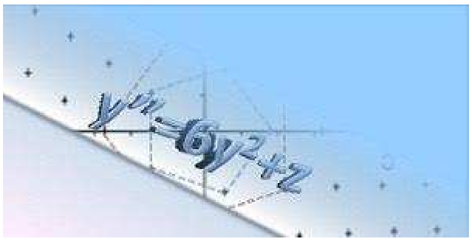
Boris Dubrovin

Isomonodromy deformation problems associated with quantum cohomology.

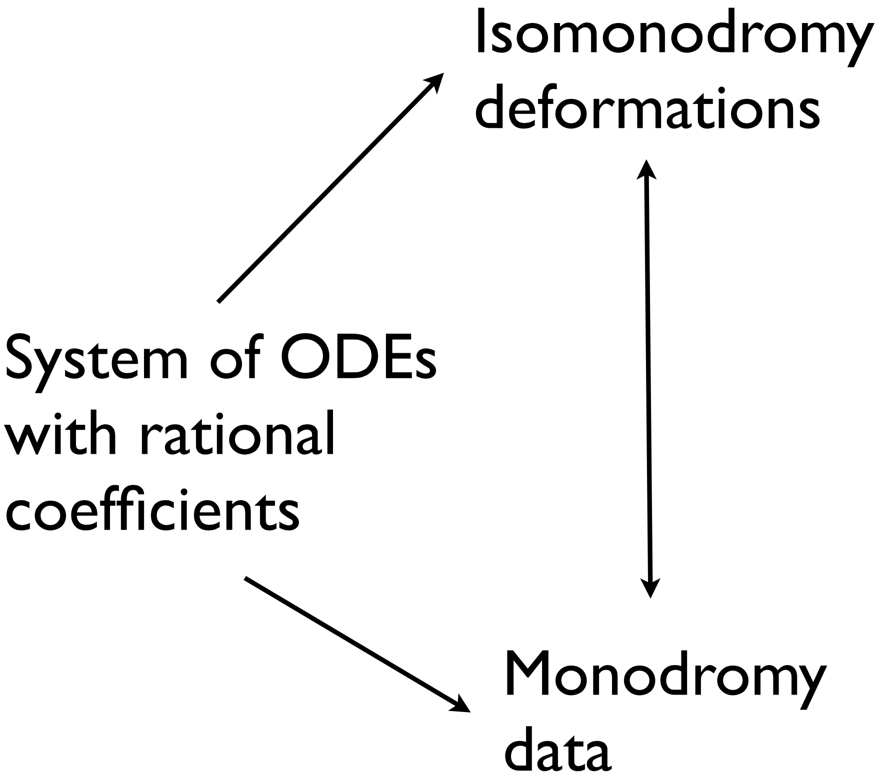
Three days on Painlevé equations
and their applications,
Roma Tre, 19.12.2014



Isomonodromy deformation problems associated with quantum cohomology

Boris DUBROVIN

SISSA, Trieste



Rational curves on a smooth projective variety X

Derived category of coherent sheaves $Der^b(X)$

Plan

- Introduction: quantum cohomology of projective plane and Painlevé-VI equation
- Semisimple Frobenius manifolds and isomonodromy deformations
- Quantum cohomology and (conjectural) description of the monodromy data

1. Quantum cohomology of $\mathbf{P}^2$ and P-VI equation

Denote $N_k =$ number of rational curves of degree k
$z \mapsto (P_0(z) : P_1(z) : P_2(z))\,, \quad \deg P_i(z) = k$
passing through $3k-1$ points

E.g., $N_1 = 1, \quad N_2 = 1, \quad N_3 = 12 \quad \text{etc.}$

Denote $$\phi = \sum_{k=1}^{\infty} \frac{N_k}{(3k-1)!} e^{k\,t}$$

Thm. (Kontsevich, '92) The function satisfies diff. equation

$$\phi''' = \frac{\phi''^2 + 54\phi'' - 33\phi' + 6\phi}{27 + 2\phi' - 3\phi''}$$

Corollary. The coefficients N_k for $k \geq 2$ are uniquely determined by $N_1 = 1$

Reduction to P-VI: change of variables $(t, \phi(t)) \mapsto (x, y(x))$

Denote $u_1,\ u_2,\ u_3$ the roots of cubic equation

$$u^3 - \phi'' u^2 - 3\left(3\phi'' + 5\phi' - 2\phi\right) u$$

$$+9{\phi''}^2 - 6\phi\phi'' - 243\phi'' + 4\ {\phi'}^2 + 243\phi' + 3\phi'\phi'' - 54\phi = 0$$

Then the function

$$y = -\frac{9{\phi''}^2 - 6\phi\phi'' + 4{\phi'}^2 + 3\phi'\phi\ ''}{3\left(9\phi'' - 9\phi' + 2\phi\right)} - u_1$$

of

$$x = \frac{u_3 - u_1}{u_2 - u_1}$$

satisfies

$$y'' = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]{y'}^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]y'$$
$$+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[9 + \frac{x(x-1)}{(y-x)^2}\right]$$

Question: what is special about this particular solution to P-VI? Can it be expressed via known functions? What are the corresponding monodromy data?

2.Frobenius manifolds

1.1 Frobenius algebra: a pair $(A, \langle\ ,\ \rangle)$

A commutative associative $\mathbb{C}$-algebra with a unit

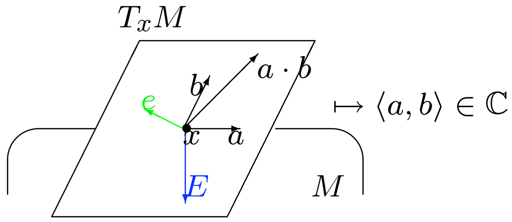
$\langle\ ,\ \rangle$ symmetric bilinear nondegenerate *invariant* form

$$\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$$

E.g., every semisimple Fr. algebra is isomorphic to

$$\begin{array}{rcl} e_i \cdot e_j & = & \delta_{ij} e_i \\ \langle e_i, e_j\rangle_{\mathbf{h}} & = & \delta_{ij} h_i \end{array} \qquad \mathbf{h} = (h_1, \dots, h_n), \quad h_i \neq 0 \quad \forall i$$

1.2 Frobenius manifold *M*: a structure of Frobenius algebra on the tangent spaces



"metric"

such that

FM1. The metric $\langle\ ,\ \rangle$ has zero curvature and $\nabla e = 0$

FM2. Define $C(X, Y, Z) := \langle X \cdot Y, Z \rangle$. Then $\nabla_W C(X, Y, Z)$ is symmetric in X, Y, Z, W

FM3. There exists a linear vector field *E*, i.e. $\nabla \nabla E = 0$

such that
$$\begin{aligned} Lie_E(\ \cdot\ ) &= \quad \cdot \\ Lie_E\langle\ ,\ \rangle &= (2-d)\langle\ ,\ \rangle \end{aligned}$$

Locally there exist *flat coordinates* $v^1, \ldots, v^n$

$$\left\langle \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta} \right\rangle = \eta_{\alpha\beta} \ (= \text{const})$$

$$e = \frac{\partial}{\partial v^1}$$

$$\exists\ F = F(v) \quad \text{such that} \quad \left\langle \frac{\partial}{\partial v^\alpha} \cdot \frac{\partial}{\partial v^\beta}, \frac{\partial}{\partial v^\gamma} \right\rangle = \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\gamma}$$

It satisfies WDVV associativity equations and quasihomogeneity

$$E = \sum \left(a^\alpha_\beta v^\beta + b^\alpha \right) \frac{\partial}{\partial v^\alpha}, \quad E\,F = (3-d)F + \text{at most quadratic terms}$$

In the example of quantum cohomology of $\mathbf{P}^2$

$$F = \frac{1}{2}\left[(v^1)^2 v^3 + v^1 (v^2)^2\right] + \sum_{k=1}^{\infty} \frac{N_k}{(3k-1)!}(v^3)^{3k-1} e^{k\,v^2}$$

$$\left\langle \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta} \right\rangle = \delta_{\alpha+\beta,4}$$

$$E = v^1 \frac{\partial}{\partial v^1} + 3\frac{\partial}{\partial v^2} - v^3 \frac{\partial}{\partial v^3}$$

WDVV associativity equation $\Leftrightarrow$ Kontsevich equation

Main ingredient of the theory of Frobenius manifolds

flat connection on $M \times \mathbb{C}^*$

$$\tilde\nabla_a b = \nabla_a b + z\, a \cdot b$$

$$\tilde\nabla_{\frac{d}{dz}} b = \partial_z b + E \cdot b - \frac{1}{z}\mu(b), \quad \mu = \frac{2-d}{2}\mathrm{id} - \nabla E$$

Observe: $\langle \mu(a), b\rangle = -\langle a, \mu(b)\rangle$

Flatness $\Rightarrow$ (local) existence of n independent

horizontal sections

$$\tilde\nabla df = 0$$

$(n = \dim M)$

On a **semisimple** Frobenius manifold existence of *canonical coordinates*

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i}$$

They also satisfy orthogonality $\left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle = 0 \quad \text{for} \quad i \neq j$

If FM3 holds true then $u^1, \dots, u^n$ are eigenvalues of the operator of multiplication by the Euler vector field

Orthonormalized idempotents

$$f_i = \frac{1}{\sqrt{\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i} \right\rangle}} \frac{\partial}{\partial u_i}, \quad i = 1, \ldots, n$$

Introduce matrices $U, \quad V$

$$U_{ij} = \langle f_i, E \cdot f_j \rangle = u_i \delta_{ij}$$

$$V_{ij} = \langle f_i, \mu(f_j) \rangle = -V_{ji}$$

(depend on the point of the Frobenius manifold)

The z-part of equations for horizontal sections of $\tilde{\nabla}$ reduces to

$$\boxed{\frac{dY}{dz} = \left(U + \frac{V}{z}\right) Y}$$

Thm. Flatness of $\tilde{\nabla}$ $\Rightarrow$ isomonodromicity

Invariant pairing

$$Y^T(-z)Y(z) = \text{const}$$

Remark. An alternative isomonodromy description for

$$(U - \lambda \cdot \mathrm{id}) \frac{d\Phi}{d\lambda} = \left(\frac{1}{2} - \nu + V \right) \Phi$$

$\Leftrightarrow$

$$\frac{d\Phi}{d\lambda} = \sum_{i=1}^{n} \frac{A_i}{\lambda - u_i} \Phi$$

Monodromy data of $\dfrac{dY}{dz} = \left(U + \dfrac{V}{z}\right) Y$

- at $z = 0$ $\quad Y_0 = (\Psi + \mathcal{O}(z))\, z^{\mu} z^{\rho}$

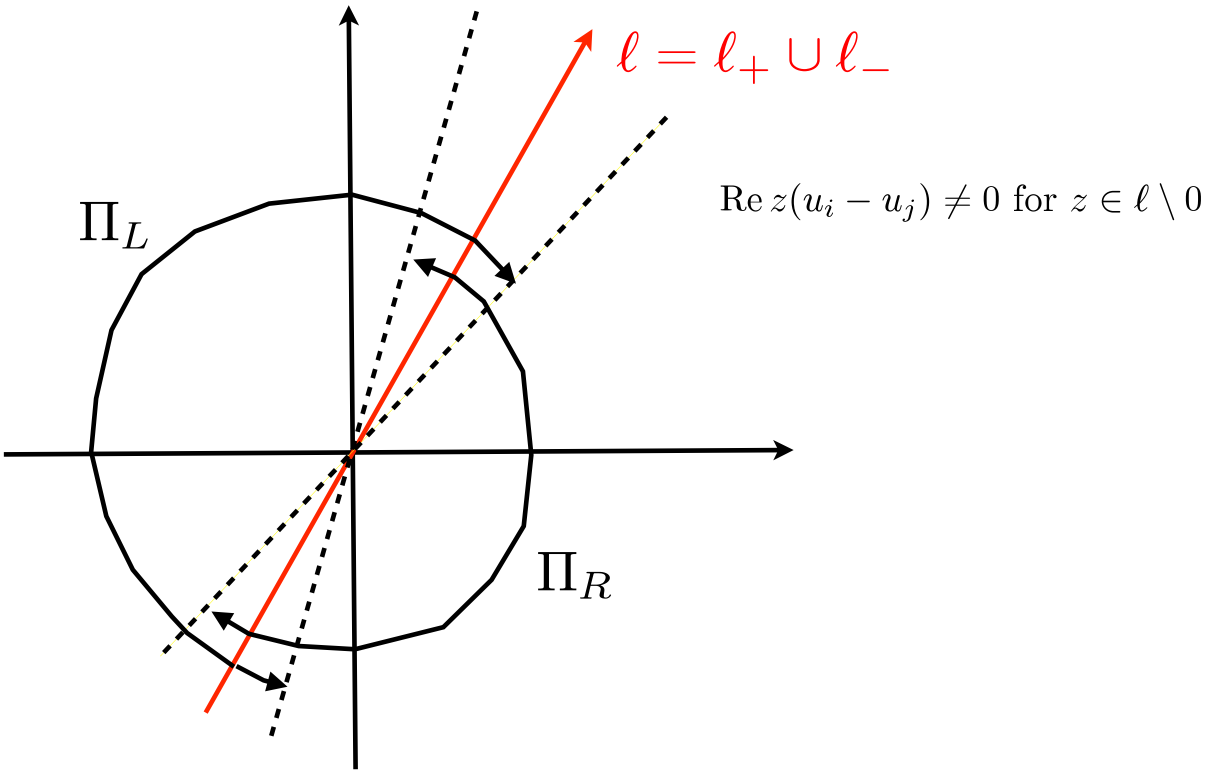
μ diagonal matrix, $\mu = \mathrm{diag}(\mu_1, \ldots, \mu_n), \quad \Psi^{-1} V \Psi = \mu$

ρ nilpotent matrix (in presence of resonances)

$\rho = \rho_0 + \rho_1 + \ldots, \quad (\rho_k)_{ij} \neq 0 \quad \text{only if} \quad \mu_i - \mu_j = k$

$$\langle \rho_k(a), b\rangle = (-1)^{k+1} \langle a, \rho_k(b)\rangle$$

- **at** $z = \infty$



$$Y_{R/L} \sim \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) e^{z\,U}, \quad |z| \to \infty, \quad z \in \Pi_{R/L}$$

Stokes matrix $Y_L(z) = Y_R(z) S, \quad z \in \ell_+$

$$S_{ii} = 1, \quad S_{ij} = 0 \quad \text{for} \quad i > j \quad (\text{after ordering } u_i)$$

- central connection matrix C

$$Y_R(z) = Y_0(z)C$$

Constraint

$$S^T = C^T \eta \, e^{\pi i \rho} e^{\pi i \mu} C$$

$\Rightarrow$ cyclic relation

$$S^T S^{-1} = C^T \eta e^{2\pi i \rho} e^{2\pi i \mu} \eta^{-1} \left(C^T\right)^{-1}$$

(monodromy around infinity = monodromy around 0)

Isomonodromicity Thm. The data (μ, ρ, S, C)

are locally independent from the point of Frobenius manifold

Reconstruction by solving a Riemann - Hilbert problem

$$(\mu, \rho, S, C) \to \big(Y_0(u,z), Y_{R/L}(u,z)\big) \to (\Psi(u), V(u)) \to Fr\,(\mu, \rho, S, C; \mu_1)$$

(depends on the choice of an eigenvalue μ_1)

Global structure

$$M = \bigcup_{\sigma \in \mathrm{Br}_n/\text{stabilizer of } (S,C)} Fr\left(\mu, \rho, S^{\sigma}, C^{\sigma}, \mu_1\right)$$

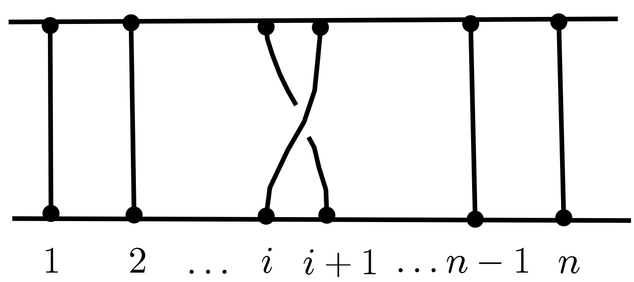
Action of the braid group Br_n $\quad S \mapsto S^{\sigma}, \quad C \mapsto C^{\sigma}$

$$S^{\sigma_i} = K_i S\, K_i, \quad C^{\sigma_i} = C\, K_i, \quad i = 1, \ldots, n-1$$

$(K_i)_{kk} = 1, \quad k \neq i,\ i+1$

$(K_i)_{ii} = -s_{i,i+1}, \quad (K_i)_{i,i+1} = (K_i)_{i+1,i} = 1$

all other entries equal 0



σ_i

Thm. Any semisimple Frobenius manifold can be obtained by this construction

Thus, semisimple Frobenius manifolds parameterized by the monodromy data (μ, ρ, S, C) satisfying above properties

Charts of the semisimple Frobenius manifold are labeled by points in the orbit of the monodromy data with respect to the action of the braid group

Quantum cohomology

X smooth projective variety, $\dim_{\mathbb{C}} X = d, \quad H^{\mathrm{odd}}(X) = 0$

$\overline{M}_{0,m,\beta}(X)$ moduli space of stable rational maps

$$f : \left(\mathbf{P}^1, x_1, \ldots, x_m\right) \to X$$

of degree $\beta = f_*[\mathbf{P}^1] \in H_2(X;\mathbb{Z})$

$x_1, \ldots, x_m \in \mathbf{P}^1$ marked points, $x_i \neq x_j$

Evaluation maps

$$\mathrm{ev}_i : \overline{M}_{0,m,\beta}(X) \to X, \quad f \mapsto f(x_i)$$

n-dimensional Frobenius manifold, $n = \dim H^*(X)$

Choose a basis $\gamma_1 = 1, \gamma_2, \ldots, \gamma_n \in H^*(X), \quad \deg \gamma_i = 2q_i$

Potential of the Frobenius manifold $M = QH^*(X)$

$$F(v) = \sum_m \sum_{\alpha_1,\ldots,\alpha_m} \frac{v^{\alpha_1}\ldots v^{\alpha_m}}{m!} \sum_{\beta \in H_2(X;\mathbb{Z})} \int_{\overline{M}_{0,m,\beta}(X)} \mathrm{ev}_1^*(\gamma_{\alpha_1})\ldots \mathrm{ev}_m^*(\gamma_{\alpha_m})$$

Thm. (Kontsevich, Manin) Triple derivatives $c_{\alpha\beta}^{\gamma}(v) = \eta^{\gamma\,\delta} \frac{\partial^3 F(v)}{\partial v^\delta \partial v^\alpha \partial v^\beta}$

are structure constants of a family of associative algebras

$\eta_{\alpha\beta} = \int_X \gamma_\alpha \wedge \gamma_\beta$ invariant inner product

Euler vector field $E = \sum_1^n \left[(1-q_\alpha)v^\alpha + \langle c_1(X), \gamma^\alpha\rangle\right] \frac{\partial}{\partial v^\alpha}$

Basis in $H^*(X)$ $\Rightarrow$ basis of horizontal sections of $\tilde{\nabla}$ near $z=0$ (hence the fundamental matrix $Y_0(z)$)

Define

$$\theta_i(z) = \sum_m \sum_{\alpha_1,\dots,\alpha_m} \frac{v^{\alpha_1}\dots v^{\alpha_m}}{m!} \sum_{\beta\in H_2(X;\mathbb{Z})} \int_{\overline{M}_{0,m+1,\beta}(X)} \mathrm{ev}_1^*(\gamma_{\alpha_1})\dots \mathrm{ev}_m^*(\gamma_{\alpha_m}) \frac{\mathrm{ev}_{m+1}^*(\gamma_i)}{1-z\,\psi_{m+1}}$$

$$\psi_{m+1} = c_1\left(\mathcal{L}_{m+1}\right), \quad \text{tautological line bundle} \qquad \begin{array}{c}\mathcal{L}_{m+1}\\ \downarrow\ T^*_{x_{m+1}}\mathbf{P}^1\\ \overline{M}_{0,m+1,\beta}(X)\end{array}$$

The basis $(\theta_1(z),\dots,\theta_n(z))\, z^\mu z^\rho$

$$\mu = \frac{1}{2}\deg - \frac{d}{2}\mathrm{id}, \quad \rho = c_1(X)\cdot$$

$$\mu,\ \rho : H^*(X)\to H^*(X)$$

Basis of horizontal sections of $\tilde{\nabla}$ near $z = \infty$

from a "basis" $E_1, \dots, E_n \quad \text{in} \quad K(X)$

When quantum cohomology is semisimple?

Conjecture (B.D. '98; A.Bayer & Yu.I.Manin '01)

Semisimplicity of $QH^*(X) \quad \Leftrightarrow$ existence of a full exceptional collection $E_1, \dots, E_n \quad \text{in} \quad Der^b(X)$ in the derived category of coherent sheafs on X

$Der^b(X)$ complexes of coherent sheaves on X

$$\dots \to C_i \to C_{i-1} \to \dots$$

up to quasiisomorphisms (introduced by J.-L.Verdier)

Ordered collection of objects $E_1, \dots, E_n \in Der^b(X)$

is *exceptional* if $Ext^k(E_i, E_j) = 0, \quad k > 0, \quad \forall\, i,\, j$

$$Hom(E_i, E_i) = \mathbb{C}, \quad i = 1, \dots, n$$
$$Hom(E_i, E_j) = 0 \quad \text{for} \quad i > j$$

It is *full* if $E_1, \dots, E_n$ generate $Der^b(X)$

Example (A.Beilinson '78) $X = \mathbf{P}^d$

Full exceptional collection $\mathcal{O},\ \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(d)$

General results: for toric varieties

Existence of full exceptional collections (Y.Kawamata, '05)

Semisimplicity of quantum cohomology (H.Iritani, '05)

Question 2: description of the monodromy data of $QH^*(X)$ in the semisimple case

Conjecture, Part 2 (B.D. '98). Basic vectors of horizontal sections of $\tilde{\nabla}$ near $z=\infty$ are in one-to-one correspondence with objects $E_1, \dots, E_n \in Der^b(X)$ of a full exceptional collection. The Stokes matrix in this basis coincides with the Gram matrix of Mukai pairing

$$S_{ij} = \chi(E_i^* \otimes E_j) = \sum_{k\geq 0} (-1)^i \dim Ext^k(E_i, E_j)$$

Motivations: S.Cecotti, C.Vafa; E.Zaslow; M.Kontsevich

Particular cases: B.D., D.Guzzetti, K.Ueda, A.Takahashi; B.Kim, C.Sabbah

Question 3: central connection matrix C

Define an operator

$$\text{vector bundles on } X \to \text{cohomology of } X$$

$$E \mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \Gamma(X) \wedge \tilde{\mathrm{ch}}(E)$$

Here $\Gamma(X)$ is the gamma-genus of TX

$\tilde{\mathrm{ch}}(E)$ is the Chern character of E

followed by the operator $(2\pi i)^{\frac{\deg}{2}}$

Gamma-genus

Recall construction of Hirzebruch characteristic classes

Given a formal series $K(t) = 1 + a_1 t + a_2 t^2 + \dots$

and a d-dimensional vector bundle E over X define

$$K(E) = K(t_1)\dots K(t_d) = 1 + a_1(t_1 + \dots + t_d) + a_2(t_1^2 + \dots + t_d^2) + a_1^2(t_1 t_2 + \dots + t_{d-1} t_d) + \dots$$

$$= 1 + a_1 c_1(E) + a_2 c_1^2(E) + (a_1^2 - 2a_2) c_2(E) + \dots \in H^*(X)$$

For gamma-genus take $K(t) = \Gamma(1-t) = 1 + \gamma\, t + \mathcal{O}(t^2)$

Notice $$\log \Gamma(1-t) = \gamma\, t + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} t^n$$

Conjecture, Part 3.

Bases of horizontal sections near a semisimple point in $QH^*(X$

- at $z = 0$: from a basis $\gamma_1, \ldots, \gamma_n \in H^*(X)$

- at $z = \infty$: from a full exceptional collection $E_1, \ldots, E_n$

  in $Der^b(X)$

Central connection matrix $C = (C_{ij})$

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \Gamma(X) \wedge \tilde{\mathrm{ch}}(E_j) = \sum_{i=1}^{n} C_{ij} \gamma_i$$

Motivations: H.Iritani '08,
L.Katzarkov, M.Kontsevich, T.Pantev '08

Verified for $X = \mathbf{P}^d$ (B.D., using results by D.Guzzetti

monodromy of $(z\,\partial_z)^{d+1}\,\Phi = [(d+1)z]^{d+1}\Phi$)

More recently: for Grassmannians (S.Galkin, V.Golyshev, H.Iritani); G.Cotti for $G_{2,4}$)

$$(z\,\partial_z)^5\Phi - 2^{10}z^5\partial_z\Phi - 2^{11}z^4\Phi = 0$$

Global structure of $QH^*(X)$

Action of the braid group Br_n on the monodromy data

$$S,\ C \mapsto S^\sigma,\ C^\sigma$$

by analytic continuation of isomonodromy deformations

corresponds to the action of Br_n on full exceptional

collections by *mutations*

$$(E_1, \ldots, E_n) \mapsto (E_1^\sigma, \ldots, E_n^\sigma)$$

(J.-M.Drézet & J.Le Potier '85;
A.Gorodentsev & A.Rudakov '87)

Question: are the charts in

$$QH^*(X) = \bigcup_{\sigma \in \mathrm{Br}_n / \text{stabilizer of } (S,C)} Fr\left(\mu, \rho, S^\sigma, C^\sigma, \mu_1\right)$$

in one-to-one correspondence with full exceptional collections in $Der^b(X)$?

Thank you!