics were described by linear differential equations of the II order.

The solutions to these ODEs define classical special functions (Bessel, Airy, hypergeometric, etc)

Over the years, they have been tabulated in handbooks (see the Bateman Project, and Handbook of Math. Functions, Abramowitz and Stegun).

Introduction

Around the middle of the 20th century, Painlevé functions, *re-appeared* in integrable systems as the **non linear special functions**.

Example: Wu, McCoy, Tracy, Barouch (1976) *Spin-spin correl. funct. 2-D Ising model*

Introduction

Painlevé functions: **the core of modern special function theory**

- Algebraic Geometry (Frobenius Manifolds, Quantum Cohomology)
- Number Theory and Combinatorics
  (Statistics of zeros of $\zeta(z)$ function, $\Re z = 1/2$. Ulam's problem [Tracy-Widom 1999 $\sim$].)
- Random Matrices
- Asymptotics of Orthogonal Polynomials
- Non linear PDEs

Introduction

Painlevé functions: **the core of modern special function theory**

Painlevé Project: We need an organization and tabulation of the properties (algebraic, analytic, asymptotic, numerical) of the Painlevé functions. NIST Digital Library of Mathematical Functions, (http://dlmf.nist.gov/)

The Sixth Painlevé Equation

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 +$$

$$-\left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx} +$$

$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right]$$

Complex constant coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$,

Singular points $\quad x = 0,\ 1,\ \infty.$

General solution depends on two integration constants $c_1,\ c_2$:

$$y(x) = y(x; c_1, c_2).$$

• **Painlevé property** = the critical points of the general solution (essential singularities, branch points) are $x = 0, 1, \infty$, determined by the equation.

• **A generic solution of PVI is not a classical function** (Painlevé Transcendent).

Classical function = Rational, Algebraic, Contour integral of rational and algebraic functions, Solution of linear homogeneous ODE with rational coefficients, Solution of algebraic ODE of I order (rational coeff), Elliptic functions, etc. [Umemura 1987-'90]

Why special functions?

Painlevé property $\Longrightarrow$ a solution $y(x)$ can be **analytically continued to the universal covering of $\mathbb{P}^1\backslash\{0,1,\infty\}$, as meromorphic function**.
[Miwa, Malgrange, (using isomonodromy deformations)]

The universal covering of $\mathbb{P}^1\backslash\{0,1,\infty\}$ is
determined only by the equation, not by the integration constants.

Fundamental property of linear equations!
This makes Painlevé transcendents special functions.

What does it mean that we know a special function?

i) Explicit critical behaviours = the behaviors of a transcendent $y(x)$ at singular points $x = 0$, 1 and ∞

$$y(x) \sim \begin{cases} y_0(x, c_1^{(0)}, c_2^{(0)}), & x \to 0 \\ y_1(x, c_1^{(1)}, c_2^{(1)}), & x \to 1 \\ y_\infty(x, c_1^{(\infty)}, c_2^{(\infty)}), & x \to \infty \end{cases}$$

$y_u(x, c_1^{(u)}, c_2^{(u)})$ = an **expansion** (convergent or asymptotic), or the **leading term**, for $x \to u \in \{0, 1, \infty\}$.

Explicit = each term of the expansion is a classical function of $(x, c_1^{(u)}, c_2^{(u)})$.

What does it mean that we know a special function?

ii) Explicit connection formulae.

Two critical points: $u,\ v \in \{0, 1, \infty\}$, $\quad u \neq v$,

and corresponding critical behaviors.

$$y(x) \sim y_u(x, c_1^{(u)}, c_2^{(u)}), \qquad x \longrightarrow u$$

$$y(x) \sim y_v(x, c_1^{(v)}, c_2^{(v)}), \qquad x \longrightarrow v$$

$$\begin{cases} c_1^{(v)} = c_1^{(v)}(c_1^{(u)}, c_2^{(u)}) \\ c_2^{(v)} = c_2^{(v)}(c_1^{(u)}, c_2^{(u)}) \end{cases} \qquad \text{Connection Formulae}$$

iii) The distribution of the movable poles.

Tabulation for PVI

Contribution to the Painlevé Project:

Tabulation of **Painlevé 6 transcendents** $y(x)$

Tables we have today provide:

- Critical behaviours with (convergent) expansions at $x = 0, 1, \infty$.
- Connection formulae.
- Correspondence:
  Critical behaviours of the $y(x)$'s $\longleftrightarrow$ Monodromy Data associated to $y(x)$.

Reference: D.G. Nonlinearity 25 (2012) 3235-3276.

Tabulation for PVI

The tables collect several results:
– M. Jimbo (1982)
– S. Shimomura (1987)
– P. Boalch (2005)
– K. Kaneko (2006)
– D. G. (2002-2006-2008-2011/12)

These tables agree with the expansions obtained by means of Power Geometry [Bruno-Goryuchkina]

How a table looks like...

	Complex power behaviours	Free Param.	Other Conditions
(36)⋆	$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=-n}^{n}c_{nm}(ax^{\sigma})^m$ $=\begin{cases}\frac{c_{1,-1}}{a}x^{1-\sigma}(1+O(x^{\sigma},x^{1-\sigma})), \\ 0<\Re\sigma<1 \\ x\left[\frac{c_{1,-1}}{a}x^{-\sigma}+c_{10}+ax^{\sigma}\right]+O(x^2), \\ \Re\sigma=0\end{cases}$ $c_{11}=1,\quad c_{10}=\frac{\sigma^2-2\beta+2\delta-1}{2\sigma^2}$ $c_{1,-1}=\frac{\left[(\sqrt{-2\beta}-\sqrt{1-2\delta})^2-\sigma^2\right]\left[(\sqrt{-2\beta}+\sqrt{1-2\delta})^2-\sigma^2\right]}{16\sigma^4}$ Basic solutions	σ $a\neq 0$	$0\leqslant\Re\sigma<1,\ \ \sigma\neq\Sigma_{\beta\delta}^k$ $2\cos\pi\sigma=p_{0x}$. $p_{0x}\neq 2\cos\pi\Sigma_{\beta\delta}^k,\pm 2$, $p_{0x}\notin(-\infty,-2]$.
(41)⋆	$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}c_{nm}(ax^{\sigma})^m$ $c_{11}=1,\quad c_{10}=\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}+(-)^k\sqrt{1-2\delta}}$ If $a=0$, $y(x)$ reduces to (42) Basic solutions	a	$\Re\sigma>-1,\ \ \sigma=\Sigma_{\beta\delta}^k\notin \boldsymbol{Z}$, $\Sigma_{\beta\delta}^k$ is (11) or (10). $p_{0x}=2\cos\pi\Sigma_{\beta\delta}^k\neq\pm 2$. $\langle M_0,M_x\rangle$ reducible.
(52)⋆	$y(x)=d_{00}+\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}d_{nm}(\tilde{a}x^{\rho})^m$ $d_{00}=\frac{\sqrt{\alpha}+(-)^k\sqrt{\gamma}}{\sqrt{\alpha}},\ \ \tilde{a}=-a\,d_{00}^2,\ \ d_{11}=1$ If $\tilde{a}=0$, $y(x)$ reduces to (53)	a	$\alpha\neq 0$. $\rho=\Omega_{\alpha\gamma}^k-1,\ \ \Re\rho>-1$, $\Omega_{\alpha\gamma}^k\notin\boldsymbol{Z},\ \ \Omega_{\alpha\gamma}^k$ is (13). $p_{0x}=-2\cos\pi\Omega_{\alpha\gamma}^k\neq\pm 2$. $\langle M_xM_0,M_1\rangle$ reducible.
(57)⋆	$y(x)=\frac{1}{a}x^{-\omega}\left(1+\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}d_{nm}(ax^{\omega})^m\right)$	a	$\alpha=0$, $\gamma\notin(-\infty,0],\ \ \sqrt{2\gamma}\notin\boldsymbol{Z}$. $\omega=\sqrt{2\gamma}\ \mathrm{sgn}(\Re\sqrt{2\gamma})$, $\Re\omega>0$. $p_{0x}=-2\cos\pi\sqrt{2\gamma}\neq\pm 2$. $\langle M_xM_0,M_1\rangle$ reducible.

	Inverse oscillatory behaviours	Free Param	Other
(50)⋆	$y(x)=\left[\sum_{n=0}^{\infty}x^n\sum_{m=-n-1}^{n+1}c_{nm}\left(e^{i\phi}x^{2i\nu}\right)^m\right]^{-1}$ $=[A\sin(2\nu\ln x+\phi)+B+O(x)]^{-1}$ $A=-\sqrt{\frac{\alpha}{2\nu^2}+B^2},\quad B=\frac{2\nu^2+\gamma-\alpha}{4\nu^2}$	ν ϕ	$\nu\in\boldsymbol{R}\backslash\{0\},\ \ 2i\nu\neq\Omega_{\alpha\gamma}^k$, $2\cosh 2\pi\nu=-p_{0x}$. $p_{0x}<-2$, $p_{0x}\neq-2\cos\pi\Omega_{\alpha\gamma}^k$.
(54)⋆	$y(x)=\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha}+(-)^k\sqrt{\gamma}}+ax^{-2i\nu}+\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n+1}c_{n+1,m}\left(ax^{-2i\nu}\right)^m\right]^{-1}$	a	$2i\nu=\Omega_{\alpha\gamma}^k\in i\boldsymbol{R}\backslash\{0\}$, $\Omega_{\alpha\gamma}^k$ is (12). $p_{0x}=-2\cos\pi\Omega_{\alpha\gamma}^k<-2$. $\langle M_xM_0,M_1\rangle$ reducible.

	Taylor expansions	Free Par	Other conditions
(42)	$y(x)=\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}+(-)^k\sqrt{1-2\delta}}x+\sum_{n=2}^{\infty}b_nx^n$ $y(x)=0$ if $\beta=0$ This is (41) when $a=0$		$\sqrt{-2\beta}+(-)^k\sqrt{1-2\delta}\notin\boldsymbol{Z}$. $p_{0x}=2\cos\pi\Sigma_{\beta\delta}^k\neq\pm 2$. $\langle M_0,M_x\rangle$ reducible.
(45)	$y(x)=\sum_{n=1}^{\lvert N\rvert}b_nx^n+ax^{\lvert N\rvert+1}+\sum_{n=\lvert N\rvert+2}^{\infty}b_n(a)x^n$ $b_1=\frac{\sqrt{-2\beta}}{N}\equiv\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}+\sqrt{1-2\delta}}$ or $\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}-\sqrt{1-2\delta}}$	a	$\sqrt{-2\beta}+\sqrt{1-2\delta}=N\neq 0$ or $\sqrt{-2\beta}-\sqrt{1-2\delta}=N\neq 0$ **and** $\sqrt{-2\beta}\in\begin{cases}\{0,-1,-2,\ldots,N\} \\ N<0 \\ \{0,\ 1,\ 2,\ldots,N\} \\ N>0\end{cases}$ or $\{\sqrt{2\alpha}+\sqrt{2\gamma},\sqrt{2\alpha}-\sqrt{2\gamma}\}$ $\cap\mathcal{N}_N\neq\emptyset$. $p_{0x}=2\cos\pi N=\pm 2$. $\langle M_0,M_x\rangle$ reducible.
(46)⋆	$y(x)=ax+a(a-1)\left(\gamma-\alpha-\frac{1}{2}\right)x^2+\sum_{n=3}^{\infty}b_n(a)x^n$	a	$2\beta=2\delta-1=0$. $p_{0x}=2$. $\langle M_0,M_x\rangle$ reducible.
(53)	$y(x)=\frac{\sqrt{\alpha}+(-)^k\sqrt{\gamma}}{\sqrt{\alpha}}+\sum_{n=1}^{\infty}b_nx^n$, Basic Taylor This is (52) when $a=0$		$\alpha\neq 0,\quad\sqrt{2\alpha}+(-)^k\sqrt{2\gamma}\notin\boldsymbol{Z}$. $p_{0x}=-2\cos\pi\Omega_{\alpha\gamma}^k\neq\pm 2$. $\langle M_xM_0,M_1\rangle$ reducible.

(61) $\star_{N=1}$	$y(x)=\sum_{n=0}^{\lvert N\rvert-1} b_n x^n + a x^{\lvert N\rvert} + \sum_{n=\lvert N\rvert+1}^{\infty} b_n(a) x^n$ $b_0=\frac{N}{\sqrt{2\alpha}}\equiv\frac{\sqrt{2\alpha}+\sqrt{2\gamma}}{\sqrt{2\alpha}}$ or $\frac{\sqrt{2\alpha}-\sqrt{2\gamma}}{\sqrt{2\alpha}}$, $\alpha\neq 0$ Basic Taylor solution when $N=1$	a	$\sqrt{2\alpha}+\sqrt{2\gamma}=N\neq 0$ or $\sqrt{2\alpha}-\sqrt{2\gamma}=N\neq 0$ **and** $\sqrt{2\alpha}\in\begin{cases}\{-1,-2,-3,\dots,N\} & N<0\\ \{1,2,3,\dots,N\} & N>0\end{cases}$ or $\{\sqrt{-2\beta}+\sqrt{1-2\delta},\ \sqrt{-2\beta}-\sqrt{1-2\delta}\}\cap\mathcal{N}_N\neq\emptyset$. $p_{0x}=-2\cos\pi N=\pm 2$. $\langle M_x M_0, M_1\rangle$ reducible.
(62)$\star$	$y(x)=a+(1-a)(\delta-\beta)x + \sum_{n=2}^{\infty} b_n(a)x^n$ Basic Taylor solution	a	$\alpha=\gamma=0$. $p_{0x}=-2$. $\langle M_x M_0, M_1\rangle$ reducible.

	Logarithmic behaviours	Free Par.	conditions Other
(43)	$y(x)=\sum_{n=1}^{\lvert N\rvert} b_n x^n + \left(a+b_{\lvert N\rvert+1}\ln x\right)x^{\lvert N\rvert+1} + \sum_{n=\lvert N\rvert+2}^{\infty} P_n(\ln x; a)x^n$ $b_1=\frac{\sqrt{-2\beta}}{N}\equiv\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}+\sqrt{1-2\delta}}$ or $\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}-\sqrt{1-2\delta}}$, $N\neq 0$.	a	$\sqrt{-2\beta}+\sqrt{1-2\delta}=N$ or $\sqrt{-2\beta}-\sqrt{1-2\delta}=N$. $2\beta=2\delta-1$ if $N=0$ **and**
(44)$\star$	$y(x)=(a\pm\sqrt{-2\beta}\ln x)x + \sum_{n=2}^{\infty} P_n(\ln x; a)x^n$, $N=0$. Basic solution when $N=0$		$\sqrt{-2\beta}\neq\begin{cases}0,-1,-2,\dots,N, & \text{if } N\leqslant 0\\ 0,1,2,\dots,N, & \text{if } N\geqslant 0\end{cases}$ $\sqrt{2\alpha}\pm\sqrt{2\gamma}\notin\mathcal{N}_N$. $p_{0x}=2\cos\pi N=\pm 2$. $\langle M_0, M_x\rangle$ reducible.
(48)$\star$	$y(x)=\left[\frac{2\beta+1-2\delta}{4}(a+\ln x)^2 + \frac{2\beta}{2\beta+1-2\delta}\right]x + \sum_{n\geqslant 2}^{\infty} P_n(\ln x; a)x^n$ Basic solution	a	$2\beta\neq 2\delta-1$. $p_{0x}=2$. no reduc. subgroups.

(59)	$y(x)=\sum_{n=0}^{\lvert N\rvert-1} b_n x^n + \left(a+b_N\ln x\right)x^{\lvert N\rvert} + \sum_{n=\lvert N\rvert+1}^{\infty} P_n(\ln x; a)x^n$ $b_0=\frac{N}{\sqrt{2\alpha}}\equiv\frac{\sqrt{2\alpha}+\sqrt{2\gamma}}{\sqrt{2\alpha}}$ or $\frac{\sqrt{2\alpha}-\sqrt{2\gamma}}{\sqrt{2\alpha}}$, $N\neq 0$	a	$\sqrt{2\alpha}+\sqrt{2\gamma}=N\neq 0$ or $\sqrt{2\alpha}-\sqrt{2\gamma}=N\neq 0$ **and** $\sqrt{2\alpha}\neq\begin{cases}0,-1,\dots,N, \text{ if } N\leqslant -1\\ 0,1,\dots,N, \text{ if } N\geqslant 1\end{cases}$ $\sqrt{-2\beta}\pm\sqrt{1-2\delta}\notin\mathcal{N}_N$. $p_{0x}=-2\cos\pi N=\pm 2$. $\langle M_x M_0, M_1\rangle$ reducible.

	Inverse logarithmic behaviours	Free Param.	Other conditions
(60)	$y(x)=\left\{a\pm\sqrt{2\alpha}\ln x+\sum_{n=1}^{\infty}P_n(\ln x; a)x^n\right\}^{-1}$ $=\pm\frac{1}{\sqrt{2\alpha}\ln x}\left[1\mp\frac{a}{\sqrt{2\alpha}\ln x}+O\left(\frac{1}{\ln^2 x}\right)\right]$	a	$\alpha=\gamma\neq 0$. $p_{0x}=-2$. $\langle M_x M_0, M_1\rangle$ reducible.
(64)	$y(x)=\left\{\frac{\alpha}{\alpha-\gamma}+\frac{\gamma-\alpha}{2}(a+\ln x)^2 + \sum_{n=1}^{\infty}P_{n+1}(\ln x; a)x^n\right\}^{-1}$ $=\frac{2}{(\gamma-\alpha)\ln^2 x}\left[1-\frac{2a}{\ln x}+O\left(\frac{1}{\ln^2 x}\right)\right]$	a	$\alpha\neq\gamma$. $p_{0x}=-2$. no reduc. subgroups.

3. Table 2: Critical behaviours when $x\to 1$

Table 2 provides the critical behaviours (7) at $x=1$. The branch cut may be taken to be $-\pi<\arg(1-x)\leqslant\pi$.

◇ The table is constructed from the table at $x=0$ by Okamoto's transformation (66), in the following way:

(a) $\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$ is given, with coefficients $\alpha, \beta, \gamma, \delta$ (or $\theta_0, \theta_x, \theta_1, \theta_\infty$).

(b) Take $\mathrm{PVI}_{\alpha',\beta',\gamma',\delta'}$ with the coefficients

$$\alpha'=\alpha,\quad \beta'=-\gamma,\quad \gamma'=-\beta,\quad \delta'=\delta,$$

$$(\text{or } \theta_0'=\theta_1, \theta_x'=\theta_x,\ \theta_1'=\theta_0,\ \theta_\infty'=\theta_\infty),$$

and the variable ξ, and compute the critical behaviours $y_0(\xi)$ for $\xi\to 0$. The critical behaviours at $x=1$ for $\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$ are then

$$y(x)=1-y_0(1-x),\qquad x\to 1.$$

This is why the behaviours in table 2 are numerated as in table 1, according to the behaviour of $y_0(\xi)$ from which they have been obtained.

Description of the Tables

First element: **critical behaviors**. Three tables collect convergent expansions of $y(x)$ as $x \to 0$, $x \to 1$, $x \to \infty$ respectively.

This is a **local problem** $\longrightarrow$ Local Analysis:

- Shimomura 1987
- Elliptic representation [D.G. 2001-2]]
- Power Geometry [Bruno- Goryuchkina (2010)]
- Analysis of Schlesinger equations [Sato-Miwa-Jimbo '79, Jimbo '82] $\to$ Method of Monodromy Preserving Deformations.

No time to describe them all. We describe Schlesinger equations...

PVI is equivalent to the **Schlesinger equations** (1912) for 2×2 matrices $A_0(x)$, $A_x(x)$, $A_1(x)$

$$\frac{dA_0}{dx}=\frac{[A_x,A_0]}{x},\quad \frac{dA_1}{dx}=\frac{[A_1,A_x]}{1-x},\quad \frac{dA_x}{dx}=-\frac{[A_1,A_x]}{x}-\frac{[A_1,A_x]}{1-x}.$$

Conditions on eigenvalues:

$$\text{Eigen}(A_0)=\pm\theta_0/2,\quad \text{Eigen}(A_1)=\pm\theta_1/2,\quad \text{Eigen}(A_{\mathbf{x}})=\pm\theta_x/2$$

$$A_0+A_1+A_x=\begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2}\end{pmatrix},\quad \theta_\infty\neq 0.$$

Eigenvalues are related to the coefficients of PVI:

$$\alpha=\frac{1}{2}(\theta_\infty-1)^2,\quad -\beta=\frac{1}{2}\theta_0^2,\quad \gamma=\frac{1}{2}\theta_1^2,\quad \left(\frac{1}{2}-\delta\right)=\frac{1}{2}\theta_2^2.$$

• The entries of the $A_i(x)$'s are simple algebraic functions of $y(x)$, $\frac{dy(x)}{dx}$ and $\int^x y(s)ds$. [Jimbo-Miwa (1981)].
• Conversely

$$y(x) = \frac{x\Big(A_0(x)\Big)_{12}}{x\Big[\Big(A_0(x)\Big)_{12} + \Big(A_1(x)\Big)_{12}\Big] - \Big(A_0(x)\Big)_{12}}$$

◇ PVI is the isomonodromy deformation equation of a 2×2 Fuchsian system [Jimbo, Miwa, Ueno (1981)]:

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0(x)}{\lambda} + \frac{A_1(x)}{\lambda - 1} + \frac{A_{\mathbf{x}}(x)}{\lambda - x}\right]\Psi.$$

The monodromy of $\Psi(x, \lambda)$ is independent of small deformations of x.

Critical behaviors

Lemma of Sato-Miwa-Jimbo '79 + Generalization on domains on universal covering of a punctured neighbourhood of $x = 0$ [methods in S. Shimomura '87, D.G. 2001].

- Let A_0^0, A_x^0 and A_1^0 be constant matrices satisfying

$$\text{Eigen}(A_0^0) = \pm\theta_0/2, \quad \text{Eigen}(A_1^0) = \pm\theta_1/2, \quad \text{Eigen}(A_{\mathsf{x}}) = \pm\theta_x/2$$

$$A_0^0 + A_1^0 + A_x^0 = \begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2} \end{pmatrix}, \quad \theta_\infty \neq 0.$$

- Let $\Lambda := A_0^0 + A_x^0$, with eigenvalues $\pm\frac{\sigma}{2}$.

$$\sigma \in \mathbb{C}\backslash(-\infty, -1] \cup [1, +\infty), \qquad \Lambda \neq 0$$

Jordan form of Λ: $\begin{pmatrix} \frac{\sigma}{2} & 0 \\ 0 & -\frac{\sigma}{2} \end{pmatrix}$ if $\sigma \neq 0$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ if $\sigma = 0$.

Critical behaviors: Complex-power behaviours

By elementary algebra we find the explicit matrices $A_0^0(\theta_0,\theta_x,\sigma,a)$, $A_x^0(\theta_0,\theta_x,\sigma,a)$, $A_1^0(\theta_1,\theta_\infty,\sigma,a)$, a=additional parameter.

Lemma: The Schlesinger equations have a *unique* solution with convergent expansions for $x\to 0$:

$$A_1(x)=A_1^0+\sum_{n=1}^{\infty}\sum_{m=-n}^{m}A_{1,m}^n x^n(ax^\sigma)^m,$$

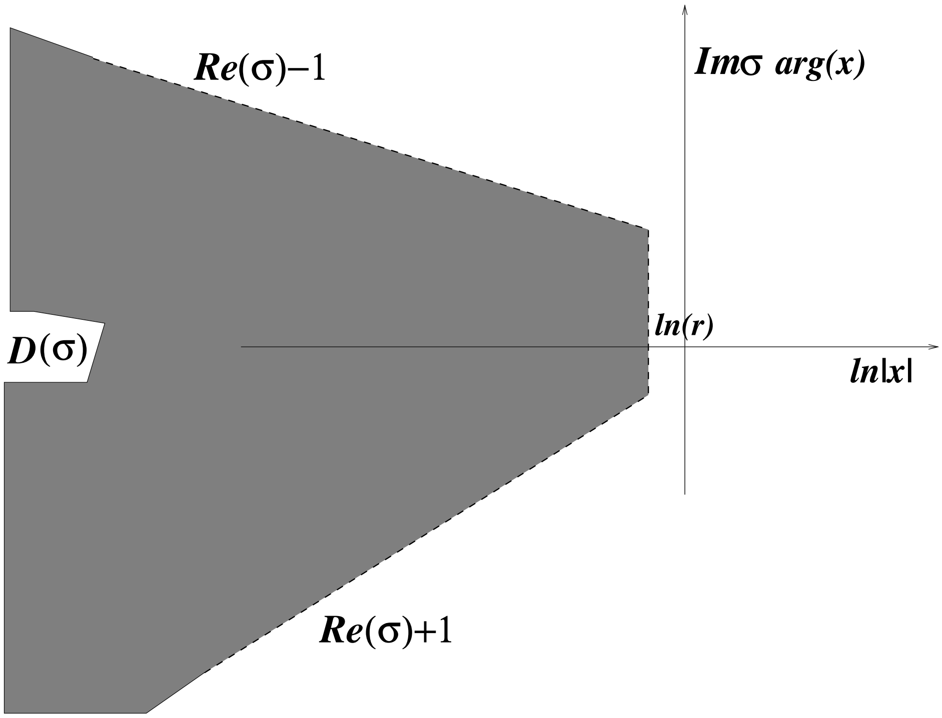
$$x^{-\Lambda}A_0(x)x^{\Lambda}=A_0^0+\sum_{n=1}^{\infty}\sum_{m=-n}^{m}A_{0,m}^n x^n(ax^\sigma)^m,$$

$$x^{-\Lambda}A_x(x)x^{\Lambda}=A_x^0+\sum_{n=1}^{\infty}\sum_{m=-n}^{m}A_{x,m}^n x^n(ax^\sigma)^m,$$

$$\sigma\neq 0$$

Expansions are convergent in a domain $\mathcal{D}(\sigma, a)$ in the universal covering of punctured neighbourhood of $x = 0$

$$|x| < r, \quad |ax^{1+\sigma}| < r, \quad |x^{1-\sigma}/a| < r.$$



[Shimoura '87, D.G. 2002 – use successive approximations for integral equations)],

Logarithmic behaviours

For $\sigma = 0$, we find logarithmic behaviours:

$$A_1(x) \longrightarrow A_1^0,$$

$$x^{-\Lambda} A_0(x) x^{\Lambda} \longrightarrow A_0^0, \qquad x \longrightarrow 0 \text{ in a sector} \qquad x^{\Lambda} = \begin{pmatrix} 1 & \ln x \\ 0 & 1 \end{pmatrix}$$

$$x^{-\Lambda} A_x(x) x^{\Lambda} \longrightarrow A_x^0,$$

Conjectural form obtained by matching methods (D.G. 2006) or direct substitution:

$$A_1(x) \sim A_1^0 + \sum_{n=1}^{\infty} P_{1,n}(\ln x + a)\ x^n,$$

$$x^{-\Lambda} A_0(x) x^{\Lambda} \sim A_0^0 + \sum_{n=1}^{\infty} P_{0,n}(\ln x + a)\ x^n, \qquad P_{u,n} \text{ polynomials.}$$

$$x^{-\Lambda} A_x(x) x^{\Lambda} \sim A_x^0 + \sum_{n=1}^{\infty} P_{x,n}(\ln x + a)\ x^n,$$

Convergence is expected in a sector with central angle in $x = 0$.

Basic expansions of PVI transcendents

This give **basic expansions** for $x \to 0$ in $\mathcal{D}(a,\sigma)$. Complex integration constants are a, σ.

- For $\sigma \neq 0$ [complex power behaviours]:

$$y(x,\sigma,a) = x(ax^{\sigma} + c_{10} + c_{1,-1}(ax^{\sigma})^{-1}) + \sum_{n=2}^{\infty} x^n \sum_{m=-n}^{n} c_{nm}(ax^{\sigma})^m,$$

$$y(x,a) = x(c_{10} + ax^{\sigma}) + \sum_{n=2}^{\infty} x^n \sum_{m=0}^{n} c_{nm}(ax^{\sigma})^m, \quad \text{if} \quad \sigma^2 = (\theta_x \pm \theta_0)^2,$$

- For $\sigma = 0$ [logarithmic behaviours]:

$$y(x,a) = \left[\frac{\theta_x^2 - \theta_0^2}{4}(a + \ln x)^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2}\right] x + \sum_{n=2}^{\infty} P_n(\ln x; a) x^n,$$

$$y(x,a) = \Big(a \pm \theta_0 \ln x\Big) x + \sum_{n=2}^{\infty} P_n(\ln x; a) x^n, \qquad \text{if} \ \ \theta_0^2 = \theta_x^2 \ \ (\alpha = \gamma)$$

Basic expansions

Remark 1: The above behaviours include Taylor series for special values of $a = 0$, or $\theta_0 = 0$.

Remark 2: For $x \to 0$ along a line

$$\Im\sigma \arg x = \Re\sigma \ln|x| + Constant, \qquad \text{in} \quad \mathcal{D}(\sigma, a),$$

$$y(x,\sigma,a) = x\Big\{A\sin(i\ \sigma\ \ln x + \phi(a)) + B\Big\} + O(x^2), \quad \sigma \neq 0$$

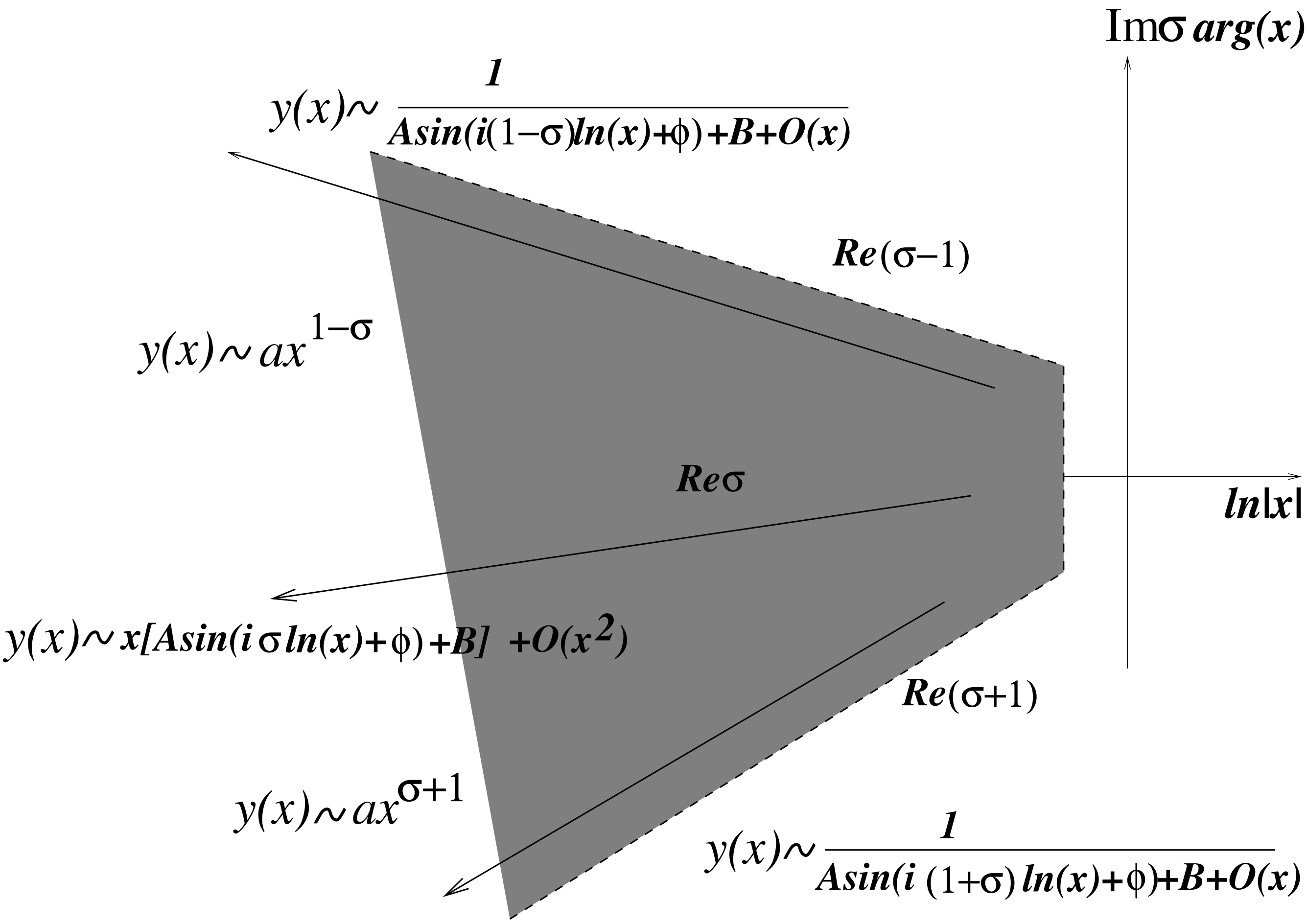
A and B are constants fixed by the coefficients of the equation.

Remark 3: For $x \to 0$ along a line

$$\Im\sigma \arg x = (\Re\sigma \pm 1)\ln|x| + Constant, \qquad \text{in} \quad \mathcal{D}(\sigma, a),$$

$$y(x,\sigma,a) = \frac{1}{A\sin(i(\sigma \pm 1)\ln x + \phi(a)) + B + O(x)}.$$

Behaviours of $y(x; a, \sigma)$ on the universal covering



i) Critical behaviours

From basic expansions $\longrightarrow$ generate other critical behaviour making use of the **symmetries** of PVI [K. Okamoto (1987)]:

$$y'(x') = \frac{x}{y(x)}, \quad x' = x,$$
$$\theta_0' = \theta_\infty - 1, \quad \theta_x' = \theta_1, \quad \theta_1' = \theta_x, \quad \theta_\infty' = \theta_0 + 1$$

Symmetry means that $y'(x')$ solves PVI with coefficients $\theta_0', \theta_x', \theta_1', \theta_\infty'$ if and only if $y(x)$ solves PVI with $\theta_0, \theta_x, \theta_1, \theta_\infty$.

• Example: from basic logarithmic behaviours

$$y(x,a) = \left[\frac{\theta_x^2-\theta_0^2}{4}(a+\ln x)^2 + \frac{\theta_0^2}{\theta_0^2-\theta_x^2}\right] x + \sum_{n=2}^{\infty} P_n(\ln x; a) x^n,$$

$$y(x,a) = \Big(a \pm \theta_0 \ln x\Big) x + \sum_{n=2}^{\infty} P_n(\ln x; a) x^n, \qquad \text{if } \theta_0^2 = \theta_x^2 \ \ (\alpha=\gamma)$$

we find

$$y'(x,a) = \frac{4}{(\theta_1^2-(\theta_\infty-1)^2)\ln^2 x}\left[1-\frac{2a}{\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \theta_1^2 \neq (\theta_\infty-1)^2$$

$$y'(x,a) = \frac{1}{(\theta_\infty-1)\ln x}\left[1-\frac{a}{(\theta_\infty-1)\ln x} + O\left(\frac{1}{\ln^2 x}\right)\right], \theta_1^2 = (\theta_\infty-1)^2$$

Description of the Table for PVI -*Nonlinearity 2012*

This procedure generates all the table of critical behaviours for $x \longrightarrow 0$. Five classes:

(1) Complex-power behaviors

(2) Inverse oscillatory bahaviors (with poles in sector)

(3) Taylor expansions

(4) Logarithmic behaviors

(5) Inverse logarithmic behaviors

When $x \longrightarrow 1$ and $x \longrightarrow \infty$ there description is similar. Use symmetries

$$y'(x') = 1 - y(x), \quad x' = 1 - x, \quad \theta_0' = \theta_1, \quad \theta_1' = \theta_0.$$

$$y'(x') = \frac{1}{x}y(x), \quad x' = \frac{1}{x}, \quad \theta_x' = \theta_1, \quad \theta_1' = \theta_x.$$

Now you can read the first column of the table...

3242 D Guzzetti

	Complex power behaviours	Free Param.	Other Conditions
(36)⋆	$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=-n}^{n}c_{nm}(ax^{\sigma})^m$ $= \begin{cases} \frac{c_{1,-1}}{a}x^{1-\sigma}(1+O(x^{\sigma},x^{1-\sigma})), & 0<\Re\sigma<1 \\ x\left[\frac{c_{1,-1}}{a}x^{-\sigma}+c_{10}+ax^{\sigma}\right]+O(x^2), & \Re\sigma=0 \end{cases}$ $c_{11}=1,\quad c_{10}=\frac{\sigma^2-2\beta+2\delta-1}{2\sigma^2}$ $c_{1,-1}=\frac{\left[(\sqrt{-2\beta}-\sqrt{1-2\delta})^2-\sigma^2\right]\left[(\sqrt{-2\beta}+\sqrt{1-2\delta})^2-\sigma^2\right]}{16\sigma^4}$ Basic solutions	σ $a\neq 0$	$0\leqslant\Re\sigma<1,\quad \sigma\neq\Sigma_{\beta\delta}^k$ $2\cos\pi\sigma=p_{0x}$. $p_{0x}\neq 2\cos\pi\Sigma_{\beta\delta}^k, \pm 2,$ $p_{0x}\notin(-\infty,-2]$.
(41)⋆	$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}c_{nm}(ax^{\sigma})^m$ $c_{11}=1,\quad c_{10}=\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}+(-)^k\sqrt{1-2\delta}}$ If $a=0$, $y(x)$ reduces to (42) Basic solutions	a	$\Re\sigma>-1,\quad \sigma=\Sigma_{\beta\delta}^k\notin Z,$ $\Sigma_{\beta\delta}^k$ is (11) or (10). $p_{0x}=2\cos\pi\Sigma_{\beta\delta}^k\neq\pm 2.$ $\langle M_0, M_x\rangle$ reducible.
(52)⋆	$d_{00}=\frac{\sqrt{\alpha}+(-)^k\sqrt{\gamma}}{\sqrt{\alpha}},\quad \tilde{a}=-a\,d_{00}^2,\quad d_{11}=1$ If $\tilde{a}=0$, $y(x)$ reduces to (53)	a	$\alpha\neq 0.$ $\Omega_{\alpha\gamma}^k\notin Z,\quad \Omega_{\alpha\gamma}^k$ is (13). $p_{0x}=-2\cos\pi\Omega_{\alpha\gamma}^k\neq\pm 2.$ $\langle M_xM_0, M_1\rangle$ reducible.
(57)⋆	$y(x)=\frac{1}{a}x^{-\omega}\left(1+\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}d_{nm}(ax^{\omega})^m\right)$	a	$\alpha=0,$ $\gamma\notin(-\infty,0],\quad \sqrt{2\gamma}\notin Z.$ $\omega=\sqrt{2\gamma}\,\mathrm{sgn}(\Re\sqrt{2\gamma}),$ $\Re\omega>0.$ $p_{0x}=-2\cos\pi\sqrt{2\gamma}\neq\pm 2.$ $\langle M_xM_0, M_1\rangle$ reducible.

Digression:

The behaviours above agree with the behaviour of PVI τ function in terms of conformal blocks

$$x(x-1)\frac{d\tau}{dx} = (x-1)\mathrm{tr}A_0A_x + x\ \mathrm{tr}A_1A_x,$$

$$\tau(x) = \mathrm{const}\sum_{n=-\infty}^{\infty} C_n(\sigma, a)x^{(\sigma+n)^2-\frac{\theta_0^2}{4}-\frac{\theta_x^2}{4}}B(\sigma+n;x)$$

$$B(\sigma+n;x) = (1-x)^{\theta_x\theta_1/2}\Big(1+\sum_{k=1}^{\infty}B_k(\sigma+n)x^k\Big)$$

= Conformal blocks of Virasoro algebra with central charge $c=1$. They are explicitly computable in closed form.

$C_n(\sigma, a)$ explicit ratio of Barnes G-functions.

O. Gamayun, N. Iorgov, Shchechkin, O. Lisovyy, J. Teschner, Y. Tykhyy, M. A. Bershtein (2012 $\sim$)).

II part: Connection Formulae

PVI is the isomonodromy deformation equation of a 2×2 Fuchsian system of ODE

$$\frac{d\Psi}{d\lambda} = \left[\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_x}{\lambda - x}\right]\Psi$$

Recall that:

$$\text{Eigen}(A_0) = \pm\theta_0/2, \quad \text{Eigen}(A_1) = \pm\theta_1/2, \quad \text{Eigen}(A_x) = \pm\theta_x/2$$

$$A_0 + A_1 + A_x = \begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2} \end{pmatrix}, \quad \theta_\infty \neq 0.$$

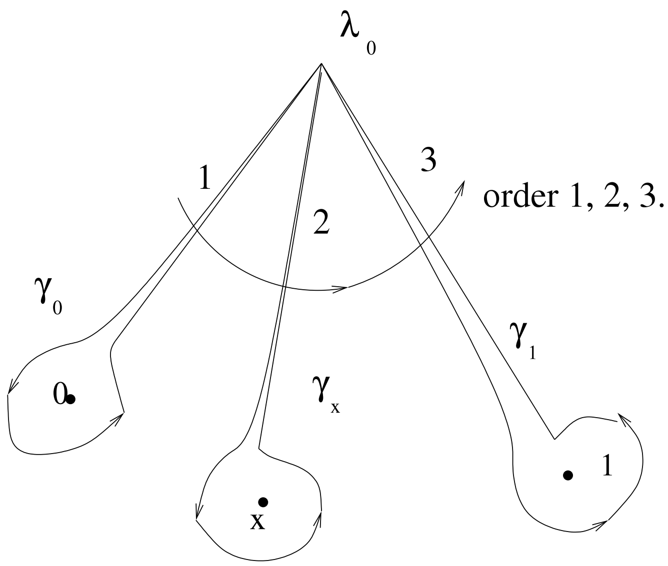
$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad -\beta = \frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \left(\frac{1}{2} - \delta\right) = \frac{1}{2}\theta_2^2.$$

Monodromy Data

$\exists\ \Psi(x,\lambda)$ with monodromy matrices M_0, M_x, M_1 independent of small deformations of x.

$$\Psi(\lambda,x) \mapsto \Psi(\lambda,x) M_0,$$
$$\Psi(\lambda,x) \mapsto \Psi(\lambda,x) M_x,$$
$$\Psi(\lambda,x) \mapsto \Psi(\lambda,x) M_1,$$

$$M_\infty :\ = M_1 M_x M_0$$



$A_0(x),\ A_1(x),\ A_x(x)$ corresponding to $M_0,\ M_x,\ M_1$ is a x-curve, given by a Painlevé VI function $y(x)$. [Jimbo-Miwa].

Monodromy Data

Monodromy data:

$$\Theta := \{(\theta_0, \theta_1, \theta_x, \theta_\infty) \in \mathbf{C}^4 \mid \theta_\infty \neq 0\} / \sim .$$

Equivalence $\sim$ is: $\theta_k \mapsto -\theta_k$, $\theta_\infty \mapsto 2 - \theta_\infty$.

$M :=$

$$\{(M_0, M_x, M_1) \mid \mathrm{Tr} M_\mu = 2\cos\pi\theta_\mu,\ \mu = 0, 1, x, \infty\}/\text{conjugation}$$

Definition: The monodromy data of the class of Fuchsian systems, with the basis of loops ordered as in figure, are elements of the set

$$\mathcal{M} := \Theta \cup M.$$

Monodromy Map

$y(x) \longrightarrow$ system $\frac{d\Psi}{d\lambda} = A(x,\lambda)\Psi \longrightarrow$ monodromy data $\mathcal{M}$.

$f : \{y(x)$ (branches of) solutions of PVI$\} \to \mathcal{M}$, Monodromy Map

- 1) f is **injective** if restricted to f^{-1}(subspace of $\mathcal{M}$ where $M_0, M_x, M_1, M_\infty \neq I$).
- 2) If the group generated by M_0, M_x, M_1 is irreducible then good **coordinates** on $\mathcal{M}$ are

$$\theta_0, \theta_x, \theta_1, \theta_\infty, \qquad p_{ij} = p_{ji} := \mathrm{Tr} M_i M_j, \qquad i \neq j \in \{0, x, 1\}$$

Only two of p_{0x}, p_{x1}, p_{01} are independent (cubic relation)

Connection Problem

If 1) and 2) hold, then for

$$y(x) = y_u(x, c_1^{(u)}, c_2^{(u)}), \quad x \to u \in \{0, 1, \infty\}$$

we **parametrize the integration constants**:

$$\begin{cases} c_1^{(u)} = c_1^{(u)}(\theta_0, \theta_x, \theta_1, \theta_\infty, p_{0x}, p_{x1}) \\ c_2^{(u)} = c_2^{(u)}(\theta_0, \theta_x, \theta_1, \theta_\infty, p_{0x}, p_{x1}) \end{cases}$$

Conversely, if 2) holds:

$$\begin{cases} p_{0x} = p_{0x}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{x1} = p_{x1}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{01} = p_{01}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \end{cases}$$

These formulae are computed **explicitly** by *asymptotic computation of monodromy matrices* M_0, M_x, M_1.

Connection Problem

Consider the formulae for $x \to u \in \{0, 1, \infty\}$:

$$\begin{cases} p_{0x} = p_{0x}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{x1} = p_{x1}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{01} = p_{01}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \end{cases}$$

and, also for $x \to v \in \{0, x, 1\}$, $v \neq u$,

$$\begin{cases} c_1^{(v)} = c_1^{(v)}(\theta_0, \theta_x, \theta_1, \theta_\infty, p_{0x}, p_{x1}) \\ c_2^{(v)} = c_2^{(v)}(\theta_0, \theta_x, \theta_1, \theta_\infty, p_{0x}, p_{x1}) \end{cases}$$

Combining, we find *explicit* **connection formulae** between integration constants at u and v

$$\begin{cases} c_1^{(v)} = c_1^{(v)}(c_1^{(u)}, c_2^{(u)}) \\ c_2^{(v)} = c_2^{(v)}(c_1^{(u)}, c_2^{(u)}) \end{cases}$$

Connection Problem

Example: [D.G. 2006] An example of Taylor expansions

$$y(x,a) = \frac{1}{1-\theta_\infty} + a\ x\ + O(x^2),$$

Integration constant in terms of monodromy data:

$$a = \frac{\theta_\infty(2s+\theta_x+1)}{2(\theta_\infty-1)}$$

where

$$s = \frac{\theta_x\left[2\cos(\pi(\theta_\infty+\theta_x)) - p_{01}\right]}{2\left[\cos(\pi(\theta_\infty-\theta_x)) - \cos(\pi(\theta_\infty+\theta_x))\right]}.$$

□

Connection Problem

Example: [Jimbo (1982), Boalch (2005)]

$$y(x,\sigma,a) \sim a x^{\sigma}, \quad x \longrightarrow 0$$

$$\sigma = \frac{1}{\pi}\mathrm{arcos}\,(p_{0x})$$

$a =$ Ratio of Trigonometric and Γ functions of $\theta_0, \theta_x, \theta_1, \theta_\infty, p_{01}, p_{x1}$

$\square$

- Please, refer to D.G. Nonlinearity 25 (2012) 3235-3276 for the collection of all connection formulae, obtained by Jimbo/Dubrovin-Mazzocco/Boalch/Kaneko/Guzzetti.

Now you can read third column of Table + Connection Formulae...

	Complex power behaviours	Free Param.	Other Conditions
(36)⋆	$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=-n}^{n}c_{nm}(ax^{\sigma})^m$	σ	$0\leqslant\Re\sigma<1,\ \ \sigma\neq\Sigma^k_{\beta\delta}$
	$=\begin{cases}\frac{c_{1,-1}}{a}x^{1-\sigma}(1+O(x^{\sigma},x^{1-\sigma})), \\ \quad 0<\Re\sigma<1 \\ x\left[\frac{c_{1,-1}}{a}x^{-\sigma}+c_{10}+ax^{\sigma}\right]+O(x^2), \\ \quad \Re\sigma=0\end{cases}$	$a\neq 0$	$2\cos\pi\sigma=p_{0x}$.
	$c_{11}=1,\quad c_{10}=\frac{\sigma^2-2\beta+2\delta-1}{2\sigma^2}$ $c_{1,-1}=\frac{\left[(\sqrt{-2\beta}-\sqrt{1-2\delta})^2-\sigma^2\right]\left[(\sqrt{-2\beta}+\sqrt{1-2\delta})^2-\sigma^2\right]}{16\sigma^4}$ Basic solutions		$p_{0x}\neq 2\cos\pi\Sigma^k_{\beta\delta},\pm 2,$ $p_{0x}\notin(-\infty,-2]$.
(41)⋆	$y(x)=\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}c_{nm}(ax^{\sigma})^m$ $c_{11}=1,\quad c_{10}=\frac{\sqrt{-2\beta}}{\sqrt{-2\beta}+(-)^k\sqrt{1-2\delta}}$ If $a=0$, $y(x)$ reduces to (42) Basic solutions	a	$\Re\sigma>-1,\ \ \sigma=\Sigma^k_{\beta\delta}\notin \boldsymbol{Z},$ $\Sigma^k_{\beta\delta}$ is (11) or (10). $p_{0x}=2\cos\pi\Sigma^k_{\beta\delta}\neq\pm 2.$ $\langle M_0,M_x\rangle$ reducible.
(52)⋆	$y(x)=d_{00}+\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}d_{nm}(\tilde{a}x^{\rho})^m$ $d_{00}=\frac{\sqrt{\alpha}+(-)^k\sqrt{\gamma}}{\sqrt{\alpha}},\ \ \tilde{a}=-a\,d_{00}^2,\ \ d_{11}=1$ If $\tilde{a}=0$, $y(x)$ reduces to (53)	a	$\rho=\Omega^k_{\alpha\gamma}-1,\ \ \Re\rho>-1,$ $\Omega^k_{\alpha\gamma}\notin\boldsymbol{Z},\ \ \Omega^k_{\alpha\gamma}$ is (13). $p_{0x}=-2\cos\pi\Omega^k_{\alpha\gamma}\neq\pm 2.$ $\langle M_xM_0,M_1\rangle$ reducible.
(57)⋆	$y(x)=\frac{1}{a}x^{-\omega}\left(1+\sum_{n=1}^{\infty}x^n\sum_{m=0}^{n}d_{nm}(ax^{\omega})^m\right)$	a	$\alpha=0,$ $\gamma\notin(-\infty,0],\ \ \sqrt{2\gamma}\notin\boldsymbol{Z}.$ $\omega=\sqrt{2\gamma}\ \mathrm{sgn}(\Re\sqrt{2\gamma}),$ $\Re\omega>0.$ $p_{0x}=-2\cos\pi\sqrt{2\gamma}\neq\pm 2.$ $\langle M_xM_0,M_1\rangle$ reducible.

Behaviour on the Universal Covering of $x = 0$

Consider again $y(x,\sigma,a)$ on $\mathcal{D}(\sigma,a)$.

$\cos\pi\sigma = p_{0x}$ is invariant for $\sigma \mapsto \sigma_N^{\pm} := \pm\sigma + 2N$, $N \in \mathbb{Z}$.

We conclude (see D.G. Comm. Pure Apl. Math (2002)) that

$$y(x,\sigma,a), \qquad \text{associated to } p_{0x}, p_{x1}, p_{01}$$

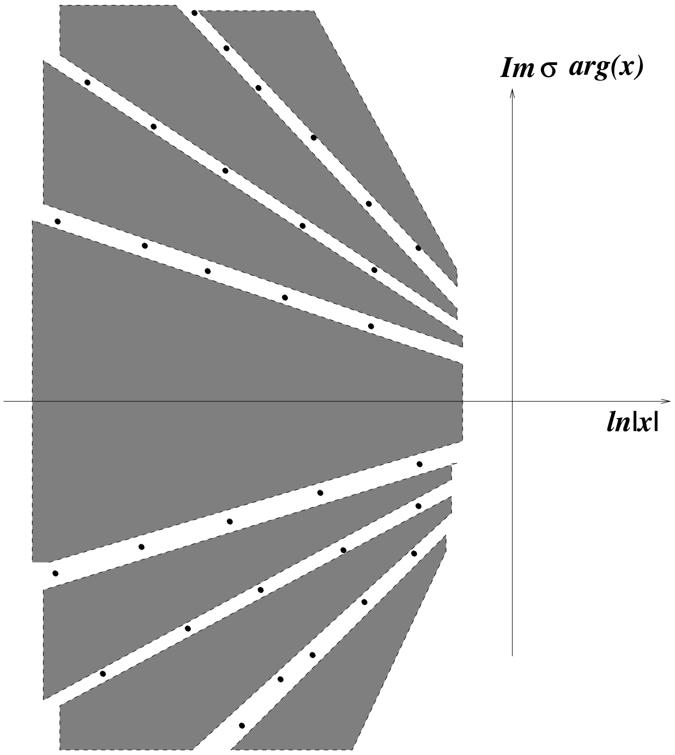
has behaviours

$$y(x,\sigma_N^{\pm},a_N^{\pm})$$

on the union of domains

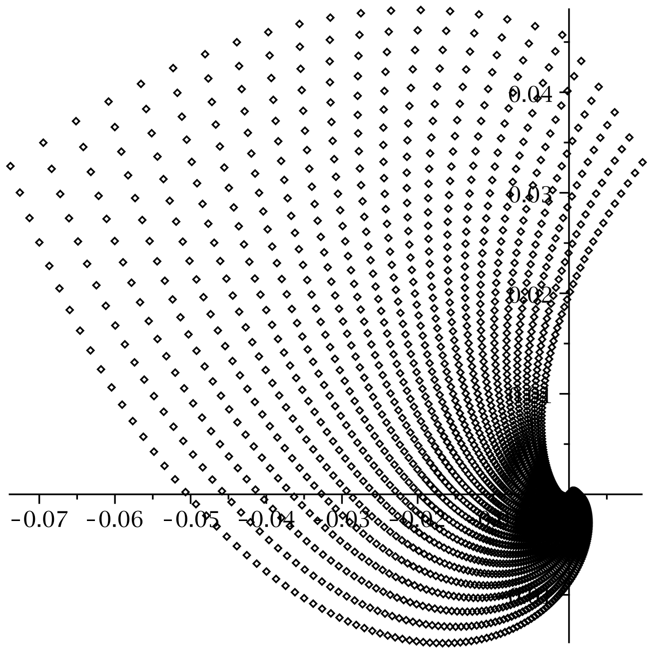
$$\bigcup_{N,\pm} \mathcal{D}(\sigma_N^{\pm},a_N^{\pm})$$

Note: also a changes to $a_N^{\pm}$.

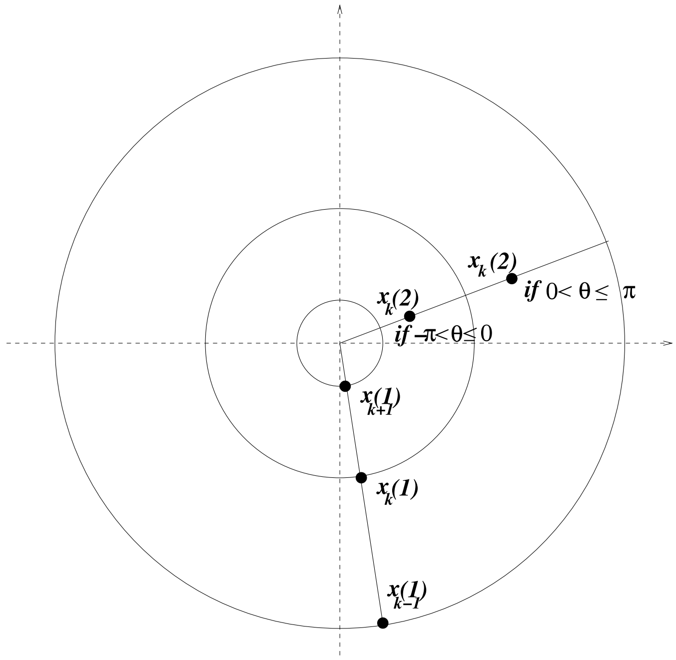


Along directions of boundaries: $y(x) = \frac{1}{\mathcal{A}_N^{\pm} \sin(i(1 \pm \sigma_N^{\pm}) \ln x + \phi_N^{\pm}) + \mathcal{B}_N^{\pm} + O(x)}$.
Dots are poles.

The poles lie along spirals in the universal covering of $\mathcal{U} = \{x \in \mathbb{C} \mid 0 < |x| < \max_N r_N^{\pm}\}$ (up to a fixed "big" N). Picture is a projection in $\mathbb{C}$.



For $\Re\sigma = 1$: two sequences of poles accumulating at $x = 0$ when we project to the x plane [D.G. Physica D (2012)]



$$y(x) = \frac{1}{\mathcal{A}\sin(\nu \ln x + \phi) + \mathcal{B} + O(x)}, \quad \nu \in \mathbb{R}, \quad \sigma = 1 + i\nu.$$

Completeness of the Table

The table is complete (= contains all possible behaviors)

if and only if

$f : \{y(x) \text{ in the table}\} \to \mathcal{M}$ is surjective.

We are able to compute explicitly formulae

$$\begin{cases} p_{0x} = p_{0x}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{x1} = p_{x1}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \\ p_{01} = p_{01}(\theta_0, \theta_x, \theta_1, \theta_\infty, c_1^{(u)}, c_2^{(u)}) \end{cases}$$

So we can study $f\left(\{y(x) \text{ in the table}\}\right) \subset \mathcal{M}$.

Completeness of the Table

We write

$$\mathcal{M} = \mathcal{M}_r \cup \mathcal{M}_i$$

$\mathcal{M}_r :=$ { monodromy data s.t. $< M_0, M_x, M_1 >$ is reducible}.
$\mathcal{M}_i :=$ { monodromy data s.t. $< M_0, M_x, M_1 >$ is irreducible}.

- All transcendents s.t. $\mathcal{M}_r$ are known, expressed in terms of Hypergeometric functions, and are contained in the table (see Reviews of Gromak-Laine-Shimomura (2002), and Clarkson (2006).
- In Appendix of *Nonlinearity 2012* structure of f(solutions in the table such that the monodromy group is irreducible) is analysed...

Conclusion

Summary: We have:

- The critical behaviors and their complete tabulation.
- The corresponding connection formulae.
- Local picture on universal covering, and asymptotic distribution of the poles.

Conformal blocks representation may help to find "closed forms" of relevant transcendents (ex. associated to Frobenius manifolds, quantum cohomology, etc) [to be done...]

For review: D.G.: Constructive Approximations (2014).

Thank you!